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Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (341) 624-9875

CREDIBLE EQUILIBRIA IN NON-FINITE GAMES AND IN GAMES WITHOUT PERFECT RECALL

Paula Corcho and José Luis Ferreira*

Abstract

Credible equilibria were defined in Ferreira, Gilboa and Maschler (1995) to handle situations of preferences changing along time in a model given by an extensive form game. This paper extends the definition to the case of infinite games and, more important, to games with non-perfect recall. These games are of great interest in possible applications of the model, but the original definition was not applicable to them. The difficulties of this extension are solved by using some ideas in the literatue of abstract systems and by proposing new ones that may prove useful in more general settings.

Key Words

Bad sets, credible equilibrium, good sets, infinite games, imperfect recall, semistable partitions, stable sets, ugly sets.

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1 Introduction

Situations in which players' preferences may change over time have been studied by many authors. A partial list of relevant works includes Strotz (1956), Pollak (1968), Phelps and Pollak (1968), Pollak (1970), Von Weizsaecker (1971), Blackorby, Nissen, Primont and Russell (1973), Hammond (1976), and Pollak (1976).

One consequence of introducing the possibility of changing preferences is that decisions may not be consistent in the classical sense: decision makers may regret having made certain choices in the past, even in situations of perfect information. Ferreira, Gilboa and Maschler (1995) propose to model these situations as extensive form games where the different information sets of a given player define a set of agents of that player. They consider only finite games and, after endowing each agent with a utility function of his own, define recursively an equilibrium concept that takes into account two important features. First the equilibrium has to be immune to the possibility of coalitional deviations by agents of one player and, second, it has to satisfy a certain time consistency, which requires that agents that appear earlier in time are considered first and that further deviations cannot include agents that play earlier than the ones in the first deviation. Agreements satisfying these requirements are called credible equilibria.

For many applications suggested in Ferreira *et al.*, the assumption of perfect recall may not be the natural one. In many game models, a player is not an individual. It can be a state, a political party or any other organization. In such cases it is natural not only to assume that the different agents within this organization may have different priorities, but to allow the possibility of imperfect recall. I.e., situations in which a given agent does not know some actions taken in the past by other agents. In this paper we take the challenge of providing an appropriate extension of credible equilibria to these situations and to the case of an infinite number of players and strategies.

There are many difficulties for such an extension. First, the definition cannot be recursive because we no longer have a finite set of agents. Second, the lack of perfect recall causes that an agent A may not know whether he is playing before or after agent B, and at the same time, agent B may not know if she plays before or after A. This again generates circularities in the definition and demands a clear discussion on the notion of time consistency.

We address the problems of circularity by using ideas from the literature related to the Theory of Social Situations, initiated by Greenberg (1990). One characteristic of this theory is that, using a generalization of the stable sets defined by von-Newmann and Morgenstern (1947), recursive definitions may be extended to infinite sets. The idea is to divide the set of agreements in a stable partition of two sets, the good and the bad, so that no element in the good set is dominated by any other good element, and every element in the bad is dominated by some good element. Of course, in our case, the domination has to do with deviations by coalitions of agents of the same player in some appropriate way.

Such a division may not be possible, as in the cases in which one has an infinite space of strategies or an acyclical dominance relation. Our model has both. The second best is to find a weaker division on the set of agreements. A semistable partition consists of three sets, the good, the bad and the ugly (Kahn and Mookherjee, 1992), where the good elements are dominated only by the bad ones, the bads are dominated by goods and the ugly set is the complement of the union of the good and the bad. This partition is always possible, and it is customary to choose the good and the ugly sets to define the equilibrium.

However, we find that, at least in our case, some ugly elements may be uglier than others. Since the ugly set may be very large (we provide an example with empty good and bad sets), it seems only natural to explore the structure of this set. We do that and propose a new relation on the ugly set based on classes of equivalence and a domination relation on the quotient set. The idea is to identify elements within a cycle. The partition on this ugly set may again not be a stable partition, but we show that the only cause for existence of such an ugly-ugly set is if the quotient set of the equivalence relation is infinite, since the new dominance relation is acyclical. Thus, to accept this set is not as problematic as to accept the whole original ugly set. The new definition uses then the original good set, the good-ugly and the ugly-ugly, and gets rid of the bad-ugly and the original bad.

Abstract stable or semistable sets have proven useful in generalizing recursive definitions and in providing insights into some difficult problems in Game Theory, such as those related to coalitional deviations. A drawback of this methodology is the difficulty to show existence theorems. We dedicate a section to discuss existence issues and provide an existence theorem for a version of credible equilibria.

It is important to notice that, at this stage, credible equilibria are just a generalization of the basic concept of Nash equilibria. Refinements like sequential, perfect and the like are still to be adjusted to the framework of changing preferences.

The paper is divided as follows. Section 2 is a brief presentation of the original model in which credible equilibria are defined. Section 3 extends the definition of credible equilibria to infinite games, but still assumes perfect recall. Section 4 drops the perfect recall assumption and studies some of the difficulties that arise by so doing. Preliminary versions of credible equilibria are provided. Section 5 studies the ugly set and proposes a partition on this set. A final version of credible equilibrium is provided. The techniques in this section may be useful for many other definitions within the theory of social situations. Section 6 discusses existence and section 7 concludes.

2 Credible Equilibria in Finite Games.

The concept of credible equilibrium was introduced by Ferreira *et al.*(1995) to handle situations in which priorities change during the conflict. In this preliminary section we present their extensive-form game model and the solution concept, called *credible equilibrium*, which generalizes the concept of Nash equilibrium. This model is that of a finite game with perfect recall. In the following sections we drop these assumptions and solve a number of conceptual and technical difficulties that arise in the process.

Let $\Gamma = (T, P, U, C, p)$ be a game in extensive-form without payments at the endpoints. Here, T is a tree, $P = \{P_0, P_1, \ldots, P_n\}$ is the players' partition on the non-final nodes of T; $U = (U_0, \ldots, U_N)$ where $U_i = \{u_{i,j}\}_{j=1}^{k_i}$ is the partition of P_i into information sets; $C = \{C(u_{i,j})\}_{i=1,\ldots,n}^{j=1,\ldots,k_i}$ is a correspondence, where $C(u_{i,j})$ is the set of choices which are available to player i at information set $u_{i,j}$; $p = \{p(u_{0,j})\}_{j=1,\ldots,k_0}$ is a vector-valued function, where $p(u_{0,j})$ is a probability distribution on chance's moves at $u_{0,j}$.

To complete the description of the model, we endow each agent $i.j$ with a von Neumann-Morgenstern utility function $h_{i,j}$, defined on lotteries over endpoints of T. Formally. the game with utilities changing during the play is a six-tuple,

$$\Gamma = (T, P, U, C, p, h).$$

where T, P, U, C, p are as above, and $h = (h_1, \ldots, h_n)$, where, for an endpoint z, $h_i(z) = (h_{i.1}(z), h_{i.2}(z), \ldots, h_{i.k_i}(z))$.

Notice that for each i we obtained different agents $i.j$ at different information sets $u_{i.j}$ of player i. The interpretation is as follows: at the beginning of play, player i believes that he will have the utility function $h_{i.j}$ when he reaches information set $u_{i.j}$.

Throughout this section we assume that the game is of perfect recall according to the standard definition (Selten, 1975).

Definition 1
A game form (T, P, U, C, p) is said to be of perfect recall if, for every i, $(i = 1, 2, \ldots, n)$ and every two information sets $u_{i.j}$ and $u_{i.k}$ of the same player i, if one node $y, y \in u_{i.k}$, comes after a choice c at $u_{i.j}$, then every node x in $u_{i.k}$ comes after the same choice c.

By Kuhn's theorem (Kuhn, 1953), in presence of perfect recall, one can restrict the analysis to behavioral strategies.

Formally, a behavioral strategy $s_{i.j}$ of agent $i.j$ is a probability distribution over the choices $c_{i.j}$ at $u_{i.j}$. We denote by $S_{i.j}$ the set of these behavioral strategies. The set of behavioral strategies for player i is $S_i = \times_{j=1,\ldots,k_i} S_{i.j}$. A n-tuple of behavioral strategies is $s = (s_1, \ldots, s_n)$, and the set of these n-tuples is $S = \times_{i=1,\ldots,n} S_i$.

Let Q be a set of agents belonging to the same player. We denote by $-Q$ the set $M \backslash Q$ where M is the set of all agents (not only of the same player). For a strategy s, we denote by s_Q the vector of strategies $(s_{i.j})_{i.j \in Q}$. For simplicity we write $(s_{-i.j}, s'_{i.j})$ to express a deviation of agent $i.j$ from $s_{i.j}$. For s and s' in S, we write $s' \succ_{i.j} s$ iff $h_{i.j}(s') > h_{i.j}(s)$.

Now, we are ready for the definitions:

Definition 2
We say that $i.j$ plays after $i.j_0$ if $i.j = i.j_0$ or if every path from $u_{i.j}$ to the root passes through $u_{i.j_0}$.

Definition 3
Let Γ be a game of perfect recall with utilities changing during the play, let s be an n-tuple of behavioral strategies, and let Q be a set of agents of player i, containing an agent $i.j_0$ and possibly some of i's agents that play after $i.j_0$. A strategy s'_Q is said to be a credible deviation from s, struck by agent $i.j_0$ using Q, if:

(i) $s' \succ_{i.j_0} s$, where $s' = (s'_Q, s_{-Q})$.

(ii) $s' \succ_{i.j} (s'_{-i.j}, s_{i.j})$ for all $i.j \in Q, i.j \neq i.j_0$.

(iii) No agent of i, whether in Q or not, that plays after $i.j_0$, can strike a credible deviation from s'.

Definition 4
Let Γ be a game of perfect recall with utilities changing during the play. A behavioral strategy profile s is called a credible equilibrium (CrE) if no agent can strike a credible

deviation from it.

Remark 1
These concepts are defined recursively. For further information concerning them see Ferreira et al. (1995).

3 Extending Credible Equilibria to Infinite Games.

In this section we extend the definition of credible equilibria to games with an infinite number of players, that is $N = \{1, 2, 3, \ldots\}$. For this purpose, we will follow the approach in Asheim (1991) where the notion introduced by von-Neumann and Morgenstern of abstract stable sets of appropriate abstract systems is used to extend recursive definitions[1] to the infinite case.

A von-Neumann and Morgenstern abstract system (AS) is a pair $(D, \succ)$ where D is an abstract set and $\succ$ is a dominance relation. The notation $f \succ d$ will be interpreted to mean that f dominates d. Let $(D, \succ)$ be an abstract system, and let $f \in D$. The dominion of f, denoted by $\Delta(f)$, is the set of all elements of D that f dominates, according to the dominance relation $\succ$.

$$\Delta(f) = \{d \in D : f \succ d\}$$
$$\Delta(F) = \cup\{\Delta(f) : F \subset D\}$$

That is, an element d in D belongs to $\Delta(F)$ if it is dominated by some element in F. A set $F \subset D$ is a von Neumann and Morgenstern abstract stable set (ASS) for the system $(D, \succ)$ if $F = D \backslash \Delta(F)$.

Let Γ be a game with utilities changing during the play. Inspired by definitions 3 and 4 an abstract system $(D, \succ)$ is introduced. For every agent, $i.j$, we define $\mathcal{Q}_{i.j}$ the set of all agents of i that play after $i.j$ and $Q_{i.j}$ a subset of $\mathcal{Q}_{i.j}$. Define $Q_{i.k}$ similarly, and let s and $\acute{s}$ be elements of $\times_{i \in N} S_i$.

$$D = \{(i.j, Q_{i.j}, s) : i.j \in Q_{i.j} \subset \mathcal{Q}_{i.j} \quad s \in \times_{i \in N} S_i,\ i \in N\ j \in \{1, \ldots k_i\}\}$$

$$(i.j, Q_{i.j}, s) \succ (i.k, Q_{i.k}, \acute{s}) \quad iff$$

(i) $i.j$ *plays after* $i.k$
(ii) $h_{i.j}(s) > h_{i.j}(\acute{s})$.
(iii) $h_{i.h}(s) > h_{i.h}(\acute{s}_{i.h}, s_{-i.h})$, for all $i.h \in Q_{i.j}$.
(iv) $s_{-Q_{i.j}} = \acute{s}_{-Q_{i.j}}$.

As in the standard definition of credible equilibrium, here a deviation from an agreement $\acute{s}$, given by agent $i.k$ to members of $Q_{i.k}$, is a set of instructions s given by $i.j$ to members of $Q_{i.j}$.

[1]The recursion being on the number of players, pure strategies, periods ...

Following the spirit of the concept of Nash equilibrium, agents of player i in $Q_{i.j}$ strike the deviation and other agents are supposed to play as agreed in $\hat{s}$. This is stated in condition (iv). Condition (i) states a time consistence requirement between agents who can strike a deviation. Conditions (ii) and (iii) are the counterparts of conditions (i) and (ii) in definition 3. Condition (ii) states that $i.j$ prefers that these instructions are obeyed, given that agents of $-Q_{i.j}$ continue to follow $\hat{s}$. Note that this implies that $i.j$ is reached with positive probability under $\hat{s}$, because prior to $i.j$ there is no distinction between elements of s and $\hat{s}$. Condition (iii) implies that each member in of $Q_{i.j}$ is reached with positive probability under s. The condition states that these members actually prefer to comply with $\hat{s}$. In the original definition, a condition was added to prevent further deviations (definition 3(iii)). For infinite games this condition is not captured in the dominance relation, but in the structure of the related stable set.

The next proposition relates the ASS of $(D, \succ)$ to the definition of credible equilibrium for finite games of perfect recall with utilities changing during the play and allows for a definition applicable to infinite games of perfect recall with utilities changing during the play.

Denote by $\Gamma^s_{i.j}$ the game obtained from Γ by fixing the strategy of every agent of every player according to s, except $i.j$ and agents of player i that play after $i.j$. Note that $\Gamma^s_{i.j}$ is a one-person game possibly with several agents, i.e., at the beginning of the game, players think that that is the game agent $i.j$ is facing, given that every agent other than him and his followers play s. Denote by $i.j_0$ an agent of player i that does not play after any other agent of that player.

Proposition 5

Let K be an ASS of $(D, \succ)$. Then, for finite games (finite number of agents and of pure strategies) of perfect recall with utilities changing during the play, we have that for all $i.j$ and $Q_{i.j} \subset \mathcal{Q}_{i.j}$ and $s \in \times_{i=1...n} S_i$, $(i.j, Q_{i.j}, s) \in K$ if and only if s is a credible equilibrium in Γ. In particular, $A = \{s : \forall i \in N, \text{ and } \forall i.j_0 \ \ (i.j_0, \mathcal{Q}_{i.j_0}, s) \in K\}$ is the set of credible equilibria of Γ.

Proof

First, if for all $i \in N$, for all $i.j$ and $Q_{i.j} \subset \mathcal{Q}_{i.j}$, $(i.j, Q_{i.j}, s) \in K$ then it is not dominated by any element in D, and no agent of i that plays after $i.j$, say $i.k$, can strike a credible deviation with $\hat{s} \succ_{i.k} s$ and $\hat{s} \succ_{i.h} (s_{i.h}, \hat{s}_{-i.h})$. This is true for all i, $i.j$ and in particular for $i.j \neq i.k$, which, by definition 4, implies that s is a credible equilibrium.

To prove the converse, let s be a credible equilibrium of Γ, so that no agent after $i.j$ can strike a credible deviation. If for some $i.j$ and $Q_{i.j}$, $(i.j, Q_{i.j}, s) \notin K$ then it is dominated by some element in K. This means that some agent that plays after $i.j$ can strike a credible deviation, i.e. $\exists \ \ i.k$, $Q_{i.k}$ and $s' = (s'_{Q_{i.k}}, s_{-Q_{i.k}})$, such that $(i.k, Q_{i.k}, s') \succ (i.j, Q_{i.j}, s)$. Since $(i.k, Q_{i.k}, s') \in K$, s' is an equilibrium in $\Gamma^{s'}_{i.k}$ and, then, s' is a deviation from s struck by $i.k$. If $i.k \neq i.j$, this contradicts the fact that s is a credible equilibrium. If $i.k = i.j$, by theorem 5.1 in Ferreira *et al.* (1995) there exists another deviation s^* struck by $i.j$ such that s^* is a credible deviation from s. But then s is no a credible equilibrium, which is a contradiction. □

The uniqueness of the abstract set stable is established in the following proposition.

Proposition 6
Let F and T be abstract stable sets of the abstract system $(D, \succ)$ associated with a finite game, as defined above, then $F = T$.

Proof
Let $\{(i.j_k, Q_{i.j_k}, s_k)\}_k$ be an infinite sequence such that

$$(i.j_{k+1}, Q_{i.j_{k+1}}, s_{k+1}) \succ (i.j_k, Q_{i.j_k}, s_k),$$

for all k. By the finiteness of the set of agents and of the set of strategies and because the dominance relation requires that $i.j_{k+1}$ play after $i.j_k$, we have that, in the tail of the sequence, $Q_{i.j_k} = Q_{i.j_{k+1}}$ and $i.j_{k+1} = i.j_k$. This tail is transitive. According to corollary 4b in Arce and Kahn (1991).[2] this is a sufficient condition for the uniqueness of the abstract stable set if it exists.

□

Since the characterization of a credible equilibrium given in Proposition 5 is not based on recursion, it can be used to formulate a general definition of the concept, covering both finite and infinite games.

Definition 7
Consider a game of perfect recall with utilities changing during the play and with an infinite or finite number of players and strategies. A sequence $s = (s_1, s_2, \ldots)$ is said to be a credible equilibrium of Γ if and only if there is an abstract stable set (ASS), K, for the associated system $(D, \succ)$ such that for all $i \in N$, for all $i.j_0$ and for all $Q_{i.j_0} \subset Q_{i.j_0}$, $(i.j_0, Q_{i.j_0}, s) \in K$.

The abstract set K is interpreted as a social norm, where every point in K equally reasonable: no one dominates any other in the social norm and any point outside the social norm is dominated by some element inside it.

A major difficulty with this approach is that when the abstract system $(D \succ)$ is not finite or $\succ$ is cyclical [3], stable sets may not exist. Kahn and Mookherjee (1992) provide examples of games with infinite action spaces where the stable set does not exist.

4 Imperfect Recall.

In this paper we follow the standard *ex ante* view on strategy choice, this implies that each agent evaluates the possible outcomes resulting from behavior and mixed strategies before the game.

The perfect recall condition on information partitions is a basic assumption in the study of extensive games. This condition expresses the idea that a player remembers what he did and what he learned and seems natural for rational players. In our model, imperfect recall means that agents may or may not have the ability to observe moves by

[2] See Appendix
[3] These cases could be an infinite game or imperfect recall

other agents of the same player. Since agents play only at one information set, the set of mixed strategies for that agent coincides with the set of behavioral strategies with and without the perfect recall assumption. The difference that imperfect recall introduces in our model is on the relation *play after*.

Ferreira *et al.* (1995) suggest some applications in which perfect recall may not be a reasonable requirement. However, they do not pursue this idea further. In this section we study this extension, and propose a notion of credible equilibrium for games without perfect recall. As we will see, one consequence of not having perfect recall is that we can no longer produce a recursive definition. When we try the ASS-approach, some difficulties arise as we cannot guarantee the existence of a unique stable partition even with a finite number of players. Hence we use the more general concept of semistability. Notice that the existence of the stable or semistable partition does not imply that the *good* set is not empty.

Another consequence of not assuming perfect recall has to do with the specification of the permitted deviations. According to condition (iii) in definition 2 of the original credible deviations, after a credible deviation, no other agent after the one who proposed the deviation should be able to find a new one. Now, this agent is not required to play with positive probability according to the first deviation, yet if he can strike a new one, that deviation was not credible. In the original framework (finiteness and perfect recall) this didn't represent any problem, since agents playing with probability zero will never gain from a deviation as they can only instruct agents after them, who, again, play with probability zero. Therefore, their actions won't change the expected payoffs when evaluated at the beginning of the play. Without perfect recall, this doesn't need to be the case. An agent that plays with probability zero may instruct agents that, according to him, may play after him and that, by changing their behavior, can make him play with positive probability. This is a very unnatural deviation to be permitted. Hence we have to prevent that possibility in the new definition.

We start by extending the relation *play after* to the case of imperfect recall and by giving an example of how this relation may be cyclical.

Definition 8

Let Γ be a game with or without perfect recall and with utilities changing during the play. We say that $i.j$ plays after $i.k$ according to strategy s if there exists a path from the root of the game passing through $y \in u_{i.k}$ and $x \in u_{i.j}$, if y comes before x in the game tree, and if both agents play with positive probability when s is played.

For a strategy s, let $i.j_0$ be an agent such that if $i.j_0$ plays after $i.h$ according to s, then $i.k$ plays after $i.j_0$ according to s.

Remark 2

If one uses this new relation plays after ... according to strategy ... in definition 2, the definition doesn't change as it refers to games of perfect recall.

The example in figure 4.1 illustrates how the lack of perfect recall may make the relation *play after* cyclical. Observe that, according to the strategy in which every agent randomizes between his strategies, agent 1.2 can play after 1.3 and agent 1.3 after 1.2, because their information sets cross each other. Notice also that, according to strategy $(R, L2, R3)$, agent 1.2 plays with probability zero. If we allowed this player to strike a deviation with agents that may play after him, he could suggest agent 1.3 to play $L3$, thus making him play with positive probability. These are the kind of situations that will prevented in the definitions in the next sections.

[Insert FIGURE 4.1]

Now we extend the definition of credible equilibrium to games of imperfect recall, in which coalitions of players communicate prior to actual play and make non-binding agreements on strategy choices. We wish to know which agreements are stable in such environments.

For a strategy $s \in \times_{i \in N} S_i$, we denote by $\mathcal{Q}_{i.j}(s)$ the set of all agents of i that play after $i.j$ according to s. Define $\mathcal{Q}_{i.k}(\hat{s})$ similarly. Inspired by definition 4 and by the discussion above, an abstract system, $(D, \succ\succ)$, is introduced with a new dominance relation $\succ\succ$. Let $Q_{i.j} \subset \mathcal{Q}_{i.j}(s)$. $Q_{i.k} \subset \mathcal{Q}_{i.k}(\hat{s})$ and $s, \hat{s} \in \times S_i$

$$D = \{(i.j, Q_{i.j}, s) : \quad i.j \in Q_{i.j} \subset \mathcal{Q}_{i.j}(s) \quad s \in \times_i S_i \; i \in N, \; j \in \{1, \ldots, k_i\}\} \qquad [4.1]$$

$$(i.j, Q_{i.j}, s) \succ\succ (i.k, Q_{i.k}, \hat{s}) \quad iff$$

(i) $i.j$ plays after $i.k$ according to $\hat{s}$.
(ii) $h_{i.j}(s) > h_{i.j}(\hat{s})$.
(iii) $h_{i.h}(s) > h_{i.h}(\hat{s}_{i.h}, s_{-i.h})$, for all $i.h \in Q_{i.j}$,
(iv) $s_{Q_{i.j}} = \hat{s}_{-Q_{i.j}}$.

An element of D, that will be called an agreement, is a specification of the moves to be taken by all members in the agreement, for a given list of moves for all other players. Condition (i) allows the analysis of games with crossing information sets as in figure 4.1. Only if agent $i.j$ is reached with positive probability under $\hat{s}$ he can strike a deviation from $\hat{s}$. Condition (ii) states that $i.j$ prefers the proposed deviation s rather than the initial strategy profile $\hat{s}$. Condition (iii) implies that when $i.h$, a member of $Q_{i.j}$, comes to play he prefers to comply with the deviation rather than with the strategy $\hat{s}$, given that all other agents in $Q_{i.j}$ follow the deviation s. Note that conditions (ii)-(iii) of the dominance relation impose requirements only on players who play after $i.j$ according to s. Also, notice that $(D, \succ\succ)$ reduces to $(D, \succ)$ for games of perfect recall.

Our objective is to determine whether an agreement is stable, i.e. whether it is never dominated or is dominated only by an agreement that is dominated by an agreement that is never dominated or... and so on.

Von Neumann and Morgenstern established sufficient conditions for the existence and uniqueness of the abstract stable set by considering properties of an acyclical dominance relation on the abstract set. Unfortunately, the dominance relation $\succ\succ$ is not acyclical, therefore we cannot use those sufficient conditions to show existence of the abstract stable set. This situation departs from the case of finite number of players and strategies and with an acyclical dominance relation, where there always exists a unique partition of the abstract system in stable and non-stable sets.

Kahn and Mookherjee (1992) provide an example of a game in which a stable partition does not exist when considering an abstract system based on the concept of Coalition-proof equilibrium. The example involves an infinite set of strategies. For this reason, these authors make a modification on the definition of stability. They show that a weaker version of stable partitions, called semistable partitions, always exists for arbitrary sets of objects and arbitrary dominance relations.

Definition 9
Let $(\mathcal{A},\succ)$ be an abstract system. A trio of subsets $\{\mathcal{G},\mathcal{B},\mathcal{U}\}$ of $\mathcal{A}$ form a semistable partition (SSP) of $\mathcal{A}$ if:

(1) $\mathcal{B}$ consists of all elements dominated by elements in $\mathcal{G}$. I.e., $x \in \mathcal{B}$ iff $\exists\ x' \in \mathcal{G}$ such that $x' \succ x$.

(2) $\mathcal{G}$ consists of all elements that are not dominated or are dominated only by elements in $\mathcal{B}$. I.e., $x' \in \mathcal{G}$ iff whenever $x \succ x'$, then $x \in \mathcal{B}$.

(3) $\mathcal{G} \cap \mathcal{B} = \emptyset$ and $\mathcal{U} = \mathcal{A}\backslash(\mathcal{G} \cup \mathcal{B})$.

$\mathcal{G}$ is called the good set, $\mathcal{B}$ the bad set and $\mathcal{U}$ the ugly set.

Ideally, we would like to have a unique stable partition (i.e $U = \emptyset$), because a semistable partition that is not stable contains an ugly set, which makes the solution concept ambiguous. Unfortunately, in our case, the abstract system $(D,\succ\succ)$ does not admit a $(\succ\succ)$-stable partition in general, as will be seen later on when studying the example in figure 5.1. A semistable partition is a weaker concept than a stable partition in the sense that the good and the bad sets do not exhaust the set of all agreements. One result on semistable partitions (Arce and Kahn, 1991) is that the ugly set contains cycles or infinite sequences of agreements dominating one another, none of which is dominated by a good agreement. We will study the ugly set of $(D,\succ\succ)$ in more detail in the next sections. Before doing that, notice that semistable partitions may not be unique. Our extensions of credible equilibria will be based on one interesting class of them: The minimal semistable partition.

Definition 10
A $\succ$-semistable partition $\{G^*(\succ), B^*(\succ), U^*(\succ)\}$ of D is minimal if it satisfies: $G^*(\succ) \subset G(\succ)$, $B^*(\succ) \subset B(\succ)$ and $U^*(\succ)$ is the complement of $G^* \cup B^*$ for every semistable partition $\{G(\succ), B(\succ), U(\succ)\}$ of D. These sets $\{G^*, B^*, U^*\}$ are called strictly good, strictly bad and strictly ugly, respectively.

Minimal semistable partitions can be constructed as follows (Kahn and Mookherjee (1991). First define the sets G_0^* and B_0^*:

$$G_0 = \{(i.j, Q_{i.j}, s) \in D : \quad not \quad \exists(i.k, Q_{i.k}, \hat{s}) \in D \quad : (i.k, Q_{i.k}, \hat{s}) \succ (i.j, Q_{i.j}, s)\}$$

$$B_0 = \{(i.j, Q_{i.j}, s) \in D : \quad \exists(i.k, Q_{i.k}, \hat{s}) \in G_0^* \quad : (i.k, Q_{i.k}, \hat{s}) \succ (i.j, Q_{i.j}, s)\}$$

Next inductively define G_k^*, B_k^* with $k = 1, \ldots$

$$G_k^* = \{(i.j, Q_{i.j}, s) \in D : if \quad (i.k, Q_{i.k}, \hat{s}) \succ (i.j, Q_{i.j}, s) \quad then \quad (i.k, Q_{i.k}, \hat{s}) \in B_{k-1}^*\}$$

$$B_k^* = \{(i.j, Q_{i.j}, s) \in D : \quad \exists(i.k, Q_{i.k}, \hat{s}) \in G_k^* \quad : (ih, H, \hat{s}) \succ (i.j, Q_{i.j}, s)\}$$

Define $G^* = \cup_{k=0}^{\infty} G_k^*$ and $B^* = \cup_{k=0}^{\infty} B_k^*$. And finally, define U^* as the complement of $G^* \cup B^*$ in D.

Once the semistable partition is defined, one can use the elements on the strictly good set to define the equilibrium. The possible existence of a non-empty ugly set opens the possibility to strong and weak versions of equilibrium.

Definition 11
Let Γ be a game of imperfect recall as defined above, with a finite or infinite number of players and strategies, let $(D, \succ\succ)$ be an abstract system as defined in [4.1], and let $\{G^*, B^*, U^*\}$ be a minimal semistable partition of D. A strategy s is said to be a strongly-credible equilibrium of Γ if for all $i.j_0$ and $Q_{i.j_0} \subset Q_{i.j_0}(s)$, $(i.j_0, Q_{i.j_0}, s) \in G^*(\succ\succ)$.

Our solution concept says that $(i.j_0, Q_{i.j_0}, s)$ is credible if it is dominated only by elements belonging to the strictly bad set. A weaker notion could also be defined as follows:

Definition 12
Let Γ, $(D, \succ\succ)$ and $\{G^*, B^*, U^*\}$ be defined as before. A strategy s is said to be a weakly-credible equilibrium of Γ if for all $i.j_0$ and $Q_{i.j_0} \subset Q_{i.j_0}(s)$, we have that $(i.j_0, Q_{i.j_0}, s) \in G^*(\succ\succ) \cup U^*(\succ\succ)$.

The weaker solution says that an element of D is credible if it is not dominated by any element in the strictly good set. These are standard ways of defining equilibria in the literature of semistability; however none of these definitions is entirely satisfactory. First, because the set G may be empty and, second, because some ugly elements may be uglier than others. The following section considers these points.

We end this section with an example of a game of imperfect recall in which the dominance relation turns to be acyclical. The example illustrates the different implications of applying the original definition of credible equilibria and the ones just proposed.

[Insert FIGURE 4.2]

Consider the game in figure 4.2. The strategy profile $(L1, L2)$ is not credible according to the original definition because agent 1.1 can instruct himself and agent 1.2 to move $(R1, R2)$. If agent 1.2 knew that agent 1.1 was moving right, he would have certainly chosen to comply. But he does not know, and if he complies, then it behooves agent 1.1 to remain at $L1$ as to get 3. So, the deviation $(R1, R2)$ is perhaps not safe, but the original definition says nothing about this.

We could conclude that the original definition of credible equilibria is not appropriate in this context, and certainly it was never intended to be. If we apply the dominance relation $\succ\succ$ in 4.2, we get the following chain of dominations:

$$(1.2, \{1.2\}, (L1, L2)) \succ\succ (1.1, \{1.1\}, (L1, R2)) \succ\succ$$

$$\succ\succ (1.1, N, (R1, R2)) \succ\succ (1.1, N, (L1, L2)).$$

Notice that the first agreement is not dominated (it would need the participation of 1.1 to go $(R1, R2)$, but this is not possible as 1.1 does not play after 1.2).

Hence $((1.1, N, (L1, L2))$, $(1.1, \{1.1\}, (L1, R2))$ are in the bad set and $(1.1, N, (R1, R2))$, and $(1.2, \{1.2\}, (L1, L2))$ are in the good set. In other words, $(R1, R2)$ and $(L1, L2)$ are credible equilibria.

5 Cycles and Equivalence Classes.

Consider the game in figure 5.1 and, for simplicity, consider only pure strategies. The following relations can be easily checked. The agreements $(1.2, N, (R2, L3))$ and $(1.3, N, (L2, R3))$ form a cycle since they dominate each other. Hence they belong to the ugly set. The agreement $(1.3, N, (R2, R3))$ (with $0 < E < \frac{1}{2}$) is also in the ugly set as it is dominated by the two agreements in the cycle before, and dominates the agreement $(1.2, N, (L2, L3))$. Finally $(1.2, N, (L2, L3))$ does not dominate any other agreement and, therefore, is also an element of the ugly set. This suggests the division of the ugly set into a new partition where the elements in a cycle that are not dominated by elements outside the cycle may be considered good. In this section we present such a division.

[Insert FIGURE 5.1]

One result on the structure of ugly sets that will prove useful is the following (Arce and Kahn (1991)): Let $(D, \succ)$ be an abstract system, and $\{G, B, U\}$ a semistable partition on it, then, if $U \neq \emptyset$, U contains a stack, i.e. an infinite sequence $\{x_i\}_i$ such that $x_{i+1} \succ x_i$ for all i. Sometimes U contains cycles, which are stacks with a finite range. The strategy in this section consists in identifying the elements within a cycle and in defining a new dominance relation on them.

Definition 13
Let $(A, \succ)$ be an abstract system, a cycle in $(A, \succ)$ is a sequence with finite range, $\{x_i\}_{i=1,\dots,n}$, such that $x_i \in A$ satisfies

$$x_1 \succ x_2 \succ \dots \succ x_1.$$

Let $\{G, B, U\}$ be a minimal semistable partition associated to $(D, \succ\succ)$. Suppose that U is a non-empty set and that it contains, at least, one cycle. As we saw in the example before, in this set some ugly elements *may be uglier than others.* We will use the methodology of stable sets to characterize these good-ugly elements in infinite games without perfect recall. Therefore, the idea is to partition the ugly set into two subsets, which can be denoted as the good-ugly and the bad-ugly, with the properties of internal stability and external stability.

From the initial binary relation $\succ\succ$ we define the following binary equivalence relation:

$$(i.j, Q_{i.j}, s) \sim (i.h, Q_{i.h}, \hat{s}) \qquad [5.1]$$

iff either $(i.j, Q_{i.j}, s) = (i.h, Q_{i.h}, \hat{s})$ or there exist sequences $\{x_i\}_i$ and $\{y_i\}_i$ in D such that

$$(i.j, Q_{i.j}, s) \succ\succ x_1 \succ\succ \dots x_p \succ\succ (i.h, Q_{i.h}, \hat{s}) \succ\succ y_1 \succ\succ \dots y_k \succ\succ (i.j, Q_{i.j}, s).$$

Proposition 14
The binary relation $\sim$ on U defined above is an equivalence relation.

Proof
It is trivial to show that the relation $\sim$ is reflexive and symmetric.

To show transitivity suppose that $(i.j, Q, s) \sim (i.h, Q_{i.h}, \hat{s})$ and $(i.h, Q_{i.h}, \hat{s}) \sim (i.k, Q_{i.k}, s')$. This implies that either $(i.j, Q_{i.j}, s) = (i.h, Q_{i.h}, \hat{s}) = (i.k, Q_{i.k}, s')$ or

$$(i.j, Q_{i.j}, s) \succ\!\succ x_1 \succ\!\succ \ldots x_p \succ\!\succ (i.h, Q_{i.h}, \hat{s}) \succ\!\succ y_1 \succ\!\succ \ldots y_k \succ\!\succ (i.j, Q_{i.j}, s).$$

and

$$(i.h, Q_{i.h}, \hat{s}) \succ\!\succ \hat{x}_1 \succ\!\succ \ldots \hat{x}_p \succ\!\succ (i.k, Q_{i.k}, s') \succ\!\succ \hat{y}_1 \succ\!\succ \ldots \hat{y}_k \succ\!\succ (i.h, Q_{i.h}, \hat{s}).$$

Then, $(i.j, Q_{i.j}, s) \succ\!\succ x_1 \succ\!\succ \ldots (i.h, Q_{i.h}, \hat{s}) \succ\!\succ \hat{x}_1 \ldots \succ\!\succ (i.k, Q_{i.k}, s')$. But this implies that $(i.j, Q_{i.j}, s) \sim (i.k, Q_{i.k}, s')$.

□

We now consider the quotient set obtained after the equivalence classes of $\sim$ and define a domination relation on it.

Definition 15
The classes of an equivalence relation $\equiv$ on $\mathcal{A}$ are the following collections $[a]$ defined as $[a] = \{x \in \mathcal{A} \ \setminus \ x \equiv a\}$. The collection $\{[a]\}_{a \in \mathcal{A}}$ is denoted $(\mathcal{A}/\equiv)$ and called the quotient set [4] *of $\mathcal{A}$ by $\equiv$.*

From the ugly set U and the initial relation $\succ\!\succ$, $(U, \succ\!\succ)$, we define a new abstract system.

$$((U/\sim), \gg)$$

where the abstract set is the quotient set of the equivalence relation [5] defined on the ugly set, and the dominance relation is defined as follows: for $[a], [b] \in (U/\sim)$

$$[a] \gg [b]$$

iff $[a] \neq [b]$ and there exist $x \in [a]$, $y \in [b]$ and a sequence $\{x_i\}_i$ with $x_i \in U$ such that, $x \succ\!\succ x_1 \succ\!\succ \ldots \succ\!\succ x_p \succ\!\succ y$.

For the general case a stable partition on $((U/\sim), \gg)$ may not exist. Consider then a minimal semistable partition, $\{G_U, B_U, U_U\}$, and the following sets:

$G_U^*(\sim) = \{x \in U : [x] \in G_U(\gg)\}$ and $U_U^*(\sim) = \{x \in U : [x] \in U_U(\gg)\}$. With these elements we can formulate a new extension of credible equilibrium that is more satisfactory than definitions 11 and 12.

Definition 16
Let Γ be a game without perfect recall, with utilities changing during the play and with a finite or infinite numbers of players. A strategy s is said to be a credible equilibrium of

[4] The reader is referred to Potter (1990), Chapter 2, for a detailed study on equivalence relations and quotient sets.

Γ *if for all* $i.j_0 \in N$, *and* $Q_{i.j_0} \subset Q_{i.j_0}(s)$, *we have that* $(i.j_0, Q_{i.j_0}, s) \in G(\succ\succ) \cup G^*_U(\sim) \cup U^*_U(\sim)$.

Clearly, for infinite games, credible equilibria are a subset of weakly credible and include the set of strongly credible. If we apply the three definitions to the example in figure 5.1 we get that the set of strongly credible equilibria is empty, while all four profiles of pure strategies are weakly credible. Of these, only $(L2, L3)$ is not credible.

At this point, it seems that one could consider again the possibility that some elements in U^*_U are uglier than others and could repeat the whole procedure for this set U^*_U. The good news is that this is not the case as the set U^*_U does not contain cycles. This is established in the next proposition.

Proposition 17
Let $((U/\sim), \gg)$ *be as defined above, then* $\gg$ *is acyclical on* $(U/\sim)$.

Proof
We use the following result: If a relation is asymmetric, irreflexive and transitive then it is acyclical. It is trivial to show that the relation $\gg$ is irreflexive and asymmetric. To show transitivity suppose that $[a], [b], [c] \in (U/\sim)$ with $[a] \gg [b]$ and $[b] \gg [c]$. These classes are different and there exist $x \in [a]$ $y, y' \in [b]$ and $z \in [c]$ such that

$$x \succ\succ x_1 \succ\succ \ldots \succ\succ x_p \succ\succ y.$$

$$y' \succ\succ y'_1 \succ\succ \ldots \succ\succ y'_p \succ\succ z.$$

Since either $y = y'$ *or* $y \succ\succ y_1 \ldots \succ\succ y'$ (because $y, y' \in [b]$), we have $x \succ\succ \ldots \succ\succ z$, which implies that $[a] \gg [c]$.

□

If the abstract set is finite and the binary relation is acyclical, then there exists a unique stable partition. Since we have eliminated the possibility of cycles, the only reason not to have a stable partition is if $(U/\sim)$ is not finite. One interesting question for future research is to find conditions that guarantee the existence of a finite quotient set $(U/\sim)$. It is clear that a finite ugly set generates a finite quotient set, but there should be infinite ugly sets whose quotient set is finite.

6 Existence

Not only in this work, but in all the literature using abstract stable sets, existence theorems are very difficult to show, even in cases where counterexamples are also hard to find. This seems the price to pay for being able to make non-recursive definitions. However this has not been a handicap for this methodology to be fruitful, providing important insights into many aspects of hard problems in Game Theory, especially, those related to the stability of coalitional deviations.

Nevertheless, we can show existence of an approximation of weakly credible equilibria for finite games with imperfect recall. The approximation is on the line of ε-equilibria.

Interestingly enough, in a very different setting, but still within the methodology of abstract stable sets, and with a different kind of proof, Greenberg, Monderer and Shitovitz (1996) were able also to prove only the existence of an ε−conservative equilibrium.

Definition 18
For any $\epsilon > 0$ define $(D, \succ\succ_\epsilon)$ as $(D, \succ\succ)$ except that conditions (ii) and (iii) in [4.1] are replaced with
(ii') $h_{i.j}(s) > h_{i.j}(\hat{s}) - \varepsilon$
(iii') $h_{i.h}(s) > h_{i.h}(\hat{s}_{i.h}, s_{-i.h}) - \varepsilon$, for all $i.h$ in $Q_{i.j}$

Then define ε−weakly credible equilibria as weakly credible equilibria in definition 12 replacing the dominance relation $\succ\succ$ with $\succ\succ_\varepsilon$. In words, an ε-weakly credible equilibrium is like the weakly credible except that agents don't deviate unless they can win more than $\varepsilon > 0$. The next proposition shows that this new equilibrium exists for an important class of games.

Proposition 19
There exist ε-weakly credible equilibria.

The proof of this proposition is established as a corollary of proposition 21. But before that, we need the following definition.

Definition 20
Define $\Gamma(\delta) = (T, P, U, C, p, h, \delta)$ as $\Gamma = (T, P, U, C, p, h)$, with the only difference that in $\Gamma(\delta)$, for every agent of every player, all pure strategies must be chosen with a positive probability that is multiple of $\delta > 0$.

Remark 3
For this definition to apply, it has to be the case that for every agent, the number of strategies divided by δ is an integer. It is always possible to define $\Gamma(\delta)$ games when the original game is finite. Within this new class of games we can apply the different definitions of credible equilibria. Also notice that, for these games, if one player plays after another according to a strategy, he plays also after him according to any other strategy, so there is no need to make a reference to the strategy.

Proposition 21
Let $\Gamma(\delta)$ be a game as defined above, if it is finite then it has at least one weakly credible equilibrium.

Proof
Start with a strategy s, if it is not a weakly credible equilibrium, then there exist $(i.j_0, Q_{i.j_0}, s)$ and $(i.j, Q_{i.j}, s')$ such that $(i.j, Q_{i.j}, s') \succ\succ (i.j_0, Q_{i.j_0}, s)$.

Now, if s' is not a weakly credible equilibrium, then we find a new dominating deviation and so on. If at some point we reach a weakly credible equilibrium we are done. If not, we can construct a sequence of deviations. Since the number of agents and strategies is finite, we must have the following cycle starting with some agent $i.h$:

$$(i.h, Q_{i.h}, s'') \succ\succ (i.k, Q_{i.k}, s''') \succ\succ \ldots \succ\succ (i.h, Q_{i.h}, s'') \succ\succ \ldots$$
$$\ldots \succ\succ (i.j, Q_{i.j}, s') \succ\succ (i.j_0, Q_{i.j_0}, s)$$

There are two possibilities. Either the elements in this cycle belong alternatively to the good and the bad sets or all of them belong to the ugly set. Consider, then, the first possibility and the case in which $(i.h, Q_{i.h}, s'')$ is in the good set. Since all strategies are completely mixed, all agents in the cycle play after each other according to any strategy (recall the remark above). This means that if s'' is not a weakly credible equilibrium, the only deviations that upset it are due to other players or to agents of i that not play after $i.h$. Fix the part of the strategy s'' for all the agents of i that play after $i.h$, i.e., fix $s''_{Q_{i.h}}$. With the existing deviation we can repeat the same process that we started with $(i.j_0, Q_{i.j_0}, s)$. Every time we do this, either we encounter a weakly credible equilibrium or else we fix a strategy for a new set of agents. In the latter situation, by the finiteness of the game, eventually, we have fixed a strategy for every agent. The strategy profile that is the result of these fixed strategies must be a weakly credible equilibrium by construction (in fact, it would be a strongly credible equilibrium, since it would belong to the good set) since no agent can find a deviation that is not in the bad set. If $(i.h, Q_{i.h}, s'')$ is in the bad set, $(i.k, Q_{i.k}, s''')$ is in the good and we can repeat the process to fix $s'''_{Q_{i.k}}$. If the elements in the cycle are in the ugly set, the process can be repeated directly with $(i.h, Q_{i.h}, s'')$ and the final strategy profile would be either in the good or in the ugly set, thus being a weakly credible equilibrium. □

Proof of Proposition 19
Since payoffs are a continuous function of the probabilities with which mixed strategies are chosen, for every ε there exists a δ small enough so that every payoff that is the result of a strategy combination in Γ is within the ε-neighborhood of a payoff in $\Gamma(\delta)$. Hence, the weakly credible equilibrium in $\Gamma(\delta)$ is an ε-weakly equilibrium in Γ. □

Remark 4
This proof works with any positive ε, but not with $\varepsilon = 0$. The reason is twofold. First, as $\varepsilon \to \infty$, the number of elements in the cycle may also go to infinity and one cannot apply the argument and, second, the set of credible equilibria need not be compact (see Ferreira et al. for an example).

We end the discussion of existence with an interesting observation. In the original paper by Ferreira *et al.*, the existence of credible equilibria was established as a consequence of having as a subset the set of agent-perfect equilibria, which is non-empty. When the assumption of perfect recall is dropped, in addition to all the difficulties we address in this work, this relation does not hold in general. The following is a counterexample.

[Insert Figure 6.1]

One can easily check that both agents moving right (dark arrows) is an agent-perfect equilibrium. The same can be said if both agents move left (white arrows). However the first cannot be a credible equilibrium, since either agent can strike a credible deviation by suggesting to go left.

7 Conclusions.

We have extended the definition of credible equilibria to infinite games and to games with non-perfect recall. By doing so we encounter a number of difficulties. The possibility of an infinite set of agreements and the fact that players may play one after the other in a cycle called for a non-recursive definition based on semistable partitions. By studying the nature of the ugly set of this semistable partition, we were able to propose a new division of the agreements that is more satisfactory than the standard in the literature.

Other problems remain open. It would be important to find necessary conditions to guarantee a finite quotient set in the equivalence relation defined on the ugly set. Another topics for further research include the extension of alternative definitions to credible equilibria like optimistic credible equilibria and its variants as defined in Ferreira *et al.* (1995). Since all these definitions are extensions of the basic concept of Nash equilibrium, and since the model of changing preferences is intrinsically dynamic, it would be of interest to study extensions of refinements of Nash equilibria to this model.

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APPENDIX

The following concepts and results are contained in Arce and Kahn (1991). The reader is referred to them for more details. Here we have selected the minimum material needed to prove the proposition 6.

Definitions:

An element is (weakly) maximal on the set if no (other) element of the set dominates it.

A stack is an infinite sequence $\{x_i\}$ such that $x_{i+1} \succ x_i$ for all i.

An element is (weakly) subdued if it is dominated by a (weakly) maximal element.

A stack is (weakly) subdued if it contains a (weakly) subdued element.

A stack $(x_i)_i$ has a transitive tail if there is an n such that for every $j > i > n$, $x_j \succ x_i$.

A stack $(x_i)_i$ is intransitive if for no i it is the case that $x_{i+2} \succ x_i$. The relation $\succ$ is (weakly) quasi-eventually-transitive (QET) if every intransitive stack is (weakly) subdued.

Theorem:

If the relation $\succ$ is weakly quasi-eventually-transitive (QET), then for any two semistable partitions $\{G, B, U\}$ and $\{G', B', U'\}$, $G \cap B' = \emptyset$.

Proof

By contradiction. Consider the semistable partitions $\{G, B, U\}$ and $\{G', B', U')$. Since $x_1 \in B'$ there exists $x_2 \in G'$ such that $x_2 \succ x_1$. The element x_2 must be in B. Thus there exists an x_3 such that $x_3 \succ x_2$ and $x_3 \in G$ and $x_3 \in B'$. Since $x_3, x_1 \in G$ they cannot dominate each other. Similarly we can find an x_4 which dominates x_3 and lives in both G' and B. x_4 cannot dominate x_2. Continuing in this way we build an infinite intransitive stack, call it $\{x_t\}$.

Weak QET implies $\exists x_j \in x_t$ which is weakly subdued. Yet because $x_j \in \{x_t\}$, it is in the good set for some semistable partitions $\{G_j, B_j, U_j\}$. This is a contradiction because by definition the maximal element which subdues x_j cannot be in the bad set of any semistable partition. Therefore it is not possible for $x_j \in G_j$ for any j. □

Corollary:

If $\succ$ is weakly QET and has a stable partition it is unique.

FIGURES

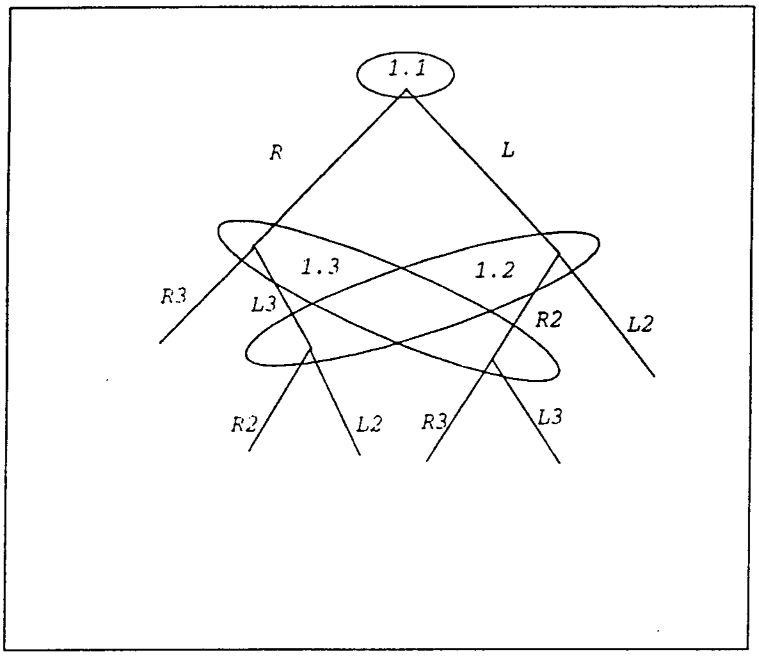

Figure 4.1

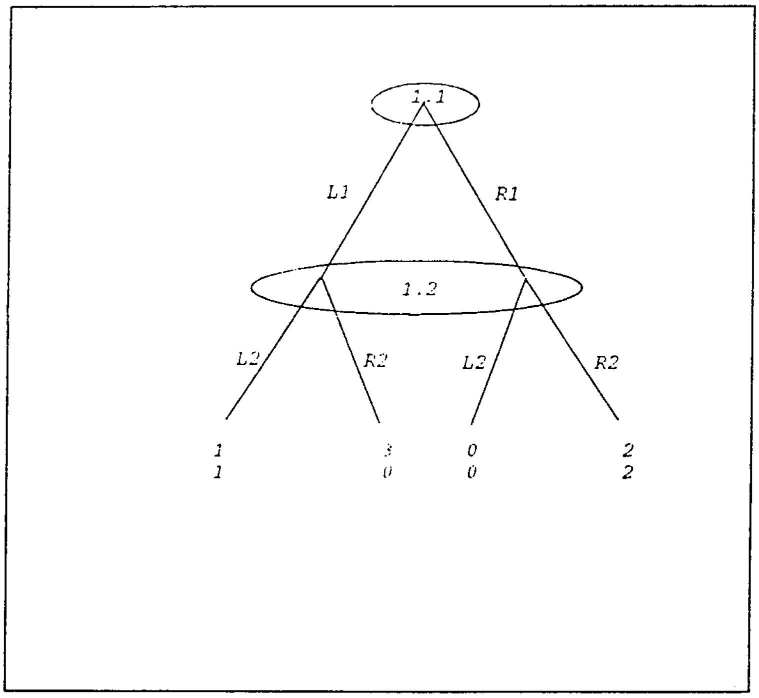

Figure 4.2

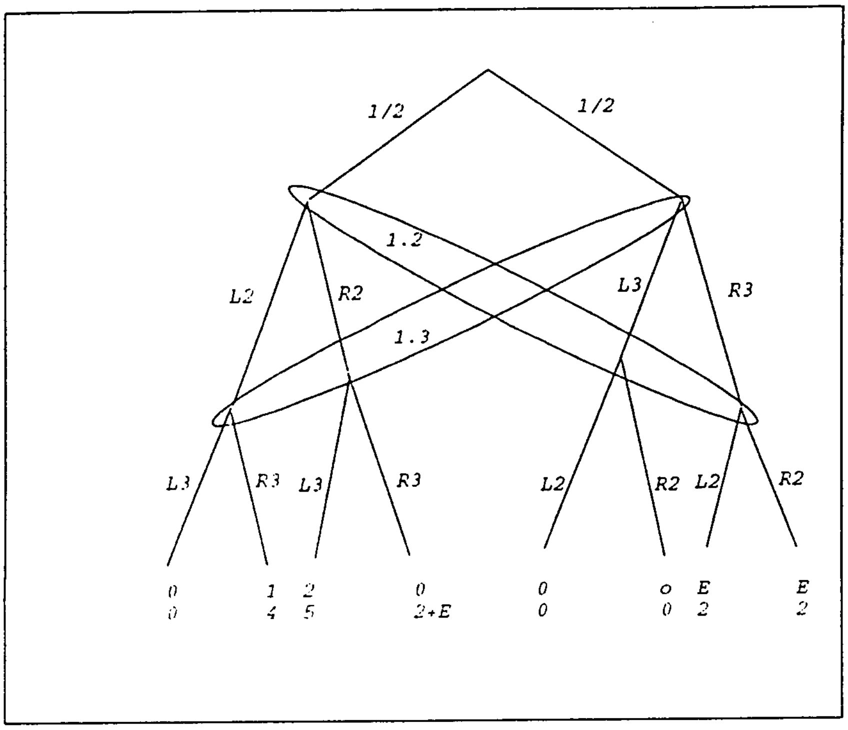

Figure 5.1

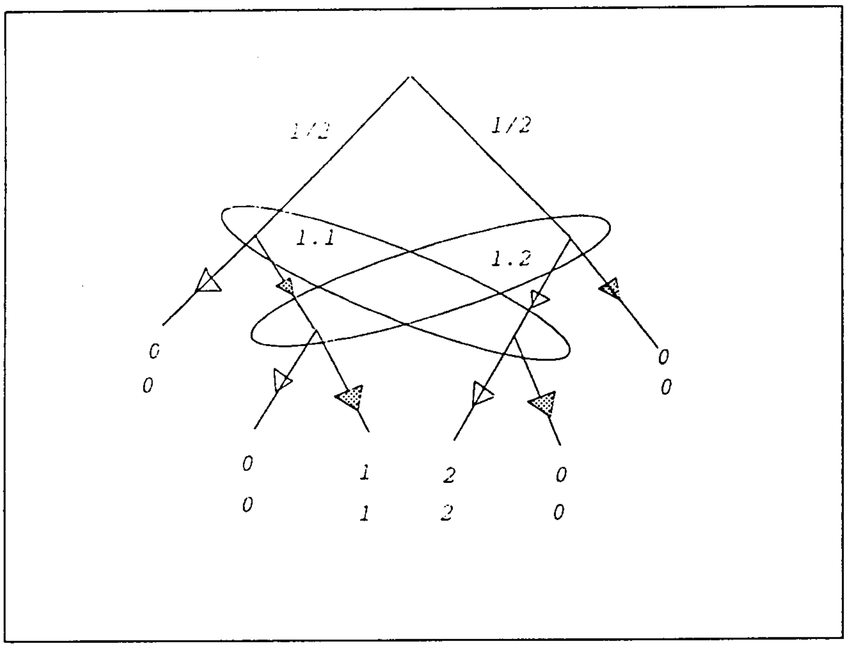

Figure 6.1