CLASSIFICATION OF ALL Δ-COHERENT PAIRS

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In a recent work we have proved that if $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ is a Δ-coherent pair of linear functionals, then at least one of them must be a classical discrete linear functional under certain conditions. In this paper we present a similar result for D_ω-coherent pairs and the description of all Δ-coherent pairs. Furthermore, by using a limit process we recover the classification of all coherent pairs of positive definite linear functionals.

KEY WORDS: orthogonal polynomials of a discrete variable, linear functionals, coherent pairs

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1. INTRODUCTION

The concepts of coherent pair and symmetric coherent pair have been introduced by A. Iserles *et al.* in [7] in the framework of the study of orthogonal polynomials associated with the Sobolev inner product

$$\langle f, g\rangle_S = \int_{\mathbb{R}} f g \, d\mu_0 + \lambda \int_{\mathbb{R}} f' g' d\mu_1, \tag{1}$$

where μ_0 and μ_1 are non-atomic positive Borel measures on the real line such that

$$\left| \int_{\mathbb{R}} x^k d\mu_i(x) \right| < \infty, \qquad k \geq 0, \qquad i = 0, 1.$$

In fact, coherence means that a relation between the MOPS (monic orthogonal polynomial sequence) $\{P_n(x)\}_n$ and $\{T_n(x)\}_n$, associated with the measures μ_0 and μ_1 respectively,

$$T_n(x) = \frac{P'_{n+1}(x)}{n+1} - \sigma_n \frac{P'_n(x)}{n}, \qquad n \geq 1,$$

where $\{\sigma_n\}_n$ is a sequence of non-zero complex numbers, is satisfied (see also [10, 11]).

In a similar way [3], we have introduced the concept of Δ-coherent pairs: Let u_0 and u_1 be weakly quasi-definite linear functionals of order M_0 and M_1, respectively, and denote by $\{P_n(x)\}_{n=0}^{M_0}$ and $\{T_n(x)\}_{n=0}^{M_1}$ the corresponding MOPS associated with u_0 and u_1, respectively. We say that (u_0, u_1) is a Δ-coherent pair of linear functionals if and only if

$$T_n(x) = \frac{\Delta P_{n+1}(x)}{n+1} - \sigma_n \frac{\Delta P_n(x)}{n}, \qquad 1 \leq n \leq \min\{M_0 - 1, M_1\},$$

where Δ stands for the forward difference operator $\Delta y(x) = y(x+1) - y(x)$ and $\{\sigma_n\}_n$ is a sequence of non-zero complex numbers. Moreover, in [3] we have proved that if (u_0, u_1) is a Δ-coherent pair of linear functionals at least one of the functionals must be a classical discrete linear functional. Δ-coherent pairs of linear functionals are a rather useful tool when dealing with approximation over discrete sets.

The aim of our present contribution is to determine all Δ-coherent pairs of linear functionals. Moreover, we introduce and characterize the D_ω-coherent pairs of linear functionals, being

$$D_\omega f = \frac{f(x+\omega) - f(x)}{\omega}, \qquad \omega \neq 0.$$

By using limit properties for linear functionals, the classification given by Meijer in the continuous case [9] is reached. In this way we cover two goals. The first one is to present the concept of coherence for two measures as an inverse problem relating two sequences of orthogonal polynomials. Despite the origin of the coherence in the framework of Sobolev inner products, notice that it is basically a problem concerning the standard orthogonality. The next step, of course, is the analysis of the corresponding Sobolev orthogonal polynomials when (u_0, u_1) is a Δ-coherent pair of positive linear functionals. Some results have been obtained when $u_0 = u_1$ is the Meixner linear functional [4]. The second goal is to recover the continuous case from the discrete case. In some sense, we follow the same approach as the stated in [2]. There, the authors have obtained the Krall-type polynomials as a limit case of perturbed classical discrete orthogonal polynomials, by adding mass points at the end of the interval of orthogonality.

The structure of the paper is as follows. In Section 2 we give the basic definitions, notations and auxiliary results. In Section 3 we present the classification of all Δ-coherent pairs. Finally, in Section 4, by using limit properties of D_ω-coherent pairs we recover the classification of coherent pairs of linear functionals given by Meijer in [9].

2. AUXILIARY RESULTS

Let $\mathbb{P}$ be the linear space of polynomials with complex coefficients and let $\mathbb{P}'$ be its algebraic dual space. We denote $\langle \boldsymbol{u}, p\rangle$ the duality bracket for $\boldsymbol{u} \in \mathbb{P}'$ and $p \in \mathbb{P}$, and we denote by $(\boldsymbol{u})_n = \langle \boldsymbol{u}, x^n\rangle$, with $n \geq 0$, the canonical moments of $\boldsymbol{u}$.

Definition 1. A linear functional $\boldsymbol{u} : \mathbb{P} \mapsto \mathbb{C}$ is said to be weakly quasi-definite if there exists $1 \leq M \leq \infty$ such that the principal submatrices $H_k = [(\boldsymbol{u}_{i+j})]_{i,j=0}^{k}$ are non singular for $1 \leq k \leq M$ and, if $M \neq \infty$, H_{M+1} is a singular matrix. We shall call M de order of the linear functional $\boldsymbol{u}$.

Note that when $M = \infty$ this definition coincides with the concept of quasi-definite linear functional given in [5, p. 16].

Given a weakly quasi-definite linear functional $\boldsymbol{u}$, there exists a family of monic polynomials $\{P_n(x)\}_{n=0}^{M}$ orthogonal with respect to $\boldsymbol{u}$, i.e. $P_n(x) = x^n +$ terms of lower degree, for every $0 \leq n \leq M$, and $\langle \boldsymbol{u}, P_n P_m\rangle = \Gamma_n \delta_{n,m}$, $\Gamma_n \neq 0$, for every $0 \leq n,\ m \leq M$. Such a sequence will be called monic orthogonal polynomial sequence (MOPS).

Definition 2. Given a complex number c, the Dirac functional $\delta_c \equiv \boldsymbol{\delta}(c)$ is defined by $\langle \boldsymbol{\delta}_c, p\rangle := p(c)$, for $p \in \mathbb{P}$.

Definition 3. Let $\boldsymbol{u}$ be a linear functional. For each polynomial p we define the linear functional $p(x)\boldsymbol{u}$ as follows, $\langle p\boldsymbol{u}, q\rangle := \langle \boldsymbol{u}, pq\rangle$, for every $q \in \mathbb{P}$. For each complex number c we introduce the linear functional $(x-c)^{-1}\boldsymbol{u}$ by means of $\left\langle (x-c)^{-1}\boldsymbol{u}, q\right\rangle := \left\langle \boldsymbol{u}, \frac{q(x)-q(c)}{x-c}\right\rangle$, for every $q \in \mathbb{P}$.

Note that $(x-c)^{-1}((x-c)\,\boldsymbol{u}) = \boldsymbol{u} - (\boldsymbol{u})_0\delta_c$, while $(x-c)((x-c)^{-1}\boldsymbol{u}) = \boldsymbol{u}$.

Definition 4. Let ω be a nonzero real number. The difference operator D_ω is defined by

$$D_\omega p\,(x) := \frac{p\,(x+\omega) - p\,(x)}{\omega}, \qquad \forall p \in \mathbb{P}. \tag{3}$$

When $\omega = 1$, D_1 becomes the forward difference operator Δ.

In what follows we shall always assume that $\omega \neq 0$.

Definition 5. For $\boldsymbol{u} \in \mathbb{P}'$, we introduce the linear functional $D_\omega \boldsymbol{u}$ as $\langle D_\omega \boldsymbol{u}, p\rangle := -\langle \boldsymbol{u}, D_\omega p\rangle$ for every $p \in \mathbb{P}$.

It is easy to check that $D_\omega[p\boldsymbol{u}] = p\,(x-\omega)\, D_\omega \boldsymbol{u} + D_\omega p\,(x-\omega)\,\boldsymbol{u}$, for every $p \in \mathbb{P}$ and $\boldsymbol{u} \in \mathbb{P}'$.

Definition 6. A linear functional u is said to be a classical discrete linear functional if u is weakly quasi-definite and there exist polynomials ϕ and ψ, with $\deg(\phi) \leq 2$ and $\deg(\psi) = 1$ such that

$$\Delta[\phi u] = \psi u. \tag{4}$$

The corresponding MOPS associated with u is said to be a classical discrete MOPS.

Classical discrete linear functionals are the corresponding to Hahn, Meixner, Kravchuk and Charlier MOPS [6]. Charlier and Meixner functionals are quasi-definite linear functionals [5] and therefore the corresponding MOPS is an infinite sequence. On the other hand, Kravchuk and Hahn linear functionals are weakly quasi-definite linear functionals because they satisfy a finite orthogonality relation and the corresponding functionals have a finite supporting set. For classical discrete polynomial sequences we give in Table 1 the polynomials ϕ and ψ appearing in the distributional equation (4).

If u is a classical discrete linear functional, then it can be represented as

$$\langle u, p\rangle = \sum_{x_i=a}^{b-1} p(x_i)\,\varrho(x_i), \qquad x_{i+1} = x_i + 1, \qquad \forall p \in \mathbb{P}, \tag{5}$$

being a, b nonnegative integer numbers (or $b = +\infty$). Explicit expressions for the weight function ϱ are given in Table 2 for each classical discrete family [12].

Let $\{R_n(x)\}_n$ be a classical discrete MOPS satisfying a three-term recurrence relation

$$\begin{gathered} xR_n(x) = R_{n+1}(x) + \beta_n R_n(x) + \gamma_n R_{n-1}(x) \\ R_{-1}(x) = 0, \qquad R_0(x) = 1, \qquad \gamma_n \neq 0, \end{gathered} \tag{6}$$

such that the linear functional u for which $\{R_n(x)\}_n$ is orthogonal satisfies the distributional equation (4). If we define a new sequence of monic polynomials $\{S_n(x)\}_n$ by means of

$$S_n(x) = \omega^n R_n\left(\frac{x}{\omega}\right), \tag{7}$$

then $\{S_n(x)\}$ also satisfies the three-term recurrence relation

$$xS_n(x) = S_{n+1}(x) + \overline{\beta}_n S_n(x) + \overline{\gamma}_n S_{n-1}(x), \tag{8}$$

with $\overline{\beta}_n = \omega\beta_n$, $\overline{\gamma}_n = \omega^2\gamma_n$, where β_n and γ_n are given in (6). Moreover, $\{S_n(x)\}_n$ is the MOPS for a linear functional u_ω which satisfies the distributional equation

$$D_\omega[\phi_\omega u_\omega] = \psi_\omega u_\omega, \qquad \text{with} \qquad \phi_\omega(x) = \phi\left(\frac{x}{\omega}\right), \qquad \psi_\omega(x) = \frac{1}{\omega}\psi\left(\frac{x}{\omega}\right). \tag{9}$$

If the linear functional u is defined by (5) then the linear functional u_ω is given by

Table 1. Polynomials in the distributional equation $\Delta[\phi u] = \psi u$ for each classical discrete family

$P_n(x)$	$\phi(x)$	$\psi(x)$
Charlier $c_n^{(\mu)}(x)$ $(\mu > 0)$	μ	$\mu - x$
Meixner $m_n^{(\gamma,\mu)}(x)$ $(\gamma > 0, \mu \in (0,1))$	$\mu(\gamma + x)$	$\mu\gamma - x(1-\mu)$
Kravchuk $k_n^{(p)}(x; N)$ $(p \in (0,1), N \in \mathbb{Z}^+)$	$N - x$	$\dfrac{Np - x}{p}$
Hahn $h_n^{(\alpha,\beta)}(x; N)$ $(\alpha, \beta > -1, N \in \mathbb{Z}^+)$	$(N-x-1)(x+\beta+1)$	$(N-1)(\beta+1)-x(\alpha+\beta+2)$

Table 2. Weight function ϱ for each classical discrete family

$P_n(x)$	$\varrho(x)$	x
Charlier $c_n^{(\mu)}(x)$ $(\mu > 0)$	$\dfrac{e^{-\mu}\,\mu^x}{\Gamma(x+1)}$	$\mathbb{N}$
Meixner $m_n^{(\gamma,\mu)}(x)$ $(\gamma > 0, \mu \in (0,1))$	$\dfrac{\mu^x(1-\mu)^\gamma\Gamma(x+\gamma)}{\Gamma(x+1)\Gamma(\gamma)}$	$\mathbb{N}$
Kravchuk $k_n^{(p)}(x; N)$ $(p \in (0,1), N \in \mathbb{Z}^+)$	$\binom{N}{x} p^x (1-p)^{N-x}$	$\{0, 1, \ldots, N\}$
Hahn $h_n^{(\alpha,\beta)}(x; N)$ $(\alpha, \beta > -1, N \in \mathbb{Z}^+)$	$\dfrac{\Gamma(N)\,\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\alpha+\beta+N+1)}$ $\times \dfrac{\Gamma(\alpha+N-x)\,\Gamma(\beta+x+1)}{\Gamma(N-x)\,\Gamma(x+1)}$	$\{0, 1, 2, \ldots, N-1\}$

$$\langle \boldsymbol{u}_\omega, p\rangle = \sum_{x_i=a}^{b-1} \varrho(x_i)\, p(\omega\, x_i), \qquad \forall p \in \mathbb{P}, \tag{10}$$

being ϕ and ψ the polynomials relative to the linear functional $\boldsymbol{u}$ in the distributional equation (4).

Moreover, if we consider

$$\overset{\vee}{S}_n(x) = S_n(x+\alpha) = \omega^n R_n\left(\frac{x}{\omega}+\alpha\right), \qquad \alpha \in \mathbb{C}, \tag{11}$$

the sequence $\{\overset{\vee}{S}_n(x)\}_n$ is orthogonal with respect to the linear functional $\overset{\vee}{\boldsymbol{u}}_\omega$ defined by

$$\langle \overset{\vee}{\boldsymbol{u}}_\omega, p\rangle = \sum_{x_i=a}^{b-1} \varrho(x_i)\, p(\omega\, x_i - \alpha), \tag{12}$$

which satisfies

$$D_\omega\left[\overset{\vee}{\phi}_\omega \overset{\vee}{\boldsymbol{u}}_\omega\right] = \overset{\vee}{\psi}_\omega \overset{\vee}{\boldsymbol{u}}_\omega, \quad \text{with} \quad \overset{\vee}{\phi}_\omega(x) = \phi_\omega(x+\alpha), \quad \overset{\vee}{\psi}_\omega(x) = \psi_\omega(x+\alpha). \tag{13}$$

Definition 7. If $\{R_n(x)\}_n$ is a classical discrete MOPS and $\{\overset{\vee}{S}_n(x)\}_n$ is the sequence of polynomials defined in (7), we shall say that $\{\overset{\vee}{S}_n(x)\}_n$ is a ω-classical discrete MOPS. If $\overset{\vee}{\boldsymbol{u}}_\omega$ is the linear functional satisfying (13), then $\overset{\vee}{\boldsymbol{u}}_\omega$ is said to be a ω-classical discrete linear functional.

Let us introduce an extension of the concept of Δ-coherent pairs, which will be used later.

Definition 8. Let $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ be two weakly quasi-definite linear functionals whose MOPS are $\{P_n(x)\}_{n=0}^{M_0}$ and $\{T_n(x)\}_{n=0}^{M_1}$, respectively. We define $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ as a D_ω-coherent pair of linear functionals if

$$T_n(x) = \frac{D_\omega P_{n+1}(x)}{n+1} - \sigma_n \frac{D_\omega P_n(x)}{n}, \qquad 1 \le n \le \min\{M_0 - 1, M_1\}, \tag{14}$$

where $\{\sigma_n\}_n$ is a sequence of non-zero complex numbers.

The following theorem characterizes the D_ω-coherent pairs of linear functionals.

Theorem 1. *Let $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ be a D_ω-coherent pair of linear functionals, whose MOPS are $\{P_n(x)\}_{n=0}^{M_0}$ and $\{T_n(x)\}_{n=0}^{M_1}$, respectively, where $M_1 > 7$. Let us consider the polynomial*

$$B_2(x) = c_1(x-\omega)\, D_\omega c_2(x-\omega) - c_2(x-\omega)\, D_\omega c_1(x-\omega) = \frac{\sigma_1\sigma_2}{t_1 t_2}(x-\xi)(x-\eta), \tag{15}$$

where

$$c_n(x) = \sigma_n \frac{T_n(x)}{t_n} - \frac{T_{n-1}(x)}{t_{n-1}}, \qquad t_n = \langle \boldsymbol{u}_1, T_n^2 \rangle, \qquad 1 \le n \le \min\{M_0 - 1, M_1\}. \tag{16}$$

One of the following situations hold

1. *If* $\xi = \eta + \omega$, *then* $\boldsymbol{u}_0$ *is a* ω*-classical discrete linear functional satisfying* $D_\omega[\widetilde{\phi}(x-\omega)\,\boldsymbol{u}_0] = \widetilde{\psi}_0(x)\,\boldsymbol{u}_0$. *Moreover,*

$$\widetilde{\phi}(x-\omega)\,\boldsymbol{u}_0 = \frac{\sigma_1\sigma_2}{t_1 t_2}(x-\xi)\,\boldsymbol{u}_1. \tag{17}$$

2. *If* $\xi \neq \eta$ *and* $\xi \neq \eta + \omega$, *then* $\boldsymbol{u}_1$ *is a* ω*-classical discrete linear functional verifying* $D_\omega[\widetilde{A}(x+\omega)\,\boldsymbol{u}_1] = \widetilde{\psi}_1(x)\,\boldsymbol{u}_1$. *Furthermore,*

$$\widetilde{A}(x)\,\boldsymbol{u}_0 = \frac{\sigma_1\sigma_2}{t_1 t_2}(x-\xi)\,\boldsymbol{u}_1, \qquad \pi_1(x)\,\boldsymbol{u}_0 = \frac{\sigma_1\sigma_2}{t_1 t_2}(x-\xi)\,D_\omega \boldsymbol{u}_1, \tag{18}$$

where $\pi_1(x) = \widetilde{\psi}_1(x) - D_\omega \widetilde{A}(x)$. *Finally, if* $\widetilde{A}(\xi) = 0$ *then* $\pi_1(\xi) = 0$.

The proof of this theorem is essentially the same as in [3] for Δ-coherent pairs following the scheme given in [9].

3. CLASSIFICATION OF ALL Δ-COHERENT PAIRS

In this Section, we shall assume that $\omega = 1$ and $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ is a Δ-coherent pair of linear functionals, being $\{P_n(x)\}_n$ and $\{T_n(x)\}_n$ the corresponding MOPS associated with $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$, respectively. From now on, only non-negative definite linear functionals will be considered.

Theorem 1 proves that if $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ is a Δ-coherent pair of linear functionals, then at least one of the functionals must be a classical discrete linear functional, i.e. Charlier, Meixner, Kravchuk or Hahn.

Let us show that the Δ-coherent pairs of linear functionals have a similar classification to the one given in [9] for the continuous case. The classification of all D_ω-coherent pairs could just be done in the same way taking into account the Definition 7.

3.1. $\boldsymbol{u}_0$ *is a classical discrete linear functional*

If the zeros ξ and η of the polynomial $B_2(x)$ in (15) are related by $\eta = \xi + 1$, by using Theorem 1 (with $\omega = 1$) we know that $\boldsymbol{u}_0$ is a classical discrete linear functional. Thus, we obtain

$$(x-\xi)\,\boldsymbol{u}_1 = \phi_0(x)\,\boldsymbol{u}_0, \tag{19}$$

$$\Delta[\phi_0(x)\,\boldsymbol{u}_0] = \psi_0(x)\,\boldsymbol{u}_0. \tag{20}$$

In Theorem 1, ξ may be complex, but in the computations below we always assume ξ to be real.

3.1.1. *Charlier case* $\left(c_n^{(\mu)}(x),\ \mu > 0\right)$. If $\boldsymbol{u}_0$ is the Charlier linear functional with $\phi_0(x) = \mu$, $\psi_0(x) = \mu - x$ then we obtain the following Δ-coherent pairs $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ where

$$\boldsymbol{u}_1 = \frac{\mu}{x-\xi}\,\boldsymbol{u}_0 + K\boldsymbol{\delta}(\xi), \qquad K \geq 0, \qquad \xi \leq 0, \tag{21}$$

i.e.,

$$\langle \boldsymbol{u}_1, p\rangle = \sum_{n=0}^{\infty} \left(\frac{p(n)-p(\xi)}{n-\xi}\right) \frac{e^{-\mu}\mu^{n+1}}{n!} + Kp(\xi), \qquad \forall p \in \mathbb{P}.$$

It is easy to check that these indeed are Δ-coherent pairs. Since

$$\begin{aligned}\left\langle \phi_0(s)\,\boldsymbol{u}_0, T_n(s)\,c_k^{(\mu)}(s)\right\rangle &= \left\langle (s-\xi)\,\boldsymbol{u}_1, T_n(s)\,c_k^{(\mu)}(s)\right\rangle \\ &= \left\langle \boldsymbol{u}_1, T_n(s)\,c_k^{(\mu)}(s)(s-\xi)\right\rangle = 0\end{aligned}$$

if $0 \leq k \leq n-2$, then the Δ-coherence of (21) follows from

$$T_n(s) = c_n^{(\mu)}(s) - \sigma_n c_{n-1}^{(\mu)}(s) = \frac{\Delta c_{n+1}^{(\mu)}(s)}{n+1} - \sigma_n \frac{\Delta c_n^{(\mu)}(s)}{n}, \qquad 1 \leq n \leq M_1,$$

where the last equality is a consequence of [12, p. 36, Eq. (2.4.17)] in the monic form.

3.1.2. *Meixner case* $\left(m_n^{(\gamma,\mu)}(x),\ \gamma > 0,\ \mu \in (0,1)\right)$. If $\boldsymbol{u}_0 \equiv \boldsymbol{u}^{(\gamma,\mu)}$ is the Meixner linear functional then $\phi_0(x) = \mu(x+\gamma)$ and $\psi_0(x) = \gamma\mu - x(1-\mu)$. From (19), $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ are Δ-coherent pairs where

$$\boldsymbol{u}_1 = \frac{1}{x-\xi}\,\boldsymbol{u}^{(\gamma+1,\mu)} + K\boldsymbol{\delta}(\xi), \qquad K \geq 0, \qquad \xi \leq 0. \tag{22}$$

Let us check that $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ is a Δ-coherent pair. Since

$$\left\langle \phi_0(s)\,\boldsymbol{u}_0, T_n(s)\,m_k^{(\gamma+1,\mu)}(s)\right\rangle = \left\langle \boldsymbol{u}_1, T_n(s)\,m_k^{(\gamma+1,\mu)}(s)(s-\xi)\right\rangle = 0$$

if $0 \leq k \leq n-2$, then

$$T_n(s) = m_n^{(\gamma+1,\mu)}(s) - \sigma_n m_{n-1}^{(\gamma+1,\mu)}(s) = \frac{\Delta m_{n+1}^{(\gamma,\mu)}(s)}{n+1} - \sigma_n \frac{\Delta m_n^{(\gamma,\mu)}(s)}{n},$$
$$1 \leq n \leq M_1,$$

where the last equality is a consequence of [12, p. 36, Eq. (2.4.16)] in the monic form.

3.1.3. *Kravchuk case* $\left(k_n^{(p)}(x;N),\ N\in\mathbb{Z}^+,\ p\in(0,1)\right)$. If we denote $\boldsymbol{u}_0\equiv \boldsymbol{u}^{(p,N)}$ the Kravchuk linear functional (weakly quasi-definite of order N) which satisfies the equation (20) with $\phi_0(x)=N-x$ and $\psi_0(x)=\dfrac{Np-x}{p}$, then from (19) we obtain that $(\boldsymbol{u}_0,\boldsymbol{u}_1)$ is a Δ-coherent pair where

$$\boldsymbol{u}_1=\frac{1}{|x-\xi|}\,\boldsymbol{u}^{(p,N-1)}+K\boldsymbol{\delta}(\xi),\qquad K\geq 0,\qquad \xi\leq 0,\qquad \text{or}\qquad \xi\geq N-1. \quad (23)$$

Let us check that $(\boldsymbol{u}_0,\boldsymbol{u}_1)$ is a Δ-coherent pair of linear functionals. Since

$$\phi_0(x)\,\boldsymbol{u}_0=(N-x)\,\boldsymbol{u}^{(p,N)}=\boldsymbol{u}^{(p,N-1)},$$

then

$$\begin{aligned}\left\langle \boldsymbol{u}^{(p,N-1)},T_n(s)\,k_r^{(p)}(s;N-1)\right\rangle &= \left\langle \phi_0(s)\,\boldsymbol{u}_0,T_n(s)\,k_r^{(p)}(s;N-1)\right\rangle\\ &= \left\langle \boldsymbol{u}_1,T_n(s)\,k_r^{(p)}(s;N-1)(s-\xi)\right\rangle=0\end{aligned}$$

if $0\leq r\leq n-2$, and we can write

$$T_n(s)=k_n^{(p)}(s;N-1)-\sigma_n k_{n-1}^{(p)}(s;N-1)=\frac{\Delta k_{n+1}^{(p)}(s;N)}{n+1}-\sigma_n\frac{\Delta k_n^{(p)}(s;N)}{n},$$

$$1\leq n\leq \min\{M_1,N-1\},$$

where the last equality is a consequence of [12, p. 36, Eq. (2.4.15)] in the monic form.

3.1.4. *Hahn case* $\left(h_n^{(\alpha,\beta)}(x;N),\ \alpha,\beta>-1,\ N\in\mathbb{Z}^+\right)$. Let us consider $\boldsymbol{u}_0=\boldsymbol{u}^{(\alpha,\beta,N)}$ the Hahn linear functional (weakly quasi-definite of order $N-1$) for which, in (20),

$$\phi_0(x)=(x+\beta+1)(x-N+1)\qquad\text{and}\qquad \psi_0(x)=(N-1)(\beta+1)-(\alpha+\beta+2)\,x.$$

Theorem 1 gives the Δ-coherent pairs $(\boldsymbol{u}_0,\boldsymbol{u}_1)$ with

$$\boldsymbol{u}_1=\frac{1}{|x-\xi|}\,\boldsymbol{u}^{(\alpha+1,\beta+1,N-1)}+K\boldsymbol{\delta}(\xi),\qquad K\geq 0,\qquad \xi\leq 0,\qquad \text{or}\qquad \xi\geq N-2, \quad (24)$$

since $\phi_0(x)\,\boldsymbol{u}_0=(x+\beta+1)(x-N+1)\,\boldsymbol{u}_0=\boldsymbol{u}^{(\alpha+1,\beta+1,N-1)}$. The Δ-coherence of $(\boldsymbol{u}_0,\boldsymbol{u}_1)$ follows in a similar way as in the aboves cases taking into account [12, p. 36, Eq. (2.4.13)] in the monic form.

3.2. u_1 *is a classical discrete linear functional*

We are looking for the Δ-coherent pairs (u_0, u_1) when u_1 is a classical discrete linear functional. Using Theorem 1 (with $\omega = 1$) we must consider the following cases:

3.2.1. *Charlier case.* If u_1 is the Charlier functional, then $\phi_1(x) = \mu$ and from (18) we obtain

$$u_0 = (x - \xi)\, u_1, \qquad \xi \le 0. \tag{25}$$

To check that (u_0, u_1) is a Δ-coherent pair, we have

$$\begin{aligned}\left\langle \phi_1(s-1)\, u_0, P_k(s)\, c_{n+1}^{(\mu)}(s) \right\rangle &= \left\langle (s-\xi)\, u_1, P_k(s)\, c_{n+1}^{(\mu)}(s) \right\rangle \\ &= \left\langle u_1, P_k(s)\, c_{n+1}^{(\mu)}(s)(s-\xi) \right\rangle = 0\end{aligned}$$

for $0 \le k \le n-1$. Thus

$$c_{n+1}^{(\mu)}(s) = P_{n+1}(s) - \sigma_n P_n(s).$$

If we apply the Δ operator to the above equality we get

$$\Delta c_{n+1}^{(\mu)}(s) = \Delta P_{n+1}(s) - \sigma_n \Delta P_n(s),$$

and now, by using [12, p. 36, Eq. (2.4.17)] in the monic form, we obtain

$$c_n^{(\mu)}(s) = \frac{\Delta P_{n+1}(s)}{n+1} - \frac{n\,\sigma_n}{n+1}\frac{\Delta P_n(s)}{n}, \qquad 1 \le n \le M_0 - 1.$$

3.2.2. *Meixner case.* Let $u_1 \equiv u^{(\gamma,\mu)}$ be the Meixner linear functional, where $\phi_1(x) = \mu(x+\gamma)$. Then, by using (18) we have in this case

$$\mu(x+\gamma-1)\, u_0 = (x-\xi)\, u_1,$$

and we can consider the following cases:

1. If $\gamma > 1$, since $(x+\gamma-1)\, u^{(\gamma-1,\mu)} = u^{(\gamma,\mu)}$ then

$$(x-\xi)\, u_1 = (x-\xi)(x+\gamma-1)\, u^{(\gamma-1,\mu)},$$

so

$$u_0 = (x-\xi)\, u^{(\gamma-1,\mu)} + L\delta(1-\gamma), \qquad L \ge 0. \tag{26}$$

From (18) and since $\pi_1(x)\, u_1 = \mu(x+\gamma-1)\, \Delta u_1$ we obtain

$$\mu(x+\gamma-1)\, \pi_1(x)\, u_0 = (x-\xi)\, \pi_1(x)\, u_1 = \mu(x-\xi)\, \pi_1(x)(x+\gamma-1)\, u^{(\gamma-1,\mu)},$$

hence

$$\pi_1(x)\, u_0 = (x-\xi)\, \pi_1(x)\, u^{(\gamma-1,\mu)} + K\delta(1-\gamma).$$

From (26) and the above equation we get

$$u_0 = (x-\xi)\, u^{(\gamma-1,\mu)}, \qquad \xi \le 0. \tag{27}$$

To check that (u_0, u_1) defines a Δ-coherent pair we prove

$$\left\langle u_0, P_k(s)\, m_{n+1}^{(\gamma-1,\mu)}(s) \right\rangle = \left\langle u^{(\gamma-1,\mu)}, P_k(s)\, m_{n+1}^{(\gamma-1,\mu)}(s)(s-\xi) \right\rangle = 0$$

for $0 \le k \le n-1$. Thus

$$m_{n+1}^{(\gamma-1,\mu)}(s) = P_{n+1}(s) - \sigma_n P_n(s).$$

Applying the Δ operator to the above relation we get

$$\Delta m_{n+1}^{(\gamma-1,\mu)}(s) = \Delta P_{n+1}(s) - \sigma_n \Delta P_n(s),$$

and now, by using [12, p. 36, Eq. (2.4.17)] in the monic form,

$$m_n^{(\gamma,\mu)}(s) = \frac{\Delta P_{n+1}(s)}{n+1} - \frac{n\,\sigma_n}{n+1}\frac{\Delta P_n(s)}{n}, \qquad 1 \le n \le M_0 - 1.$$

2. If $\gamma = 1$ then $\xi = 0$ and we obtain

$$u_0 = u_1 + K\boldsymbol{\delta}(0), \qquad K \ge 0. \tag{28}$$

Since

$$\begin{aligned}
&\left\langle u_0, \left(m_{n+1}^{(1,\mu)}(s) - (n+1)\frac{\mu}{\mu-1}\, m_n^{(1,\mu)}(s) \right) P_k(s) \right\rangle \\
&\qquad = \left\langle u_1, \left(m_{n+1}^{(1,\mu)}(s) - (n+1)\frac{\mu}{\mu-1}\, m_n^{(1,\mu)}(s) \right) P_k(s) \right\rangle \\
&\qquad\quad + K \left(m_{n+1}^{(1,\mu)}(0) - (n+1)\frac{\mu}{\mu-1}\, m_n^{(1,\mu)}(0) \right) \\
&\qquad = \left\langle u_1, \left(m_{n+1}^{(1,\mu)}(s) - (n+1)\frac{\mu}{\mu-1}\, m_n^{(1,\mu)}(s) \right) P_k(s) \right\rangle = 0,
\end{aligned}$$

for $0 \le k \le n-1$, using the relation

$$m_{n+1}^{(1,\mu)}(0) - (n+1)\frac{\mu}{\mu-1}\, m_n^{(1,\mu)}(0) = 0.$$

Then, we can write

$$m_{n+1}^{(1,\mu)}(s) - (n+1)\frac{\mu}{\mu-1} m_n^{(1,\mu)}(s) = P_{n+1}(s) - \sigma_n P_n(s), \qquad 1 \le n \le M_0 - 1.$$

Applying the Δ operator and taking into account the so-called Δ-difference representation for Meixner polynomials [6] we get

$$m_n^{(1,\mu)}(s) = \frac{\Delta P_{n+1}(s)}{n+1} - \frac{n\,\sigma_n}{n+1}\frac{\Delta P_n(s)}{n}, \qquad 1 \le n \le M_0 - 1,$$

the Δ-coherence of the pairs $(\boldsymbol{u}_0, \boldsymbol{u}_1)$.

3.2.3. *Kravchuk case.* Let $\boldsymbol{u}_1 \equiv \boldsymbol{u}^{(p,N)}$ be the Kravchuk linear functional, being $\phi_1(x) = (N - x)$. Then, we have

$$(x - \xi)\,\boldsymbol{u}_1 = (N + 1 - x)\,\boldsymbol{u}_0, \qquad \xi \le 0.$$

Since $\boldsymbol{u}_1 = \boldsymbol{u}^{(p,N)} = (N + 1 - x)\,\boldsymbol{u}^{(p,N+1)}$ then

$$\boldsymbol{u}_0 = |x - \xi|\,\boldsymbol{u}^{(p,N+1)}, \qquad \xi \le 0, \qquad \text{or} \qquad \xi \ge N + 1.$$

follows in a similar way as in the Meixner case. Again we check the Δ-coherence finding

$$\left\langle \boldsymbol{u}_0, P_k(s)\,k_{n+1}^{(p)}(s; N+1)\right\rangle = \left\langle \boldsymbol{u}^{(p,N+1)}, P_k(s)\,k_{n+1}^{(p)}(s; N+1)(s - \xi)\right\rangle = 0$$

if $0 \le k \le n - 1$, whence

$$k_{n+1}^{(p)}(s; N+1) = P_{n+1}(s) - \sigma_n P_n(s), \qquad 1 \le n \le \min\{M_0 - 1, N\}.$$

Applying the Δ operator to the above expression, and using [12, p. 36, Eq. (2.4.15)] in the monic form we obtain

$$k_n^{(p)}(s; N) = \frac{\Delta P_{n+1}(s)}{n+1} - \frac{n\,\sigma_n}{n+1}\frac{\Delta P_n(s)}{n}, \qquad 1 \le n \le N.$$

3.2.4. *Hahn case.* Let $\boldsymbol{u}_1 \equiv \boldsymbol{u}^{(\alpha,\beta,N)}$ be the Hahn linear functional, where $\phi_1(x) = (x + \beta + 1)(N - x - 1)$. Then, we have

$$(x - \xi)\,\boldsymbol{u}_1 = (x + \beta)(N - x)\,\boldsymbol{u}_0,$$

and we can distinguish the following cases:

1. If $\alpha, \beta > 0$, since $(x + \beta)(N - x)\,\boldsymbol{u}^{(\alpha-1,\beta-1,N+1)} = \boldsymbol{u}^{(\alpha,\beta,N)} = \boldsymbol{u}_1$ then $(x-\xi)\,\boldsymbol{u}_1 = (x-\xi)(x+\beta)(N-x)\,\boldsymbol{u}^{(\alpha-1,\beta-1,N+1)}$ so, repeating the arguments used in the Meixner case,

$$\boldsymbol{u}_0 = |x - \xi|\,\boldsymbol{u}^{(\alpha-1,\beta-1,N+1)}, \qquad \xi \le 0, \qquad \text{or} \qquad \xi \ge N.$$

To check that $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ is a Δ-coherent pair we compute

$$\left\langle \boldsymbol{u}_0, P_k(s)\, h_{n+1}^{(\alpha-1,\beta-1)}(s;N+1)\right\rangle$$
$$= \left\langle \boldsymbol{u}^{(\alpha-1,\beta-1,N+1)}, P_k(s)\, h_{n+1}^{(\alpha-1,\beta-1)}(s;N+1)(s-\xi)\right\rangle = 0$$

if $0 \le k \le n-1$. Then

$$h_{n+1}^{(\alpha-1,\beta-1)}(s;N+1) = P_{n+1}(s) - \sigma_n P_n(s), \qquad 1 \le n \le \min\{M_0-1, N\}.$$

If we apply the Δ operator to the above expression, and using [12, p. 36, Eq. (2.4.13)] in the monic form we obtain

$$h_n^{(\alpha,\beta)}(s;N) = \frac{\Delta P_{n+1}(s)}{n+1} - \frac{n\,\sigma_n}{n+1}\,\frac{\Delta P_n(s)}{n}, \qquad 1 \le n \le \min\{M_0-1, N-1\}.$$

2. If $\beta = 0$ then $\xi = 0$ and

$$\boldsymbol{u}_0 = \boldsymbol{u}^{(\alpha-1,0,N+1)} + K\boldsymbol{\delta}(0), \qquad K \ge 0, \qquad \alpha > 0. \tag{29}$$

In this case it can be proved that $(\boldsymbol{u}_0, \boldsymbol{u}_1)$ is a Δ-coherent pair using the same technique as in the Meixner case taking into account the following formula

$$\begin{aligned} h_n^{(\alpha,0)}(s;N) = {} & \frac{\Delta h_{n+1}^{(\alpha-1,0)}(s;N+1)}{n+1} \\ & - \frac{(\alpha+n)(n-N)}{(2n+\alpha)(2n+\alpha+1)}\,\Delta h_n^{(\alpha-1,0)}(s;N+1), \end{aligned} \tag{30}$$

valid for $0 \le n \le N-1$. From (29) we have

$$\left\langle \boldsymbol{u}_0, \left(h_{n+1}^{(\alpha-1,0)}(s;N+1) - \frac{(n+1)(\alpha+n)(n-N)}{(2n+\alpha)(2n+\alpha+1)}\, h_n^{(\alpha-1,0)}(s;N+1)\right) P_k(s)\right\rangle$$
$$= \left\langle \boldsymbol{u}^{(\alpha-1,0,N+1)}, \left(h_{n+1}^{(\alpha-1,0)}(s;N+1) - \frac{(n+1)(\alpha+n)(n-N)}{(2n+\alpha)(2n+\alpha+1)}\, h_n^{(\alpha-1,0)}(s;N+1)\right) P_k(s)\right\rangle$$
$$= 0$$

for $0 \le k \le n-1$, since

$$h_{n+1}^{(\alpha-1,0)}(0;N+1) - \frac{(n+1)(\alpha+n)(n-N)}{(2n+\alpha)(2n+\alpha+1)}\, h_n^{(\alpha-1,0)}(0;N+1) = 0.$$

Thus, for $1 \le n \le \min\{M_0-1, N\}$,

$$h_{n+1}^{(\alpha-1,0)}(s;N+1)-\frac{(n+1)(\alpha+n)(n-N)}{(2n+\alpha)(2n+\alpha+1)}h_n^{(\alpha-1,0)}(s;N+1)=P_{n+1}(s)-\sigma_n P_n(s).$$

Applying the Δ operator to the above equation and using (30) we obtain

$$h_n^{(\alpha,0)}(s;N)=\frac{\Delta P_{n+1}(s)}{n+1}-\frac{n\,\sigma_n}{n+1}\frac{\Delta P_n(s)}{n},\quad 1\le n\le \min\{M_0-1,N-1\}.$$

4. THE LIMIT TRANSITION

In [9] Meijer proved that if $(\boldsymbol{u}_0,\boldsymbol{u}_1)$ is a coherent pair of linear functionals (2) then at least one of the functionals has to be a classical continuous one, i.e. Laguerre or Jacobi functional. He gave the classification of coherent pairs of linear functionals which can be represented by distribution functions. The transition from Hahn to Jacobi, Meixner to Laguerre, Kravchuk to Hermite and Charlier to Hermite polynomials is described in [12]. In this section, limit relations between Hahn to Jacobi, Meixner to Laguerre, Kravchuk to Hermite and Charlier to Hermite linear functionals are obtained. From Section 3 and using these limit relations we are able to recover the classification given in [9].

It is clear that the difference operator D_ω defined in (3) converges to the derivative operator $\mathcal{D}=d/dx$ when $\omega\to 0$. Given a linear functional $\boldsymbol{u}$ the linear functional $\mathcal{D}\boldsymbol{u}$ is defined [8] as $\langle\mathcal{D}\boldsymbol{u},p\rangle=-\langle\boldsymbol{u},\mathcal{D}p\rangle$, $\forall p\in\mathbb{P}$.

Definition 9. A sequence of linear functionals $\{\boldsymbol{u}_n\}_n$ converges to $\boldsymbol{u}\in\mathbb{P}'$ if and only if $\{\langle\boldsymbol{u}_n,p\rangle\}_n$ converges to $\langle\boldsymbol{u},p\rangle$, $\forall p\in\mathbb{P}$.

From the above definition, $D_\omega\boldsymbol{u}$ converges to $\mathcal{D}\boldsymbol{u}$ when $\omega\to 0$.

Taking limit when $\omega\to 0$ in the definition (15) of the polynomial $B_2(x)$, it converges to a new polynomial $B(x)$ of degree 2 which has two zeros, ζ and ϑ.

4.1. *Hahn to Jacobi*

Let us consider the (finite) Hahn MOPS $\{h_n^{(\alpha,\beta)}(x;N+1)\}_{n=0}^{N}$ and let $\omega=2/N$. The polynomials

$$\check{S}_n(x)=\omega^n h_n^{(\alpha,\beta)}\left(\frac{x+1}{\omega};N+1\right)\tag{31}$$

are orthogonal with respect to a linear functional $\check{\boldsymbol{u}}_\omega$ which satisfies an equation of type (13) with

$$\check{\phi}_\omega(x)=(1-x)\,(1+x+\omega(\beta+1)),\qquad \check{\psi}_\omega(x)=\beta-\alpha-x\,(2+\alpha+\beta).\tag{32}$$

If we take the limit when $\omega\to 0$ ($N\to\infty$) in (13) we obtain

$$\mathcal{D}\left[(1-x^2)\,\boldsymbol{v}\right]=(\beta-\alpha-x\,(2+\alpha+\beta))\,\boldsymbol{v},\tag{33}$$

which is the distributional equation of the Jacobi linear functional $v = v_J^{(\alpha,\beta)}$ given by

$$\left\langle v_J^{(\alpha,\beta)}, p \right\rangle = \int_{-1}^{1} p(x) \frac{(1-x)^\alpha (1+x)^\beta \Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\,\Gamma(\beta+1)}\, dx, \qquad \forall p \in \mathbb{P}. \tag{34}$$

Therefore, $\check{u}_\omega \to v_J^{(\alpha,\beta)}$ when $\omega \to 0$.

4.1.1. u_0 *is a* ω*-Hahn linear functional.* Let $u_0 = \check{u}_\omega$ be the linear functional satisfying (13) with ϕ_ω and ψ_ω given in (32). From (17) we get

$$(1-x)\,(1+x+\omega(\beta+1))\, u_0 = (x-\xi)\, u_1. \tag{35}$$

When $\omega \to 0$ the above equation becomes

$$\left(1-x^2\right) v_J^{(\alpha,\beta)} = (x-\zeta)\, v_1. \tag{36}$$

Thus, we obtain that

$$\left(v_J^{(\alpha,\beta)}, \frac{v_J^{(\alpha+1,\beta+1)}}{|x-\zeta|} + K\delta(\zeta) \right)$$

is a coherent pair of linear functionals (see [9, p. 332]) where $K \geq 0$, $\alpha > -1$, $\beta > -1$ and $|\zeta| \geq 1$.

4.1.2. u_1 *is a* ω*-Hahn linear functional.* Let $u_1 = \check{u}_\omega$ be the linear functional satisfying (13) with ϕ_ω and ψ_ω given in (32). From (18) it follows

$$\begin{aligned} (1-x+\omega)\,(1+x+\omega\beta)\, u_0 &= (x-\xi)\, u_1, \\ (\beta\,(1-x+\omega) - \alpha(1+x) - 2\omega)\, u_0 &= (x-\xi)\, D_\omega u_1. \end{aligned} \tag{37}$$

If we take the limit when $\omega \to 0$ in these equations we obtain

$$(1-x^2)\, v_0 = (x-\zeta)\, v_J^{(\alpha,\beta)}, \qquad (\beta-\alpha-(\alpha+\beta)\,x)\, v_0 = (x-\zeta)\, \mathcal{D} v_J^{(\alpha,\beta)}, \tag{38}$$

and three different situations appear [9, p. 333]

1. If $\alpha > 0$ and $\beta > 0$, then $\left(|x-\zeta| v_J^{(\alpha-1,\beta-1)}, v_J^{(\alpha,\beta)}\right)$ is a coherent pair of linear functionals, being $|\zeta| > 1$.
2. If $\alpha = 0$ and $\beta > 0$, then $\left(v_J^{(0,\beta-1)} + K\delta(1), v_J^{(0,\beta)}\right)$ is a coherent pair of linear functionals, with $K \geq 0$.
3. If $\alpha > 0$ and $\beta = 0$, $\left(v_J^{(\alpha-1,0)} + K\delta(-1), v_J^{(\alpha,0)}\right)$ is a coherent pair of linear functionals, where $K \geq 0$.

4.2. *Meixner to Laguerre*

Let us consider the Meixner MOPS $\left\{m_n^{(\gamma,\mu)}(x)\right\}_n$. If $\omega \neq 0$, the polynomials

$$S_n(x) = \omega^n m_n^{(\alpha+1,1-\omega)}\left(\frac{x}{\omega}\right) \tag{39}$$

are orthogonal with respect to a linear functional $\boldsymbol{u}_\omega$ which satisfies an equation of type (9) with

$$\phi_\omega(x) = (1-\omega)((\alpha+1)\,\omega + x), \qquad \psi_\omega(x) = (\alpha+1)(1-\omega) - x. \tag{40}$$

If we take the limit when $\omega \to 0$ in the distributional equation (9) for the linear functional $\boldsymbol{u}_\omega$ we obtain

$$\mathcal{D}[x\boldsymbol{v}] = (\alpha+1-x)\,\boldsymbol{v}, \tag{41}$$

which is the distributional equation satisfied by the Laguerre linear functional $\boldsymbol{v} = \boldsymbol{v}_L^{(\alpha)}$ defined as

$$\left\langle \boldsymbol{v}_L^{(\alpha)}, p\right\rangle = \int_0^\infty p(x)\,\frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}\,dx, \qquad \forall p \in \mathbb{P}, \tag{42}$$

where $\Gamma(z)$ denotes the usual gamma function [1]. So, $\boldsymbol{u}_\omega \to \boldsymbol{v}_L^{(\alpha)}$ when $\omega \to 0$.

4.2.1. $\boldsymbol{u}_0$ *is a ω-Meixner linear functional.* Let $\boldsymbol{u}_0 = \boldsymbol{u}_\omega$ be the linear functional satisfying (13) with ϕ_ω and ψ_ω given in (40). From (17) we get

$$(1-\omega)((\alpha+1)\,\omega + x)\,\boldsymbol{u}_0 = (x-\xi)\,\boldsymbol{u}_1. \tag{43}$$

When $\omega \to 0$ this equation yields

$$x\boldsymbol{v}_L^{(\alpha)} = (x-\zeta)\,\boldsymbol{v}_1. \tag{44}$$

Thus,

$$\left(\boldsymbol{v}_L^{(\alpha)}, \frac{\boldsymbol{v}_L^{(\alpha+1)}}{x-\zeta} + K\boldsymbol{\delta}(\zeta)\right)$$

is a coherent pair of linear functionals, where $\alpha > -1$, $K \geq 0$ and $\zeta \leq 0$ (see [9, p. 331]).

4.2.2. $\boldsymbol{u}_1$ *is a ω-Meixner linear functional.* Let $\boldsymbol{u}_1 = \boldsymbol{u}_\omega$ be the linear functional satisfying (13) with ϕ_ω and ψ_ω given in (40). From (18)

$$(1-\omega)(x+\omega\alpha)\,\boldsymbol{u}_0 = (x-\xi)\,\boldsymbol{u}_1, \qquad (\alpha - \alpha\omega - x)\,\boldsymbol{u}_0 = (x-\xi)\,D_\omega \boldsymbol{u}_1. \tag{45}$$

If we take the limit in the above equations when $\omega \to 0$ we obtain

$$x v_0 = (x-\zeta)\, v_L^{(\alpha)}, \qquad (\alpha - x)\, u_0 = (x-\zeta)\, \mathcal{D} v_L^{(\alpha)}. \tag{46}$$

We must distinguish two situations (see [9, p. 331]):

1. If $\alpha > 0$, then $\left((x-\zeta)\, v_L^{(\alpha-1)}, v_L^{(\alpha)}\right)$ is a coherent pair of linear functionals, where $\zeta < 0$.

2. If $\alpha = 0$, then $\left(v_L^{(0)} + K\boldsymbol{\delta}(0), v_L^{(0)}\right)$ is a coherent pair of linear functionals, being $K \geq 0$.

4.3. *Kravchuk and Charlier to Hermite*

In case of Kravchuk (finite) MOPS $\left\{k_n^{(p)}(x;N)\right\}_{n=0}^{N}$, let us consider $\omega = \frac{1}{\sqrt{2p(1-p)\,N}}$. Then, the polynomials

$$\check{S}_n(x) = \omega^n k_n^{(p)}\left(\frac{x}{\omega} + Np; N\right) \tag{47}$$

are orthogonal with respect to a linear functional $\check{u}_\omega$ which satisfies an equation of type (13) with

$$\check{\phi}_\omega(x) = 1 - 2\omega p x, \qquad \check{\psi}_\omega(x) = -2\,x. \tag{48}$$

If we take the limit in (13) when $\omega \to 0$ ($N \to \infty$), we obtain

$$\mathcal{D} v = -2 x v \tag{49}$$

which is the distributional equation satisfied by the Hermite linear functional $v = v_H$ given by

$$\langle v_H, p\rangle = \int_{-\infty}^{+\infty} p\,(x)\, \frac{e^{-x^2}}{\sqrt{\pi}}\, dx, \qquad \forall p \in \mathbb{P}, \tag{50}$$

and then $\check{u}_\omega \to v_H$ when $\omega \to 0$.

Let us consider Charlier MOPS $\{c_n^{(\mu)}(x)\}_n$ and let $\omega = 1/\sqrt{2\mu}$ ($\mu > 0$). The polynomials

$$\check{S}_n(x) = \omega^n c_n^{(\mu)}\left(\frac{x}{\omega} + \mu\right) \tag{51}$$

are orthogonal with respect to a linear functional $\check{u}_\omega$ which satisfies an equation of type (13) with

$$\check{\phi}_\omega(x) = 1, \qquad \check{\psi}_\omega(x) = -2x. \tag{52}$$

If we take the limit in (13) when $\omega \to 0$ we obtain again (49). Thus $\check{u}_\omega \to v_H$, when $\omega \to 0$.

Therefore, Kravchuk and Charlier linear functionals go in the limit to the Hermite linear functional. From Theorem 1 and using the limit relations between Kravchuk and Charlier to Hermite linear functional we obtain that for Hermite linear functional there cannot exist coherent pairs (see [7, p. 13] and also [9, p. 333]).

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