

# On Semiclassical Linear Functionals: The Symmetric Companion

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## Abstract

In this paper we analyze the symmetric companion of a quasi-definite linear functional. We focus our attention in semiclassical linear functionals and we analyze the class of the symmetric companion in terms of the starting linear functional. We present some examples for classes zero and one. Finally, we consider symmetric linear functionals related with perturbation by the addition of Dirac masses.

Keywords: Linear functionals, Orthogonal polynomials, Stieltjes functions, Semiclassical functionals, Finite perturbations.

## 1 Introduction

Given a sequence of monic polynomials  $(p_n)$  which are orthogonal with respect to a linear functional  $\mathcal{U}$  it is a natural question to ask how the polynomials behave when a perturbation in the linear functional  $\mathcal{U}$  is introduced. In the literature several situations have been considered. For instance, in [5] (see Chapter 1) the case of a perturbation  $\mathcal{V} = x\mathcal{U}$  is analyzed and the explicit expression for the corresponding sequences of orthogonal polynomials is given. This has been extended to any polynomial perturbation  $\mathcal{V} = h(x)\mathcal{U}$  (see [15]).

In a different way, a perturbation  $\mathcal{V} = \mathcal{U} + M\delta_c$  via the addition of a Dirac linear functional has been introduced in [6] and subsequently generalized in [10].

On the other hand, polynomial mappings appear in sieved process when we modify a sequence of monic orthogonal polynomials  $(p_n)$  in such a way that we are interested to describe a sequence of monic polynomials  $(q_n)$  such that  $q_{nk}(x) = p_n(h(x))$  where  $h(x)$  is a polynomial of degree  $k$ . If  $k = 2$ , this problem was extensively studied in [11] and [12]. In particular, if  $h(x) = x^2$  the connection with the so-called symmetrization problem is quite natural.

In a new direction (see [7]) the generation of non-symmetric sequences of orthogonal polynomials from a given sequence of monic orthogonal polynomials is introduced. The general framework of this problem is described in [14].

After such preliminary comments we will give some basic definitions and tools which are needed for the comprehension of the article.

### 1.1 Basic definitions and tools

Let  $\mathbb{P}$  be the linear space of polynomials with complex coefficients.  $\mathbb{P}_n$  will denote the linear subspace of polynomials of degree at most  $n$ .

Let  $\mathcal{U}$  be a linear functional on  $\mathbb{P}$ . The complex numbers  $(\mathcal{U}_n)$ ,  $\mathcal{U}_n = \langle \mathcal{U}, x^n \rangle$ , where  $\langle \cdot, \cdot \rangle$  means the duality bracket, are said to be the sequence of moments associated with the linear functional  $\mathcal{U}$ .

The set of linear functionals defined on  $\mathbb{P}$  is said to be the algebraic dual space of  $\mathbb{P}$ .

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**Definition 1.1** Let  $h \in \mathbb{P}$ . The linear functional  $h\mathcal{U}$  such that  $\langle h\mathcal{U}, p \rangle = \langle \mathcal{U}, hp \rangle$ ,  $p \in \mathbb{P}$ , is said to be the left-multiplication of  $\mathcal{U}$  by a polynomial  $h$ .

**Definition 1.2** Let  $c \in \mathbb{C}$ . The linear functional  $\delta_c$  such that  $\langle \delta_c, p \rangle = p(c)$ ,  $p \in \mathbb{P}$ , is said to be the Dirac functional at  $c$ .

**Definition 1.3** Let  $c \in \mathbb{C}$ . The linear functional  $(z - c)^{-1}\mathcal{U}$  such that  $\langle (z - c)^{-1}\mathcal{U}, p \rangle = \langle \mathcal{U}, \Theta_c p \rangle$ , where  $\Theta_c(p) = \frac{p(z) - p(c)}{z - c}$  is said to be the left-multiplication of  $\mathcal{U}$  by the rational function  $(z - c)^{-1}$ .

**Definition 1.4** Let  $h = \sum_{i=0}^n h_i z^i$  a polynomial of degree  $n$ . The polynomial

$$(\mathcal{U}h)(z) = (1, z, \dots, z^n) \begin{pmatrix} \mathcal{U}_0 & \cdots & \mathcal{U}_n \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{U}_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \end{pmatrix},$$

is said to be the right-multiplication of the linear functional  $\mathcal{U}$  by  $h$ .

**Definition 1.5** Let  $\mathcal{U}$  be a linear functional. The linear functional  $\mathcal{D}\mathcal{U}$  such that  $\langle \mathcal{D}\mathcal{U}, p \rangle = -\langle \mathcal{U}, p' \rangle$ ,  $p \in \mathbb{P}$ , is said to be the derivative of the linear functional  $\mathcal{U}$ .

**Lemma 1.1** For every polynomial  $p$ ,

$$\langle \mathcal{U}, \Theta_c p \rangle = (\mathcal{U}\Theta_0 p)(c).$$

**Proof:** It is well known that for  $q(z) = \sum_{k=0}^m q_k z^k$ ,  $q_m \neq 0$ ,

$$(\mathcal{U}q)(z) = (1, z, \dots, z^m) \begin{pmatrix} \mathcal{U}_0 & \cdots & \mathcal{U}_m \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{U}_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_m \end{pmatrix}.$$

Thus, if  $p(z) = \sum_{k=0}^m p_k z^k$ ,  $p_m \neq 0$ ,

$$(\mathcal{U}\Theta_0 p)(z) = (1, z, \dots, z^{m-1}) \begin{pmatrix} \mathcal{U}_0 & \cdots & \mathcal{U}_{m-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{U}_0 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}.$$

In other words

$$(\mathcal{U}\Theta_0 p)(c) = (1, c, \dots, c^{m-1}) \begin{pmatrix} \mathcal{U}_0 & \cdots & \mathcal{U}_{m-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{U}_0 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}.$$

On the other hand

$$\Theta_c p = (1, c, \dots, c^{m-1}) \begin{pmatrix} 1 & \cdots & z^{m-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}.$$

Thus,

$$\langle \mathcal{U}, \Theta_c p \rangle = (1, c, \dots, c^{m-1}) \begin{pmatrix} \mathcal{U}_0 & \cdots & \mathcal{U}_{m-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{U}_0 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix},$$

and our result follows. ■

**Lemma 1.2** For every polynomial  $p$

$$\langle \mathcal{U}, \Theta_c^2 p \rangle = (\mathcal{U} \Theta_0 p)'(c).$$

**Proof:** According to the above Lemma

$$\langle \mathcal{U}, \Theta_c^2 p \rangle = [\mathcal{U} \Theta_c (\Theta_0 p)](c).$$

If  $p$  is a polynomial of degree  $m$  as above, then

$$(\mathcal{U} \Theta_0 p)(z) = (1, z, \dots, z^{m-1}) \begin{pmatrix} \mathcal{U}_0 & \cdots & \mathcal{U}_{m-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{U}_0 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}.$$

Thus,

$$\begin{aligned} (\mathcal{U} \Theta_0 p)'(c) &= (0, 1, \dots, (m-1)c^{m-2}) \begin{pmatrix} \mathcal{U}_0 & \cdots & \mathcal{U}_{m-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{U}_0 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} \\ &= (1, 2c, \dots, (m-1)c^{m-2}, mc^{m-1}) \begin{pmatrix} 0 & \mathcal{U}_0 & \cdots & \mathcal{U}_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{U}_0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} \\ &= (1, 2c, \dots, mc^{m-1}) \begin{pmatrix} \mathcal{U}_0 & \cdots & \mathcal{U}_{m-2} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{U}_0 \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix} \\ &= (1, 2c, \dots, (m-1)c^{m-2}) \begin{pmatrix} \mathcal{U}_0 & \cdots & \mathcal{U}_{m-2} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{U}_0 \end{pmatrix} \begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix}. \end{aligned}$$

On the other hand, let  $h(z) = \Theta_0 p(z) = \sum_{j=0}^{m-1} p_{j+1} z^j$ . Then,

$$(\Theta_c h)(z) = (1, z, \dots, z^{m-2}) \begin{pmatrix} 1 & c & \cdots & c^{m-2} \\ 0 & 1 & \cdots & c^{m-3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix},$$

and consequently,

$$(\mathcal{U} \Theta_c h)(z) = (1, z, \dots, z^{m-2}) \begin{pmatrix} \mathcal{U}_0 & \mathcal{U}_1 & \cdots & \mathcal{U}_{m-2} \\ 0 & \mathcal{U}_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mathcal{U}_1 \\ 0 & 0 & \cdots & \mathcal{U}_0 \end{pmatrix} \begin{pmatrix} 1 & c & \cdots & c^{m-2} \\ 0 & 1 & \cdots & c^{m-3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix}.$$

The evaluation of this polynomial at  $c$  yields

$$\begin{aligned} (\mathcal{U} \Theta_c h)(c) &= (1, c, \dots, c^{m-2}) \begin{pmatrix} 1 & c & \cdots & c^{m-2} \\ 0 & 1 & \cdots & c^{m-3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \mathcal{U}_0 & \mathcal{U}_1 & \cdots & \mathcal{U}_{m-2} \\ 0 & \mathcal{U}_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mathcal{U}_1 \\ 0 & 0 & \cdots & \mathcal{U}_0 \end{pmatrix} \begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix} \\ &= (1, 2c, \dots, (m-1)c^{m-2}) \begin{pmatrix} \mathcal{U}_0 & \cdots & \mathcal{U}_{m-2} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{U}_0 \end{pmatrix} \begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix}. \end{aligned}$$

Thus, the statement follows. ■

**Definition 1.6** A linear functional  $\mathcal{U}$  is said to be quasi-definite if the principal submatrices of the Hankel matrix  $(u_{i+j})_{i,j=0}^{\infty}$  are nonsingular.

**Proposition 1.1**  $\mathcal{U}$  is a quasi-definite linear functional if and only if there exists a sequence of monic polynomials  $(p_n)$  with  $\deg p_n = n$  such that

$$i) \quad \langle \mathcal{U}, p_n p_m \rangle = 0, \quad n \neq m.$$

$$ii) \quad \langle \mathcal{U}, p_n^2 \rangle \neq 0, \quad \text{for every } n \in \mathbb{N}.$$

Such a sequence is said to be a sequence of monic orthogonal polynomials with respect to the linear functional  $\mathcal{U}$ .

From the above result, we can deduce.

**Theorem 1.1 (Favard's Theorem)**  $(p_n)$  is a sequence of monic orthogonal polynomials with respect to a quasi-definite linear functional if and only if there exist sequences of complex numbers  $(\beta_n)$  and  $(\gamma_n)$  with  $\gamma_n \neq 0$  for every  $n \in \mathbb{N}$  such that

$$\begin{aligned} xp_n(x) &= p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \quad n \geq 1, \\ p_0 &= 1, \quad p_1(x) = x - \beta_0. \end{aligned}$$

Let  $\mathcal{U}$  be symmetric and quasi-definite linear functional. This means that

$$\langle \mathcal{U}, x^{2n+1} \rangle = 0, \quad n = 0, 1, \dots,$$

and the principal submatrices of the Hankel matrix  $(u_{i+j})_{i,j=0}^{\infty}$  are nonsingular.

Let  $(q_n)$  be the corresponding sequence of monic orthogonal polynomials. They satisfy a three-term recurrence relation

$$\begin{aligned} xq_n(x) &= q_{n+1}(x) + s_n q_{n-1}(x), \quad n \geq 1, \\ q_0 &= 1, \quad q_1(x) = x. \end{aligned}$$

It is very well known (see [5] Chapter 1, Sections 8 and 9) that

$$q_{2n}(x) = p_n(x^2), \quad q_{2n+1}(x) = x r_n(x^2),$$

where  $(p_n)$  and  $(r_n)$  are sequences of monic orthogonal polynomials related to quasi-definite linear functionals  $\mathcal{V}$  and  $x\mathcal{V}$ , respectively, where  $\langle \mathcal{V}, x^n \rangle = \langle \mathcal{U}, x^{2n} \rangle$ .

The linear functional  $\mathcal{U}$  is said to be the symmetrized linear functional associated with the linear functional  $\mathcal{V}$ . We will use the notation  $\mathcal{U} = \mathcal{S}\mathcal{V}$ . Furthermore, taking into account the three-term recurrence relations for  $(p_n)$  and  $(r_n)$

$$\begin{aligned} xp_n(x) &= p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \quad n \geq 1, \\ p_0 &= 1, \quad p_1(x) = x - \beta_0, \end{aligned}$$

and

$$\begin{aligned} xr_n(x) &= r_{n+1}(x) + \zeta_n r_n(x) + \eta_n r_{n-1}(x), \quad n \geq 1, \\ r_0 &= 1, \quad r_1(x) = x - \zeta_0, \end{aligned}$$

we get

$$\begin{cases} \beta_n = s_{2n} + s_{2n+1}, & n \geq 1, & \beta_0 = s_1, \\ \gamma_n = s_{2n-1}s_{2n}, & n \geq 1. \end{cases} \quad \text{and} \quad \begin{cases} \zeta_n = s_{2n+1} + s_{2n+2}, & n \geq 0, \\ \eta_n = s_{2n}s_{2n+1}, & n \geq 1. \end{cases}$$

On the other hand, taking into account the Stieltjes function  $S_{\mathcal{U}}(z) = -\sum_{n=0}^{\infty} \frac{\mathcal{U}_n}{z^{n+1}}$  associated with a linear functional  $\mathcal{U}$  with the moments  $\mathcal{U}_n = \langle \mathcal{U}, x^n \rangle$ , we get

$$S_{\mathcal{V}}(z) = -\sum_{n=0}^{\infty} \frac{\mathcal{V}_n}{z^{n+1}} = -\sum_{n=0}^{\infty} \frac{\mathcal{U}_{2n}}{z^{n+1}}.$$

Thus,

$$S_{\mathcal{V}}(z^2) = -\sum_{n=0}^{\infty} \frac{\mathcal{U}_{2n}}{z^{2n+2}} = \frac{1}{z} S_{\mathcal{U}}(z),$$

i.e.

$$S_{\mathcal{U}}(z) = z S_{\mathcal{V}}(z^2).$$

In the last decades, some attention was paid to the so-called semiclassical linear functionals. They have been introduced by J. Shohat [16] for the weight functions  $\rho$  satisfying a Pearson differential equation, i.e., there exist polynomials  $\sigma$  and  $\tau$  such that  $D(\sigma\rho) = \tau\rho$  together with some extra boundary conditions at the ends of the support interval of  $\rho$ .

In 1985, E. Hendriksen and H. Van Rossum [8] from a different point of view, consider formal power series  $f$  satisfying first order linear differential equations with polynomial coefficients and state the connection between the Padé denominators of  $f$  and the polynomial solutions of some second order differential equation, where the coefficients are also polynomials (the so-called Laguerre-Perron Theorem). Later on [9], the same authors introduce the word semiclassical for weight functions and rewrote the Laguerre-Perron Theorem in terms of weight functions. In 1987, P. Maroni [13] stated the connection between such a kind of polynomials and quasi-orthogonality for their derivatives. Finally, P. Maroni [15] proved the following characterization result for semiclassical orthogonal polynomials (i.e. semiclassical linear functionals).

**Theorem 1.2** *The following statement are equivalent*

i)  $\mathcal{U}$  is a quasi-definite linear functional such that

$$D(\sigma\mathcal{U}) = \tau\mathcal{U}, \tag{1.1}$$

where  $\sigma, \tau$  are polynomials with  $\deg \sigma = t > 0$  and  $\deg \tau = p \geq 1$ .

ii) The Stieltjes function  $S(z) = -\sum_{n=0}^{\infty} \frac{\mathcal{U}_n}{z^{n+1}}$  satisfies

$$\sigma(z)S'_{\mathcal{U}}(z) = C(z)S_{\mathcal{U}}(z) + D(z), \tag{1.2}$$

where

$$\begin{aligned} C(z) &= -\sigma'(z) + \tau(z), \\ D(z) &= -(\mathcal{U}\Theta_0\sigma)'(z) + (\mathcal{U}\Theta_0\tau)(z). \end{aligned}$$

iii) There exists a non-negative integer number  $s$  such that

$$\sigma(z)p'_{n+1}(z) = \sum_{k=n-s}^{n+t} a_{n,k}p_k(z), \quad n \geq s,$$

with  $a_{n,n-s} \neq 0$ ,  $n \geq s+1$ .

P. Maroni [15] introduced the concept of the class of a semiclassical linear functional taking into account that the Pearson equation (1.1) holds for an infinite family of pairs of polynomials  $(\sigma, \tau)$ . Indeed, if

$$D(\sigma_1\mathcal{U}) = \tau_1\mathcal{U}, \quad \text{i.e.} \quad \sigma_1 D\mathcal{U} + \sigma_1'\mathcal{U} = \tau_1\mathcal{U},$$

multiplying by a polynomial  $\pi(z)$  in both hand sides

$$\pi\sigma_1 D\mathcal{U} + \pi\sigma'_1 \mathcal{U} = \pi\tau_1 \mathcal{U},$$

or, equivalently

$$D(\pi\sigma_1 \mathcal{U}) = (\pi\tau_1 + \pi'\sigma_1)\mathcal{U}.$$

Thus,  $\{(\pi\sigma_1, \pi\tau_1 + \pi'\sigma_1)\}$  for every polynomial  $\pi$  is a set of pairs of polynomials associated with the Pearson equation (1.1). Thus,

**Definition 1.7** If  $(\sigma, \tau)$  is the pair of polynomials of minimum degree such that (1.1) holds, we define the class of  $\mathcal{U}$  as  $s = \max\{\deg \sigma - 2, \deg \tau - 1\}$ .

**Theorem 1.3**  $\mathcal{U}$  is a semiclassical linear functional of class  $s$  if and only if

$$|\sigma'(c) - \tau(c)| + |\langle \mathcal{U}, \Theta_c^2 \sigma - \Theta_c \tau \rangle| > 0,$$

for every zero  $c$  of the polynomial  $\sigma$ .

**Proof:** Let  $\sigma(c) = 0$ . Taking into account  $\sigma D\mathcal{U} = (\tau - \sigma')\mathcal{U}$ , if  $\tau(c) - \sigma'(c) \neq 0$  then we can not simplify the above equation.

Now, we consider the case  $\tau(c) - \sigma'(c) = 0$ . Take

$$\sigma(z) = (z - c)\sigma_c(z), \quad \tau(z) - \sigma'(z) = (z - c)\tau_c(z).$$

Then,

$$\sigma_c(z) D\mathcal{U} = \tau_c(z)\mathcal{U} + M\delta_c,$$

where

$$\begin{aligned} M &= \langle \sigma_c(z) D\mathcal{U}, 1 \rangle - \langle \tau_c(z) \mathcal{U}, 1 \rangle \\ &= \langle D\mathcal{U}, \sigma_c(z) \rangle - \langle \mathcal{U}, \tau_c(z) \rangle \\ &= \left\langle D\mathcal{U}, \frac{\sigma(z) - \sigma(c)}{z - c} \right\rangle - \left\langle \mathcal{U}, \frac{\tau(z) - \sigma'(z) - \tau(c) + \sigma'(c)}{z - c} \right\rangle \\ &= \left\langle D\mathcal{U}, \frac{\sigma(z) - \sigma(c)}{z - c} \right\rangle - \left\langle \mathcal{U}, \frac{\tau(z) - \tau(c)}{z - c} \right\rangle + \left\langle \mathcal{U}, \frac{\sigma'(z) - \sigma'(c)}{z - c} \right\rangle. \end{aligned}$$

But the above expression becomes

$$\begin{aligned} M &= -\langle \mathcal{U}, \Theta_c \tau \rangle - \left\langle \mathcal{U}, D \left( \sigma'(c) + \frac{\sigma''(c)}{2!}(z - c) + \dots \right) - \sigma''(c) - \frac{\sigma'''(c)(z - c)}{2!} - \dots \right\rangle \\ &= -\langle \mathcal{U}, \Theta_c \tau \rangle - \left\langle \mathcal{U}, -\frac{1}{2}\sigma''(c) - \frac{1}{3!}\sigma'''(c)(z - c) - \dots \right\rangle \\ &= -\langle \mathcal{U}, \Theta_c \tau \rangle + \left\langle \mathcal{U}, \frac{1}{2}\sigma''(c) + \frac{1}{3!}\sigma'''(c)(z - c) - \dots \right\rangle \\ &= -\langle \mathcal{U}, \Theta_c \tau \rangle + \left\langle \mathcal{U}, \frac{\sigma(z) - \sigma(c) - \sigma'(c)(z - c)}{(z - c)^2} \right\rangle \\ &= \langle \mathcal{U}, \Theta_c^2 \sigma \rangle - \langle \mathcal{U}, \Theta_c \tau \rangle. \end{aligned}$$

Thus, if  $\tau(c) - \sigma'(c) = 0$ , we can simplify the Pearson equation, i.e., we can reduce the class if and only if  $M = 0$ , or, equivalently,

$$\langle \mathcal{U}, \Theta_c^2 \sigma \rangle = \langle \mathcal{U}, \Theta_c \tau \rangle.$$

■

**Corollary 1.1**  $(\sigma, \tau)$  is the pair of polynomials of minimum degree for (1.1) if and only if the polynomials  $(\sigma, C, D)$  in (1.2) are coprime, i.e., their greatest common divisor is 1.

**Proof:** According to Theorem 1.2 it is enough to prove that if  $c$  is a zero of  $\sigma$ , then either  $C(c) \neq 0$  or  $C(c) = 0$  and  $D(c) \neq 0$ . But it follows from Lemma 1.1 and Lemma 1.2. ■

## 1.2 The aim of the paper

In this paper we will analyze two problems concerning symmetric linear functionals.

In Section 2, we prove that if  $\mathcal{V}$  is a semiclassical linear functional of class  $s$ , then the linear functional  $\mathcal{U}$  such that  $\mathcal{U} = \mathcal{S}\mathcal{V}$  is also semiclassical and the class of  $\mathcal{U}$  is  $2s$ ,  $2s + 1$  or  $2s + 3$  (see Theorem 2.1 below). In Section 3 we give some examples of symmetric linear functionals of class 0 and 1.

In Section 4, we consider a perturbation of  $\mathcal{U}$ ,  $\tilde{\mathcal{U}} = \mathcal{U} + M\delta_0$  which preserves the symmetry, and we analyze the relation between the parameters  $\varsigma_n$  for both linear functionals in such a way that we can deduce an algorithm for the parameters of the three-term recurrence relation satisfied by the sequence of monic orthogonal polynomials corresponding to  $\tilde{\mathcal{V}}$ , where  $\tilde{\mathcal{U}} = \mathcal{S}\tilde{\mathcal{V}}$ .

## 2 Main Theorem

From  $S_{\mathcal{U}}(z) = zS_{\mathcal{V}}(z^2)$  we get

$$S'_{\mathcal{U}}(z) = S_{\mathcal{V}}(z^2) + 2z^2 S'_{\mathcal{V}}(z^2).$$

But if the linear functional  $\mathcal{V}$  is semiclassical, then

$$\sigma(z)S'_{\mathcal{V}}(z) = C(z)S_{\mathcal{V}}(z) + D(z), \quad (2.3)$$

where  $C$  and  $D$  are the polynomials introduced in (1.2).

Taking into account the change of variables  $z$  by  $z^2$  in the above equation

$$\sigma(z^2)S'_{\mathcal{V}}(z^2) = C(z^2)S_{\mathcal{V}}(z^2) + D(z^2).$$

Thus,

$$\sigma(z^2) [S'_{\mathcal{U}}(z) - S_{\mathcal{V}}(z^2)] = 2zC(z^2)S_{\mathcal{U}}(z) + 2z^2D(z^2),$$

i.e.

$$z\sigma(z^2)S'_{\mathcal{U}}(z) = [2z^2C(z^2) + \sigma(z^2)] S_{\mathcal{U}}(z) + 2z^3D(z^2). \quad (2.4)$$

Several situations can be considered in order to the polynomial coefficients in (2.4) yield the minimal condition:

**First:** If  $\sigma(0) = 0$ , i.e.  $\sigma(z) = zE(z)$  then (2.4) becomes

$$zE(z^2)S'_{\mathcal{U}}(z) = [2C(z^2) + E(z^2)] S_{\mathcal{U}}(z) + 2zD(z^2). \quad (2.5)$$

Again, two cases must be discussed

**1.1)**  $2C(0) + E(0) = 0$ , i.e.  $2C(z) + E(z) = zG(z)$ . Then,

$$E(z^2)S'_{\mathcal{U}}(z) = zG(z^2)S_{\mathcal{U}}(z) + 2D(z^2). \quad (2.6)$$

Let  $\alpha \neq 0$  be such that  $\sigma(\alpha^2) = E(\alpha^2) = 0$ .

If  $D(\alpha^2) \neq 0$ , the above equation can not be simplified, because the polynomial coefficients are coprime.

If  $D(\alpha^2) = 0$ , then the above equation can be simplified if and only if  $G(\alpha^2) = 0$ . But this means that  $C(\alpha^2) = 0$ . In other words, the coefficients in (2.3) are not coprime, a contradiction. As a conclusion, (2.6) cannot be simplified. Thus, we get

$$\begin{aligned} \tilde{\sigma}(z) &= E(z^2), \\ \tilde{\tau}(z) &= z [G(z^2) + 2E'(z^2)]. \end{aligned} \quad (2.7)$$

**1.2)**  $2C(0) + E(0) \neq 0$ . Let  $\alpha \neq 0$  such that  $E(\alpha^2) = 0$ .

If  $2C(\alpha^2) + E(\alpha^2) \neq 0$ , the equation (2.5) cannot be simplified.

If  $2C(\alpha^2) + E(\alpha^2) = 0$ , then  $C(\alpha^2) = 0$ . From the minimal condition in (2.3),  $D(\alpha^2) \neq 0$ . Thus, (2.5) cannot be simplified. In such a case

$$\begin{aligned}\tilde{\sigma}(z) &= zE(z^2), \\ \tilde{\tau}(z) &= 2[E(z^2) + z^2E'(z^2) + C(z^2)].\end{aligned}\tag{2.8}$$

**Second:** If  $\sigma(0) \neq 0$ , let  $\alpha \neq 0$  be such that  $\sigma(\alpha^2) = 0$ . Then, two cases must be discussed.

**2.1)** If  $C(\alpha^2) = 0$  then from (2.3)  $D(\alpha^2) \neq 0$  and thus (2.4) cannot be simplified.

**2.2)** If  $C(\alpha^2) \neq 0$ , then (2.4) cannot be simplified.

As a conclusion, if  $\sigma(0) \neq 0$ ,

$$\begin{aligned}\tilde{\sigma}(z) &= z\sigma(z^2), \\ \tilde{\tau}(z) &= 2[\sigma(z^2) + z^2\sigma'(z^2) + z^2C(z^2)].\end{aligned}\tag{2.9}$$

Now, we can discuss the class for the semiclassical linear functional  $\mathcal{U}$  taking into account the three possible situations analyzed as above.

Notice that if  $s$  is the class of  $\mathcal{V}$  it means that

- (a) Either  $\deg \sigma = s + 2$  and  $\deg \tau < s + 1$ .
- (b) or  $\deg \sigma < s + 2$  and  $\deg \tau = s + 1$ .
- (c) or  $\deg \sigma = s + 2$  and  $\deg \tau = s + 1$ .

**Proposition 2.1** *If (a) holds then*

i)  $\tilde{s} = 2s$  for (2.7).

ii)  $\tilde{s} = 2s + 1$  for (2.8).

iii)  $\tilde{s} = 2s + 3$  for (2.9).

**Proof:** i) Let  $\sigma(z) = a_0z^{s+2} + \text{lower degree terms}$  and  $\tau(z) = b_0z^s + \text{lower degree terms}$ . Then  $E(z) = a_0z^{s+1} + \dots$ . Since

$$C(z) = -\sigma'(z) + \tau(z) = -(s+2)a_0z^{s+1} + \dots.$$

Thus,  $G(z) = -(2s+3)a_0z^s + \dots$ . Taking into account (2.7) we get

$$\begin{aligned}\tilde{\tau}(z) &= z[-(2s+3)a_0z^{2s} + \dots + 2(s+1)a_0z^{2s} + \dots] \\ &= -a_0z^{2s+1} + \dots,\end{aligned}$$

and then  $\deg \tilde{\sigma} = 2s+2$  and  $\deg \tilde{\tau}(z) = 2s+1$ . As a consequence  $\tilde{s} = \max\{\deg \tilde{\sigma} - 2, \deg \tilde{\tau} - 1\} = 2s$ .

ii)  $\deg \tilde{\sigma} = 2s + 3$ .

$$\tilde{\tau}(z) = 2[a_0z^{2s+2} + \dots + a_0(s+1)z^{2s+2} + \dots - (s+2)a_0z^{2s+2} + \dots].$$

Thus,  $\deg \tilde{\tau}(z) \leq 2s + 1$ . Then,  $\tilde{s} = \max\{\deg \tilde{\sigma} - 2, \deg \tilde{\tau} - 1\} = 2s + 1$ .

iii)  $\deg \tilde{\sigma} = 2s + 5$ ,

$$\begin{aligned}\tilde{\tau}(z) &= 2[a_0z^{2s+4} + \dots + a_0(s+2)z^{2s+4} + \dots - (s+2)a_0z^{2s+4} + \dots] \\ &= 2a_0z^{2s+4}.\end{aligned}$$

Thus,  $\deg \tilde{\tau} = 2s + 4$ . Then,  $\tilde{s} = \max\{\deg \tilde{\sigma} - 2, \deg \tilde{\tau} - 1\} = 2s + 3$ . ■

**Proposition 2.2** *If b) holds, then*

i)  $\tilde{s} = 2s$  for (2.7).

ii)  $\tilde{s} = 2s + 1$  for (2.8).

iii)  $\tilde{s} = 2s + 3$  for (2.9).

**Proof:** Let  $\sigma(z) = a_0 z^{s+1} + \text{lower degree terms}$ , and  $\tau(z) = b_0 z^{s+1} + \text{lower degree terms}$ , where eventually  $a_0 = 0$ .

i)  $E(z) = a_0 z^s + \text{lower degree terms}$ . Since

$$C(z) = -\sigma'(z) + \tau(z) = b_0 z^{s+1} + \dots.$$

Thus,  $G(z) = 2b_0 z^s + \text{lower degree terms}$ . Taking into account (2.7) we get  $\deg \tilde{\sigma} = 2s + 2$  as well as

$$\begin{aligned} \tilde{\tau}(z) &= z [2b_0 z^{2s} + \dots + 2a_0 s z^{2s-2} + \dots] \\ &= 2b_0 z^{2s+1} + \dots. \end{aligned}$$

Thus,  $\deg \tilde{\tau} = 2s + 1$ . Then,  $\tilde{s} = \max\{\deg \tilde{\sigma} - 2, \deg \tilde{\tau} - 1\} = 2s$ .

ii)  $\deg \tilde{\sigma} \leq 2s + 1$ ,

$$\begin{aligned} \tilde{\tau}(z) &= 2 [2a_0 z^{2s} + \dots + s a_0 z^{2s} + \dots + b_0 z^{2s+2} + \dots] \\ &= 2b_0 z^{2s+2} + \dots. \end{aligned}$$

Thus,  $\deg \tilde{\tau} = 2s + 2$ . Then,  $\tilde{s} = \max\{\deg \tilde{\sigma} - 2, \deg \tilde{\tau} - 1\} = 2s + 1$ .

iii)  $\deg \tilde{\sigma} \leq 2s + 3$ ,

$$\tilde{\tau}(z) = 2 [2a_0 z^{2s+2} + \dots + (s+1)a_0 z^{2s+2} + \dots + b_0 z^{2s+4} + \dots].$$

Thus,  $\deg \tilde{\tau} = 2s + 4$ . Then,  $\tilde{s} = \max\{\deg \tilde{\sigma} - 2, \deg \tilde{\tau} - 1\} = 2s + 3$ . ■

**Proposition 2.3** *If c) holds, then*

i)  $\tilde{s} = 2s$  for (2.7).

ii)  $\tilde{s} = 2s + 1$  for (2.8).

iii)  $\tilde{s} = 2s + 3$  for (2.9).

**Proof:** Let  $\sigma(z) = a_0 z^{s+2} + \text{lower degree terms}$  and  $\tau(z) = b_0 z^{s+1} + \text{lower degree terms}$ , with  $a_0 \neq 0$ , and  $b_0 \neq 0$ .

i)  $E(z) = a_0 z^{s+1} + \text{lower degree terms}$ . Since

$$C(z) = -\sigma'(z) + \tau(z) = [b_0 - a_0(s+2)]z^{s+1} + \dots.$$

Thus,  $G(z) = [2b_0 - (2s+3)a_0]z^s + \text{lower degree terms}$ . Taking into account (2.7) we get  $\deg \tilde{\sigma} \leq 2s$  as well as

$$\begin{aligned} \tilde{\tau}(z) &= z [(2b_0 - (2s+3)a_0)z^{2s} + \dots + 2a_0(s+1)z^{2s} + \dots] \\ &= z[2b_0 - a_0]z^{2s} + \dots. \end{aligned}$$

Thus,  $\tilde{s} = \max\{\deg \tilde{\sigma} - 2, \deg \tilde{\tau} - 1\} = 2s$ .

ii)  $\deg \tilde{\sigma} = 2s + 3$ ,

$$\begin{aligned} \tilde{\tau}(z) &= 2 [a_0 z^{2s+2} + \dots + a_0(s+1)z^{2s+2} + \dots + [b_0 - a_0(s+2)]z^{2s+2} + \dots] \\ &= 2[b_0 z^{2s+2} + \dots]. \end{aligned}$$

Thus,  $\deg \tilde{\tau} = 2s + 2$ . Then,  $\tilde{s} = \max\{\deg \tilde{\sigma} - 2, \deg \tilde{\tau} - 1\} = 2s + 1$ .

iii)  $\deg \tilde{\sigma} = 2s + 5$ ,

$$\begin{aligned}\tilde{\tau}(z) &= 2 [a_0 z^{2s+4} + \dots + a_0(s+2)z^{2s+4} + \dots + [b_0 - a_0(s+2)]z^{2s+4} + \dots] \\ &= (b_0 + a_0)z^{2s+4} + \dots.\end{aligned}$$

Thus,  $\tilde{s} = \max\{\deg \tilde{\sigma} - 2, \deg \tilde{\tau} - 1\} = 2s + 3$ . ■

As a conclusion, we can summarize the above results in Propositions 2.1-2.3 as follows.

**Theorem 2.1** *Let  $\mathcal{V}$  be a semiclassical linear functional of class  $s$  and  $\mathcal{U}$  be its symmetrized linear functional. Then  $\mathcal{U}$  is a semiclassical linear functional of class  $\tilde{s}$  with*

i)  $\tilde{s} = 2s$  if (2.7) holds.

ii)  $\tilde{s} = 2s + 1$  if (2.8) holds.

iii)  $\tilde{s} = 2s + 3$  if (2.9) holds.

### 3 Examples

#### 3.1 Classical symmetric linear functionals

Taking into account Theorem 2.1, symmetric classical linear functionals can appear if and only if the linear functional  $\mathcal{V}$  is classical together with the facts  $\sigma(0) = 0$  and  $2[\tau(0) - \sigma'(0)] + E(0) = 0$ , where  $\sigma(z) = zE(z)$ .

Thus, because for a classical linear functional  $\deg \sigma \leq 2$  and  $\deg \tau = 1$  we will consider three cases.

a) Let  $\sigma(z) = z$ ,  $\tau(z) = az + b$ ,  $a \neq 0$ .

Thus, according to Theorem 2.1,

$$2(b - 1) + 1 = 0 \Rightarrow b = \frac{1}{2}.$$

On the other hand,

$$zG(z) = 2C(z) + E(z) = 2 \left[ az + \frac{1}{2} - 1 \right] + 1 = 2az.$$

Thus, from (2.7)

$$\begin{aligned}\tilde{\sigma}(z) &= 1, \\ \tilde{\tau}(z) &= 2az.\end{aligned}$$

Hence, up to a linear change in the variable,  $\mathcal{U}$  is the Hermite linear functional.

b) Let  $\sigma(z) = z(z - 1)$ ,  $\tau(z) = az + b$ ,  $a \neq 0$ .

Thus, according to Theorem 2.1,

$$2(b + 1) - 1 = 0 \Rightarrow b = -\frac{1}{2}.$$

On the other hand,

$$zG(z) = 2C(z) + E(z) = 2 \left[ az - \frac{1}{2} - 2z + 1 \right] + z - 1 = (2a - 3)z.$$

Thus, from (2.7)

$$\begin{aligned}\tilde{\sigma}(z) &= z^2 - 1, \\ \tilde{\tau}(z) &= z[2a - 3 + 2] = (2a - 1)z.\end{aligned}$$

Hence, up to a linear change in the variable,  $\mathcal{U}$  is the Gegenbauer linear functional.

c) Let  $\sigma(z) = z^2$ ,  $\tau(z) = az + b$ ,  $a \neq 0$ .

Thus, according to Theorem 2.1,

$$2b = 0 \Rightarrow b = 0.$$

On the other hand,

$$zG(z) = 2C(z) + E(z) = 2[az - 2z] + z = (2a - 3)z.$$

Thus, from (2.7)

$$\begin{aligned}\tilde{\sigma}(z) &= z^2, \\ \tilde{\tau}(z) &= z[(2a - 3) + 2] = (2a - 1)z.\end{aligned}$$

This means that  $D(z^2\mathcal{U}) = (2a - 1)z\mathcal{U}$ . In other words, for  $n \in \mathbb{N}$ ,

$$\begin{aligned}\langle z^2\mathcal{U}, nz^{n-1} \rangle &= (1 - 2a) \langle \mathcal{U}, z^{n+1} \rangle \Rightarrow n\mathcal{U}_{n+1} = (1 - 2a)\mathcal{U}_{n+1}, \\ (n + 2a - 1)\mathcal{U}_{n+1} &= 0.\end{aligned}$$

Taking into account that  $\mathcal{U}$  is a symmetric linear functional, the above condition yields

$$2(m + a)\mathcal{U}_{2m+2} = 0.$$

Thus  $\mathcal{U}_{2p} = 0$  for every  $p \in \mathbb{N}$  up to at most one  $p_0 \in \mathbb{N}$  such that  $p_0 = -a$  (if  $a \in \mathbb{Z}^-$ ) or  $\mathcal{U}_{2p} = 0$  for every  $p \in \mathbb{N}$ . In both cases,  $\mathcal{U}$  is not quasi-definite.

### 3.2 Semiclassical symmetric linear functional of class $\tilde{s} = 1$

Taking into account Theorem 2.1, this kind of semiclassical linear functionals can appear if and only if the linear functional  $\mathcal{V}$  is classical together with the facts  $\sigma(0) = 0$ ,  $2[\tau(0) - \sigma'(0)] + E(0) \neq 0$ .

a) Let  $\sigma(z) = z$ ,  $\tau(z) = az + b$ ,  $a \neq 0$ .

Thus  $E(z) = 1$  and  $2[b - 1] + 1 \neq 0$ . As a consequence,  $b \neq \frac{1}{2}$ .

On the other hand, from (2.8)

$$\begin{aligned}\tilde{\sigma}(z) &= z, \\ \tilde{\tau}(z) &= 2[1 + az^2 + b - 1] \\ &= 2(az^2 + b).\end{aligned}$$

The linear functional  $\mathcal{U}$  is, up to a linear change in the variable, the Hermite-Chihara linear functional (see [5]).

Indeed, from

$$D(z\mathcal{U}) = 2(az^2 + b)\mathcal{U},$$

we get

$$-n\mathcal{U}_n = 2a\mathcal{U}_{n+2} + 2b\mathcal{U}_n \Rightarrow 2a\mathcal{U}_{n+2} = -(2b + n)\mathcal{U}_n.$$

Because the linear functional  $\mathcal{U}$  is symmetric,

$$\begin{aligned}2a\mathcal{U}_{2p+2} &= -2(b + p)\mathcal{U}_{2p}, \quad p \in \mathbb{N}, \\ a\mathcal{U}_{2p+2} &= -(b + p)\mathcal{U}_{2p}, \Rightarrow \mathcal{U}_{2p} = (-1)^p \frac{(b)_p}{a^p} \mathcal{U}_0,\end{aligned}$$

as well as  $\mathcal{U}_{2p+1} = 0$ ,  $p \in \mathbb{N}$ .

Notice that is  $-b \in \mathbb{N}$ ,  $\mathcal{U}_{2k} = 0$  for  $k \geq -b$ .  $\mathcal{U}$  is not a quasi-definite linear functional.

b) Let  $\sigma(z) = z(z-1)$ ,  $\tau(z) = az + b$ ,  $a \neq 0$ .

Thus,  $E(z) = z - 1$  and  $2[b+1] - 1 \neq 0$ , i.e.  $b \neq \frac{1}{2}$ .

On the other hand, from (2.8)

$$\begin{aligned}\tilde{\sigma}(z) &= (z^2 - 1)z, \\ \tilde{\tau}(z) &= 2[z^2 - 1 + z^2 + az^2 + b - 2z^2 + 1] \\ &= 2[az^2 + b].\end{aligned}$$

The linear functional  $\mathcal{U}$  is up to a linear change in the variable, a generalized Gegenbauer linear functional (with an extra knot at  $z = 0$ ).

Indeed

$$D(z^3 - z)\mathcal{U} = (2az^2 + 2b)\mathcal{U},$$

yields

$$n(\mathcal{U}_{n+2} - \mathcal{U}_n) = 2a\mathcal{U}_{n+2} + 2b\mathcal{U}_n,$$

i.e.

$$(2a - n)\mathcal{U}_{n+2} = -(n + 2b)\mathcal{U}_n.$$

Because the linear functional  $\mathcal{U}$  is symmetric

$$\begin{aligned}2(a - p)\mathcal{U}_{2p+2} &= -(b + p)\mathcal{U}_{2p}, \quad p \in \mathbb{N}, \\ (p - a)\mathcal{U}_{2p+2} &= \left(p + \frac{b}{2}\right)\mathcal{U}_{2p}, \Rightarrow \mathcal{U}_{2p} = \frac{\left(\frac{b}{2}\right)_p}{(-a)_p}\mathcal{U}_0.\end{aligned}$$

Notice that we assume  $a \notin \mathbb{N}$ ; otherwise if  $a \in \mathbb{N}$  and  $-\frac{b}{2} \notin \mathbb{N}$ , then  $\mathcal{U}_{2k} = 0$ ,  $k \leq a$ .  $\mathcal{U}$  is not a quasi-definite linear functional.

c) Let  $\sigma(z) = z^2$ ,  $\tau(z) = az + b$ ,  $b \neq 0$ .

Thus  $E(z) = z$  and  $b \neq 0$ . On the other hand, from (2.8)

$$\begin{aligned}\tilde{\sigma}(z) &= z^3, \\ \tilde{\tau}(z) &= 2[z^2 + z^2 + az^2 + b - 2z^2] \\ &= 2(az^2 + b).\end{aligned}$$

This linear functional is the usual symmetrized of the Bessel linear functional.

These three examples are the unique symmetrized linear functionals in the eight canonical semiclassical functional equations given in [4]. See also [1] in a more general framework of Laguerre-Hahn linear functionals. An example of non-symmetric semiclassical linear functional of class 1 associated with the Jacobi linear functional is presented in [2].

### 3.3 The symmetric companion with respect to the linear functional $\mathcal{J}(\alpha, \alpha+1)(\mu)$

The symmetric companion of the polynomial sequence  $\{p_n^{(\alpha, \alpha+1)}(x; \mu)\}$  orthogonal with respect to the linear functional  $\mathcal{J}(\alpha, \alpha+1)(\mu)$  given in [2]:

$$p_0^{(\alpha, \alpha+1)}(x; \mu) = 1, \quad p_1^{(\alpha, \alpha+1)}(x; \mu) = x - \beta_0,$$

$$p_{n+2}^{(\alpha, \alpha+1)}(x; \mu) = (x - \beta_{n+1})p_{n+1}^{(\alpha, \alpha+1)}(x; \mu) - \gamma_{n+1}p_n^{(\alpha, \alpha+1)}(x; \mu), \quad n \geq 0,$$

where

$$\beta_0 = -\frac{\mu - 1}{\mu - 2\alpha + 3}, \quad \beta_{n+1} = (-1)^n \frac{\mu(\mu - 2n - 2\alpha - 4) + (-1)^{n+1}(2\alpha + 1)}{(2n + 2\alpha + 3 - \mu)(2n + 2\alpha + 5 - \mu)},$$

$$\gamma_{2n+1} = \frac{2(n+\alpha+1)(2n+1-\mu)}{(4n+2\alpha+3-\mu)^2}, \quad \gamma_{2n+2} = \frac{(2n+2)(2n+2\alpha+3-\mu)}{(4n+2\alpha+5-\mu)^2}, \quad n \geq 0,$$

is the sequence  $\{q_n(x)\}$  orthogonal with respect to the linear functional  $\mathcal{U}$  such that  $\mathcal{U} = \mathcal{S}\mathcal{J}(\alpha, \alpha+1)(\mu)$  fulfils the following three term recurrence relation

$$q_0(x) = 1, \quad q_1(x) = x,$$

$$q_{n+2}(x) = xq_{n+1}(x) - \varsigma_{n+1}q_n(x), \quad n \geq 0,$$

with

$$\begin{aligned} \varsigma_{4n+1} &= -\frac{2n+1-\mu}{4n+2\alpha+3-\mu}, & \varsigma_{4n+2} &= -\frac{2(n+\alpha+1)}{4n+2\alpha+3-\mu}, \\ \varsigma_{4n+3} &= \frac{2n+2\alpha+3-\mu}{4n+2\alpha+5-\mu}, & \varsigma_{4n+4} &= \frac{2n+2}{4n+2\alpha+5-\mu}. \end{aligned}$$

Moreover,  $\mathcal{U}$  is of class  $\tilde{s} = 3$  if and only if  $\mu \neq \frac{1}{2}$ . If  $\mu = \frac{1}{2}$  then  $\mathcal{U}$  of class  $\tilde{s} = 2$ .

### 3.4 The symmetric companion with respect to the linear functional $\mathcal{V}$

The symmetric companion of the polynomial sequence  $\{p_n(x)\}$  orthogonal with respect to the linear functional  $\mathcal{V}$  given in [3]:

$$p_0(x) = 1, \quad p_1(x) = x - \beta_0,$$

$$p_{n+2}(x) = (x - \beta_{n+1})p_{n+1}(x) - \gamma_{n+1}p_n(x), \quad n \geq 0,$$

where

$$\begin{aligned} \beta_n &= (-1)^{n+1}, \quad \gamma_{2n+1} = -\frac{2(n+\alpha+1)(2n+2\alpha+2p+3)}{(4n+2\alpha+2p+3)(4n+2\alpha+2p+5)}, \\ \gamma_{2n+2} &= -\frac{(2n+2)(2n+2p+3)}{(4n+2\alpha+2p+5)(4n+2\alpha+2p+7)}, \quad n \geq 0, \end{aligned}$$

is the sequence  $\{q_n(x)\}$  orthogonal with respect to the linear functional  $\mathcal{U}$  such that  $\mathcal{U} = \mathcal{S}\mathcal{V}$  fulfils the following three term recurrence relation

$$q_0(x) = 1, \quad q_1(x) = x,$$

$$q_{n+2}(x) = xq_{n+1}(x) - \varsigma_{n+1}q_n(x), \quad n \geq 0,$$

with

$$\begin{aligned} \varsigma_{4n+1} &= -\frac{2n2\alpha+2p+3}{4n+2\alpha+2p+3}, & \varsigma_{4n+2} &= \frac{2(n+\alpha+2)}{4n+2\alpha+2p+5}, \\ \varsigma_{4n+3} &= \frac{2n+2p+3}{4n+2\alpha+2p+5}, & \varsigma_{4n+4} &= -\frac{2n+2}{4n+2\alpha+2p+7}. \end{aligned}$$

## 4 Recurrence coefficients

Let  $\mathcal{U}$  be a quasi-definite and symmetric linear functional. We will denote as in Section 1 ( $q_n$ ) the corresponding sequence of monic orthogonal polynomials.

We introduce the linear functional  $\tilde{\mathcal{U}} = \mathcal{U} + M\delta_0$  which is also symmetric. In [10] a necessary and sufficient condition in order to  $\tilde{\mathcal{U}}$  be quasi-definite is given.

Indeed, this is  $1 + MK_n(0, 0) \neq 0$  for every  $n \in \mathbb{N}$ , where

$$K_n(x, y) = \sum_{j=0}^n \frac{q_j(x)q_j(y)}{\langle \mathcal{U}, q_j^2(x) \rangle}.$$

**Remark 4.1** If  $\mathcal{U} = \mathcal{S}\mathcal{V}$ , then  $\tilde{\mathcal{U}} = \mathcal{S}\tilde{\mathcal{V}}$  where  $\tilde{\mathcal{V}} = \mathcal{V} + M\delta_0$ .

On the other hand, if we denote  $(\tilde{q}_n)$  the sequence of monic polynomials orthogonal with respect to  $\tilde{\mathcal{U}}$ , assuming the quasi-definiteness of such a functional, we get

$$\begin{aligned}\tilde{q}_n(x) &= q_n(x) - \frac{Mq_n(0)}{1 + MK_{n-1}(0,0)}K_{n-1}(x,0), \\ \tilde{q}_n(0) &= \frac{q_n(0)}{1 + MK_{n-1}(0,0)}.\end{aligned}\tag{4.10}$$

If

$$x\tilde{q}_n(x) = \tilde{q}_{n+1}(x) + \tilde{\varsigma}_n\tilde{q}_{n-1}(x),\tag{4.11}$$

then

$$\tilde{\varsigma}_n = \frac{\langle \tilde{\mathcal{U}}, \tilde{q}_n^2 \rangle}{\langle \tilde{\mathcal{U}}, \tilde{q}_{n-1}^2 \rangle}.$$

But

$$\begin{aligned}\langle \tilde{\mathcal{U}}, \tilde{q}_n^2(x) \rangle &= \langle \tilde{\mathcal{U}}, \tilde{q}_n(x)q_n(x) \rangle \\ &= \langle \mathcal{U}, \tilde{q}_n(x)q_n(x) \rangle + M\tilde{q}_n(0)q_n(0) \\ &= \langle \mathcal{U}, q_n^2(x) \rangle - \frac{Mq_n(0)}{1 + MK_{n-1}(0,0)}\langle \mathcal{U}, K_{n-1}(x,0)q_n(x) \rangle + M\tilde{q}_n(0)q_n(0) \\ &= \langle \mathcal{U}, q_n^2(x) \rangle \frac{1 + MK_n(0,0)}{1 + MK_{n-1}(0,0)}.\end{aligned}$$

Thus we get

**Proposition 4.1**

$$\tilde{\varsigma}_n = \varsigma_n \frac{[1 + MK_n(0,0)][1 + MK_{n-2}(0,0)]}{[1 + MK_{n-1}(0,0)]^2}, \text{ for } n \geq 1.$$

Taking into account that  $\mathcal{U}$  is a symmetric linear functional

$$K_{2m+1}(0,0) = K_{2m}(0,0),$$

and thus

$$\begin{aligned}\tilde{\varsigma}_{2m} &= \varsigma_{2m} \frac{1 + MK_{2m}(0,0)}{1 + MK_{2m-2}(0,0)} = \varsigma_{2m}\gamma_{2m}, \\ \tilde{\varsigma}_{2m+1} &= \varsigma_{2m+1} \frac{1 + MK_{2m-2}(0,0)}{1 + MK_{2m}(0,0)} = \frac{\varsigma_{2m+1}}{\gamma_{2m}}.\end{aligned}\tag{4.12}$$

On the other hand, from the Christoffel-Darboux formula

$$\begin{aligned}K_{2m}(x,0) &= \frac{1}{\langle \mathcal{U}, q_{2m}^2 \rangle} \frac{q_{2m+1}(x)q_{2m}(0)}{x} \\ &= \frac{1}{\langle \mathcal{U}, q_{2m}^2 \rangle} r_m(x^2)p_m(0).\end{aligned}$$

As a conclusion

$$K_{2m}(0,0) = \frac{1}{\langle \mathcal{U}, q_{2m}^2 \rangle} r_m(0)p_m(0).$$

On the other hand, taking into account

$$\gamma_{2m} = 1 + \frac{Mq_{2m}^2(0)}{1 + MK_{2m-2}(0,0)} \frac{1}{\langle \mathcal{U}, q_{2m}^2 \rangle},$$

as well as eventually the three-term recurrence relation for the sequence  $(q_n)$  for  $x = 0$  we get

$$\begin{aligned}\gamma_{2m} &= 1 + \frac{\varsigma_{2m-1}}{\varsigma_{2m}} \frac{M q_{2m-2}^2(0)}{1 + M K_{2m-2}(0, 0)} \frac{1}{\langle \mathcal{U}, q_{2m-2}^2 \rangle} \\ &= 1 + \frac{\varsigma_{2m-1}}{\varsigma_{2m}} \frac{M [K_{2m-2}(0, 0) - K_{2m-4}(0, 0)]}{1 + M K_{2m-2}(0, 0)} \\ &= 1 + \frac{\varsigma_{2m-1}}{\varsigma_{2m}} \left[ 1 - \frac{1}{\gamma_{2m-2}} \right],\end{aligned}$$

or, equivalently,

$$\tilde{\varsigma}_{2m} = \gamma_{2m} \varsigma_{2m} = \varsigma_{2m-1} + \varsigma_{2m} - \frac{\varsigma_{2m-1} \varsigma_{2m-2}}{\tilde{\varsigma}_{2m-2}}.$$

For the parameters of odd order

$$\tilde{\varsigma}_{2m+1} = \frac{\varsigma_{2m+1}}{\gamma_{2m}} = \frac{\varsigma_{2m+1} \varsigma_{2m}}{\tilde{\varsigma}_{2m}}.$$

Thus, we get

**Proposition 4.2** *The sequence of parameters  $(\tilde{\varsigma}_n)$  can be obtained recursively as follows*

$$\begin{cases} \tilde{\varsigma}_{2m} &= \varsigma_{2m-1} + \varsigma_{2m} - \tilde{\varsigma}_{2m-1}, \quad m \geq 1, \quad \tilde{\varsigma}_1 = \frac{\varsigma_1}{1 + M \mathcal{U}_0^{-1}} \\ \tilde{\varsigma}_{2m+1} &= \frac{\varsigma_{2m} \varsigma_{2m+1}}{\tilde{\varsigma}_{2m}}, \quad m \geq 1. \end{cases}$$

**Remark 4.2** *Notice that if  $(\tilde{p}_n)$  and  $(\tilde{r}_n)$  are the sequences of monic polynomials orthogonal with respect to  $\tilde{\mathcal{V}}$  and  $x\tilde{\mathcal{V}}$ , respectively, then we easily deduced that  $x\tilde{\mathcal{V}} = x\mathcal{V}$ . Thus,  $\tilde{r}_n(x) = r_n(x)$  for every  $n \in \mathbb{N}$ , or equivalently,  $\tilde{q}_{2n+1}(x) = q_{2n+1}(x)$  for every  $n \in \mathbb{N}$ . This means that the polynomials of odd degree for a symmetric linear functional do not change under the perturbation  $\tilde{\mathcal{U}} = \mathcal{U} + M\delta_0$ .*

## 5 Other interesting examples

### 5.1 Symmetrization of $\mathcal{L}(\alpha) + \lambda\delta_0$

Let  $\mathcal{L}(\alpha)$  where  $-\alpha \notin \mathbb{N}$  the Laguerre linear functional and consider the linear functional  $\tilde{\mathcal{L}}(\alpha; \lambda) = \mathcal{L}(\alpha) + \lambda\delta_0$ .

In [10] it is proved that  $\tilde{\mathcal{L}}(\alpha; \lambda)$  is a quasi-definite linear functional if and only if  $\lambda \neq \lambda_n = -[K_n(0, 0)]^{-1}$ , where  $K_n(\cdot, \cdot)$  denotes the  $n$ -th kernel polynomial associated with the linear functional  $\mathcal{L}(\alpha)$ .

Furthermore, the parameters of the corresponding three-term recurrence relations are (see [10])

$$\begin{aligned}\beta_0 &= \frac{1}{\lambda + 1} \left( \mu + \frac{1}{2} \right), \quad \beta_{n+1} = 2(n+1) + \left( \mu + \frac{1}{2} \right) \left( \frac{\lambda}{\lambda - \lambda_{2n}} + 1 - \frac{\lambda}{\lambda - \lambda_{2n+2}} \right), \quad n \geq 0, \\ \gamma_{n+1} &= \left[ \left( \mu + \frac{1}{2} \right) \left( 1 - \frac{\lambda}{\lambda - \lambda_{2n}} \right) + n \right] \left[ \left( \mu + \frac{1}{2} \right) \frac{\lambda}{\lambda - \lambda_{2n}} + n + 1 \right].\end{aligned}$$

If  $\mathcal{U}$  denotes the symmetric companion of  $\tilde{\mathcal{L}}(\alpha; \lambda)$ , i.e.  $\mathcal{S}[\tilde{\mathcal{L}}] = \mathcal{U}$ , then taking into account Proposition 4.2 the recurrence coefficients of  $\mathcal{U}$  are

$$\begin{aligned}\varsigma_{2n+1} &= \left( \mu + \frac{1}{2} \right) \frac{\lambda}{\lambda - \lambda_{2n}} + n, \\ \varsigma_{2n+2} &= \left( \mu + \frac{1}{2} \right) \left( 1 - \frac{\lambda}{\lambda - \lambda_{2n}} \right) + n + 1.\end{aligned}$$

## 5.2 Symmetrization of $\mathcal{B}(\alpha) + \lambda\delta_0$

Let  $\mathcal{B}(\alpha)$  ( $-\alpha \notin \mathbb{N}$ ) the Bessel linear functional and consider the linear functional  $\tilde{\mathcal{B}}(\alpha; \lambda) = \mathcal{B}(\alpha) + \lambda\delta_0$ .

In [10] it is proved that  $\tilde{\mathcal{B}}(\alpha; \lambda)$  is a quasi-definite linear functional if and only if  $\lambda \neq \lambda_n = -[K_n(0, 0)]^{-1}$  where  $K_n(\cdot, \cdot)$  denotes the  $n$ -th kernel polynomial associated with the linear functional  $\mathcal{B}(\alpha)$ .

Furthermore, the parameters of the corresponding three-term recurrence relations are (see [10])

$$\begin{aligned}\beta_0 &= -\frac{1}{\alpha(\lambda + 1)}, \\ \beta_{n+1} &= \frac{1 - \alpha}{(n + \alpha)(n + \alpha + 1)} + \frac{\lambda}{(n + \alpha + 1)(\lambda - \lambda_{n+1})} - \frac{\lambda}{(n + \alpha)(\lambda - \lambda_n)}, \quad n \geq 0, \\ \gamma_{n+1} &= -\frac{\left[\frac{\lambda(2n+2\alpha+1)}{\lambda-\lambda_n} - n - 1\right] \left[\frac{\lambda(2n+2\alpha-1)}{\lambda-\lambda_n} - n - 2\alpha + 1\right]}{(2n + 2\alpha - 1)(n + \alpha)^2(2n + 2\alpha + 1)}.\end{aligned}$$

If  $\mathcal{U}$  denotes the symmetric companion of  $\tilde{\mathcal{B}}(\alpha; \lambda)$ , then taking into account Proposition 4.2 we get

$$\begin{aligned}s_1 &= -\frac{1}{\alpha(\lambda + 1)}, \quad s_{2n+2} = -\frac{\left[\frac{\lambda}{\lambda-\lambda_n} - \frac{n+1}{(2n+2\alpha-1)}\right]}{n + 1}, \\ s_{2n+3} &= \frac{\left[\frac{\lambda}{\lambda-\lambda_n} - \frac{n+2\alpha-1}{(2n+2\alpha-1)}\right]}{n + \alpha}.\end{aligned}$$

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