Δ-COHERENT PAIRS AND ORTHOGONAL POLYNOMIALS OF A DISCRETE VARIABLE

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In this paper we define the concept of Δ-coherent pair of linear functionals. We prove that if (u_0, u_1) is a Δ-coherent pair of linear functionals then at least one of them must be a classical discrete linear functional under certain conditions. Examples related to Meixner and Hahn linear functionals are given.

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1 INTRODUCTION

The concepts of coherent pair and symmetric coherent pair have been introduced by A. Iserles *et al.* in Ref. [9] in the framework of the study of orthogonal polynomials associated with the Sobolev inner product

$$\langle f, g \rangle_S = \int_{\mathbb{R}} fg \, \mathrm{d}\mu_0 + \lambda \int_{\mathbb{R}} f'g' \, \mathrm{d}\mu_1,$$

where μ_0 and μ_1 are non-atomic positive Borel measures on the real line such that

$$\left| \int_{\mathbb{R}} x^k \, \mathrm{d}\mu_i(x) \right| < \infty, \quad k \geq 0, \quad i = 0, 1.$$

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In fact, coherence means that a relation between the MOPS (monic orthogonal polynomial sequence) $\{\mathrm{P}_n(x)\}_n$ and $\{\mathrm{T}_n(x)\}_n$, associated with the measures μ_0 and μ_1 respectively,

$$\mathrm{T}_n(x) = \frac{\mathrm{P}'_{n+1}(x)}{n+1} - \sigma_n \frac{\mathrm{P}'_n(x)}{n}, \quad n \geq 1,$$

where $\{\sigma_n\}_n$ is a sequence of non-zero complex numbers, is satisfied.

The description of the measures satisfying the coherence condition was an open problem in Ref. [9]. A first result is given in Ref. [15] where the complete set of coherence pairs when one of the measures is a classical one (Hermite, Laguerre, Jacobi) is described. Later on, in Ref. [16] it is proved that both measures involved in a coherent pair must be semiclassical.

Finally, in Ref. [20] H. G. Meijer gave the complete classification of the coherent pairs of measures which is, essentially, the same stated in Ref. [15]. As a conclusion, the Meijer's result shows that coherent pairs of measures constitute a restrictive class, helpful in order to obtain properties of polynomials orthogonal with respect to Sobolev inner products [14, 19, 22, 23].

The aim of this paper is to analyze the extension of the above definition of coherent pairs (based in the use of the derivative operator) to the situation when the forward difference operator Δ is considered. Thus, orthogonal polynomials of a discrete variable [21] appear in a natural way. Our main result states that if (u_0, u_1) is a Δ-coherent pair of linear functionals, one of them must be a classical discrete linear functional. In this situation, the support of the functional can be finite (Kravchuk, Hahn) or infinite (Meixner, Charlier) and the extension of the Meijer's result involves more than a merely formal analogy. First, the concept of weakly quasi-definite linear functional (including both finite and infinite supporting sets for the functional) is introduced in order to cover both situations. Second, using a limit process we can recover the Meijer's classification from the complete description of the Δ-coherent pairs which we have already stated in Ref. [5]. Third, we compare the complexity of the analysis for the Δ operator with the derivative operator, because the situation, in our case, is more interesting taking into account that not only quasi-definite linear functionals are involved.

The outline of the paper is as follows: in Section 2 we give the basic definitions and results which will be helpful in the following sections. In Section 3 we introduce the concept of Δ-coherent pair of linear functionals, and we prove that if (u_0, u_1) is a Δ-coherent pair, both u_0 and u_1 are semiclassical discrete linear functionals. In Section 4 we prove that if (u_0, u_1) is a Δ-coherent pair of linear functionals, at least one of them must be a classical discrete linear functional, where u_0 and u_1 are required to have a restriction on their orders. Finally, in Section 5 we present two examples of Δ-coherent pair of linear functionals related to the Meixner and the Hahn linear functional, as examples of quasi-definite and weakly quasi-definite linear functionals, respectively.

2 NOTATIONS AND BASIC RESULTS

Let $\mathbb{P}$ be the linear space of polynomials with complex coefficients and let $\mathbb{P}'$ be its algebraic dual space. We denote by $\langle u, f \rangle$ the duality bracket for $u \in \mathbb{P}'$ and $f \in \mathbb{P}$, and we denote by $(u)_n = \langle u, x^n \rangle$, with $n \geq 0$, the canonical moments of u.

DEFINITION 2.1 *A linear functional u is said to be weakly quasi-definite if there exists $0 \leq M \leq \infty$ such that the principal submatrices $\mathbf{H}_k := [(u)_{i+j}]_{i,j=0}^{k}$ are nonsingular for*

$0 \leq k \leq M$ *and, if* $M \neq \infty$, $\mathbf{H}_{M+1}$ *is a singular matrix. M is said to be the order of the linear functional u.*

Remark 1 Note that when $M = \infty$ this definition coincides with the concept of quasi-definite linear functional given in Ref. [6, p. 16]. In what follows, we shall have in mind this fact despite the notation associated to the order of the linear functional.

Given a weakly quasi-definite linear functional u of order M, there exists a family of monic polynomials $\{\mathrm{P}_n(x)\}_{n=0}^{M}$ orthogonal with respect to u, *i.e.* $\mathrm{P}_n(x) = x^n +$ terms of lower degree, for every $0 \leq n \leq M$, and $\langle u, \mathrm{P}_n \mathrm{P}_m \rangle = \Gamma_n \delta_{n,m}$, $\Gamma_n \neq 0$, for every $0 \leq n, m \leq M$. Such a sequence will be called monic orthogonal polynomial sequence (MOPS).

Remark 2 Given a weakly quasi-definite linear functional u of order $M < \infty$, it is possible to build a unique finite family of monic polynomials $\{\mathrm{P}_n(x)\}_{n=0}^{M+1}$ such that

$$\begin{aligned} &\langle u, x^m\ \mathrm{P}_n(x)\rangle = 0 \quad 0 \leq m \leq n-1, \quad 1 \leq n \leq M+1, \\ &\langle u, x^n\ \mathrm{P}_n(x)\rangle \neq 0 \quad 0 \leq n \leq M, \end{aligned}$$

and $\langle u, x^{M+1}\ \mathrm{P}_{M+1}(x)\rangle = 0$. Thus, the sequence of orthogonal polynomials is $\{\mathrm{P}_n(x)\}_{n=0}^{M}$. Note that this sequence of orthogonal polynomials does not generate $\mathbb{P}$ when $M < \infty$.

In the most important occurrences of orthogonal polynomials, the linear functional u satisfies an extra condition [6, p. 13].

DEFINITION 2.2 *A linear functional u is called positive-definite if its moments are all real and* $\det(\mathbf{H}_k) > 0$, *for every* $k \geq 0$.

DEFINITION 2.3 *Given a complex number c, the Dirac functional* δ_c *is defined by* $\langle \delta_c, p(x)\rangle := p(c)$, *for every* $p \in \mathbb{P}$.

DEFINITION 2.4 *Given a linear functional u and a polynomial p, we define the linear functional pu as* $\langle pu, q\rangle := \langle u, pq\rangle$, *for every* $q \in \mathbb{P}$. *For each complex number c, the linear functional* $(x-c)^{-1}u$ *is given by*

$$\langle (x-c)^{-1}u, q\rangle := \left\langle u, \frac{q(x)-q(c)}{x-c}\right\rangle$$

for every $q \in \mathbb{P}$.

Note that

$$(x-c)^{-1}((x-c)u) = u - (u)_0 \delta_c, \quad \text{for every } u \in \mathbb{P}', \tag{2.1}$$

while $(x-c)((x-c)^{-1}u) = u$.

DEFINITION 2.5 *The forward and backward difference operators* Δ *and* ∇ *are defined by*

$$\Delta p(x) := p(x+1) - p(x) \qquad \nabla p(x) := p(x) - p(x-1),$$

for every $p \in \mathbb{P}$, *respectively.*

Let p and q be two polynomials. Then, we have

$$\Delta\nabla = \nabla\Delta, \qquad \Delta = \nabla + \Delta\nabla, \qquad \Delta p(x) = \nabla p(x+1),$$
$$\Delta(p(x)q(x)) = q(x)\Delta p(x) + p(x+1)\Delta q(x).$$

DEFINITION 2.6 *For $u \in \mathbb{P}'$, the linear functional Δu is defined as $\langle \Delta u, p\rangle = -\langle u, \Delta p\rangle$, for every $p \in \mathbb{P}$.*

PROPOSITION 2.7 *For $u \in \mathbb{P}'$ and $p \in \mathbb{P}$, we have $\Delta[p(x)u] = p(x-1)\Delta u + \Delta p(x-1)u$. In particular, we obtain $(x-c)\Delta u = \Delta[(x+1-c)u] - u$.*

DEFINITION 2.8 *A linear functional u is said to be a classical discrete linear functional if u is weakly quasi-definite and there exist polynomials ϕ and ψ, with $\deg(\phi) \leq 2$ and $\deg(\psi) = 1$, such that*

$$\Delta[\phi(x)u] = \psi(x)u. \tag{2.2}$$

The corresponding MOPS associated with u is said to be a classical discrete MOPS.

Classical discrete orthogonal polynomials can be characterized by means of the Hahn's property in the following way [see Ref. 7].

PROPOSITION 2.9 *Let $\{\mathrm{P}_n(x)\}_{n=0}^{M}$ be the MOPS associated with a weakly quasi-definite linear functional u of order $M \geq 1$. The sequence $\{\mathrm{P}_n(x)\}_{n=0}^{M}$ is a classical discrete MOPS if and only if $\{\mathrm{Q}_n(x)\}_{n=0}^{M\ 1}$ defined by*

$$\mathrm{Q}_n(x) := \frac{\Delta \mathrm{P}_{n+1}(x)}{n+1}, \quad 0 \leq n \leq M-1, \tag{2.3}$$

is also a MOPS. Furthermore, if u satisfies $\Delta[\phi(x)u] = \psi(x)u$ then $\{\mathrm{Q}_n(x)\}_{n=0}^{M\ 1}$ is *orthogonal with respect to the functional $\breve{u} := \phi(x)u$.*

Note that weakly positive-definite classical discrete functionals are associated with a measure $\varrho(x)$ whose support is a countable set. In such a sense, we shall only consider here orthogonal polynomials of a discrete variable on $[a, b-1]$ with weight $\varrho(x)$,

$$\sum_{x_i=a}^{b\ 1} \mathrm{P}_m(x_i)\mathrm{P}_n(x_i)\varrho(x_i) = d_n^2\delta_{n,m}, \quad x_{i+1} = x_i + 1,$$

provided that the interval (a, b) is contained in $\mathbb{R}$ and the function $\varrho(x)$ satisfies

$$\Delta(\sigma(x)\varrho(x)) = \psi(x)\varrho(x), \quad \sigma(x) := \phi(x) - \psi(x), \tag{2.4}$$

with

$$\sigma(x)\varrho(x)x^{\ell}\Big|_{x=a,b} = 0, \quad \ell = 0, 1, \ldots \tag{2.5}$$

Thus, classical discrete linear functionals u satisfying (2.2) can be represented as

$$\langle u, p\rangle = \sum_{x_i=a}^{b\ 1} p(x_i)\ \varrho(x_i), \qquad \text{for every } p \in \mathbb{P},$$

where $\varrho(x)$ is a weight function satisfying (2.4) and (2.5). In this situation, classical discrete linear functionals are the corresponding to Hahn, Meixner, Kravchuk and Charlier MOPS [21]. Charlier and Meixner functionals are quasi-definite linear functionals and therefore the corresponding MOPS are infinite sequences. On the other hand, Kravchuk and Hahn linear functionals are weakly quasi-definite linear functionals and the corresponding MOPS are finite. Some applications of these finite families can be found in *e.g.* [12, 21].

In Ref. [11] another systematic study of (positive-definite) orthogonal polynomials of a discrete variable is given from the second order linear difference equation they satisfy. Moreover, a self-contained overview of classical discrete polynomials has been done in Ref. [1], where finite orthogonal sequences are not considered (only quasi-definite functionals are studied).

In the aforementioned classifications [1, 11] there appear other families of orthogonal polynomials, which are outside the scope of this paper since the Hahn's property they satisfy can not be written in terms of the forward difference operator Δ.

Next we introduce the concept of semiclassical discrete linear functional [see Ref. 17].

DEFINITION 2.10 *A linear functional u is said to be a semiclassical discrete linear functional if u is weakly quasi-definite and there exist two polynomials ϕ and ψ such that*

$$\Delta[\phi(x)u] = \psi(x)u \tag{2.6}$$

where $\deg(\phi) = t \geq 0$ *and* $\deg(\psi) = p \geq 1$. *A MOPS with respect to a semiclassical discrete functional u is called a semiclassical discrete MOPS.*

It is possible to associate with (2.6) a nonnegative integer $s = \max\{\deg(\psi) - 1, \deg(\phi) - 2\}$, but a semiclassical discrete functional u satisfies an infinite number of equations as (2.6). It is enough to multiply both sides of (2.6) by a polynomial f with $\deg(f) = q$ and from Proposition 2.7 we have $\Delta[f(x+1)\phi(x)u] = (\phi(x)\Delta f(x) + f(x)\psi(x))u$. So u fulfills also $\Delta[\phi_1(x)u] = \psi_1(x)u$, where $\phi_1(x) = f(x+1)\phi(x)$ and $\psi_1(x) = \phi(x)\Delta f(x) + f(x)\psi(x)$. From (2.6) we have $s_1 = \max\{p_1 - 1, t_1 - 2\} \leq s + q$. Hence, we can associate with a semiclassical discrete functional u a set of nonnegative integer numbers $h(u)$.

DEFINITION 2.11 *Let u be a semiclassical discrete functional. The minimum of the set $h(u)$ is called the class of u. When s is the class of u, then the sequence of polynomials orthogonal with respect to u is said to be of class s.*

Note that classical discrete polynomials are semiclassical of class 0.

DEFINITION 2.12 *Let $\{\mathrm{P}_n(x)\}_{n=0}^{M}$ be the MOPS associated with the weakly quasi-definite linear functional u of order M. The family of linear functionals $\{\alpha_n\}_n$ defined by $\langle \alpha_n,\ \mathrm{P}_m\rangle = \delta_{nm}$, $0 \leq n, m \leq M$, is called the dual basis of $\{\mathrm{P}_n(x)\}_{n=0}^{M}$.*

In fact,

$$\alpha_n = \frac{\mathrm{P}_n(x)}{\langle u, \mathrm{P}_n^2(x)\rangle} u, \quad 0 \le n \le M. \tag{2.7}$$

An immediate consequence of the above equation can be stated as follows.

PROPOSITION 2.13 *Let* $\{\mathrm{P}_n(x)\}_{n=0}^{M}$ *be the MOPS associated with the weakly quasi-definite linear functional* u *of order* $M \ge 1$ *and let* $\{\mathrm{Q}_n(x)\}_{n=0}^{M\;1}$ *as in (2.3). If we denote by* $\{\alpha_n\}_{n=0}^{M}$ *and* $\{\breve{\alpha}_n\}_{n=0}^{M\;1}$ *the corresponding dual bases, then* $\Delta\breve{\alpha}_n = -(n+1)\alpha_{n+1}$, $0 \le n \le M-1$.

3 Δ-COHERENT PAIRS

DEFINITION 3.1 *Let* u_0 *and* u_1 *be two weakly quasi-definite linear functionals of order* M_0 *and* M_1*, whose MOPS are* $\{\mathrm{P}_n(x)\}_{n=0}^{M_0}$ *and* $\{\mathrm{T}_n(x)\}_{n=0}^{M_1}$ *respectively, with* $M_0 \ge 2$ *and* $M_1 \ge 1$. *The pair* (u_0, u_1) *is called a* Δ*-coherent pair of linear functionals if*

$$\mathrm{T}_n(x) = \frac{\Delta\mathrm{P}_{n+1}(x)}{n+1} - \sigma_n \frac{\Delta\mathrm{P}_n(x)}{n}, \quad 1 \le n \le \min\{M_0 - 1, M_1\}, \tag{3.1}$$

where $\{\sigma_n\}_n$ *is a sequence of non-zero complex numbers.*

Example 3.2 Let $\mathbb{N}$ be the set of positive integers. Monic Kravchuk polynomials $\mathrm{k}_n^{(p)}(x; N)$ have the following representation in terms of hypergeometric series [Ref. 21, p. 51]

$$\mathrm{k}_n^{(p)}(x;N) = (-p)^n (N-n+1)_n \; {}_2\mathrm{F}_1\left(\begin{matrix} -n, & -x \\ & -N \end{matrix} \,\middle|\, \frac{1}{p}\right),$$
$$0 < p < 1, \quad N \in \mathbb{N}, \quad 0 \le n \le N, \; N \ge 2, \tag{3.2}$$

where $(a)_n$ denotes the Pochhammer symbol [2], which is defined by

$$(a)_0 = 1, \qquad (a)_n = a(a+1) \cdots (a+n-1), \quad n \ge 1.$$

Since Kravchuk polynomials satisfy [see, for example, Ref. 10]

$$\mathrm{k}_n^{(p)}(x;N) = \frac{\Delta\mathrm{k}_{n+1}^{(p)}(x;N)}{n+1} - \sigma_n \frac{\Delta\mathrm{k}_n^{(p)}(x;N)}{n}, \quad \text{with } \sigma_n = np, \quad 1 \le n \le N-1, \tag{3.3}$$

if we denote by $u^{(p,N)}$ the Kravchuk linear functional of order N (the binomial distribution from probability theory) given by

$$\langle u^{(p,N)}, r\rangle = \sum_{s=0}^{N} \binom{N}{s} p^s \, (1-p)^{N\;s} \, r(s), \qquad 0 < p < 1, \quad N \in \mathbb{N}, \quad \text{for every } r \in \mathbb{P}, \tag{3.4}$$

we have that $(u^{(p,N)}, u^{(p,N)})$ is a Δ-coherent pair of linear functionals, if $N \ge 2$.

Proposition 3.3 *Let* (u_0, u_1) *be a* Δ*-coherent pair of linear functionals and let* $\{\alpha_n^{(0)}\}_{n=0}^{M_0}$ *and* $\{\alpha_n^{(1)}\}_{n=0}^{M_1}$ *be the dual bases of* u_0 *and* u_1*, respectively, with* $M_0 \geq 2$ *and* $M_1 \geq 1$*. If we denote by* $\{\breve{\alpha}_n^{(0)}\}_{n=0}^{M_0\;1}$ *the dual basis corresponding to* $\{\mathrm{Q}_n(x)\}_{n=0}^{M_0\;1}$ *defined in* (2.3)*, then we have*

$$\breve{\alpha}_n^{(0)} = \alpha_n^{(1)} - \sigma_{n+1}\alpha_{n+1}^{(1)}, \quad 0 \leq n \leq \min\{M_0 - 2, M_1 - 1\}, \tag{3.5}$$

$$(n+1)\,\alpha_{n+1}^{(0)} = \sigma_{n+1}\Delta\alpha_{n+1}^{(1)} - \Delta\alpha_n^{(1)}, \quad 0 \leq n \leq \min\{M_0 - 2, M_1 - 1\}. \tag{3.6}$$

Proof Let $\mathbb{P}_{M_1}$ be the space of polynomials of degree at most M_1 and let $\mathbb{P}'_{M_1}$ be its algebraic dual space. Equation (3.5) is a consequence of (3.1) and using that $\{\alpha_n^{(1)}\}_{n=0}^{M_1}$ is a basis of $\mathbb{P}'_{M_1}$. Applying the Δ operator to (3.5) and using Proposition 2.13, (3.6) is obtained. ■

Theorem 3.4 *Let* (u_0, u_1) *be a* Δ*-coherent pair of linear functionals and let* $\{\mathrm{P}_n(x)\}_{n=0}^{M_0}$ *and* $\{\mathrm{T}_n(x)\}_{n=0}^{M_1}$ *be the corresponding MOPS associated with* u_0 *and* u_1*, respectively, with* $M_0 \geq 2$ *and* $M_1 \geq 1$*. Then*

(i) *The functional* u_1 *is a semiclassical discrete linear functional of class at most* 1*. That is, there exist two polynomials* ϕ_1 *and* ψ_1*, of degree at most* 3 *and* 2*, respectively, such that*

$$\Delta[\phi_1(x)u_1] = \psi_1(x)u_1. \tag{3.7}$$

Their explicit expressions are

$$\phi_1(x) = 2\frac{\mathrm{P}_2(x+1)}{\langle u_0, \mathrm{P}_2^2\rangle}\,\mathrm{c}_1(x) - \frac{\mathrm{P}_1(x+1)}{\langle u_0, \mathrm{P}_1^2\rangle}\,\mathrm{c}_2(x), \tag{3.8}$$

$$\psi_1(x) = \frac{\mathrm{P}_1(x)}{\langle u_0, \mathrm{P}_1^2\rangle}\Delta\mathrm{c}_2(x-1) - 2\frac{\mathrm{P}_2(x)}{\langle u_0, \mathrm{P}_2^2\rangle}\Delta\mathrm{c}_1(x-1) + \Delta\phi_1(x-1), \tag{3.9}$$

where

$$\mathrm{c}_{n+1}(x) := \sigma_{n+1}\frac{\mathrm{T}_{n+1}(x)}{\langle u_1, \mathrm{T}_{n+1}^2\rangle} - \frac{\mathrm{T}_n(x)}{\langle u_1, \mathrm{T}_n^2\rangle}, \quad 0 \leq n \leq \min\{M_0 - 2, M_1 - 1\}. \tag{3.10}$$

(ii) *There exist polynomials* A_3 *and* B_2 *of degree at most* 3 *and* 2*, respectively, such that*

$$\mathrm{A}_3(x)u_0 = \mathrm{B}_2(x)u_1, \tag{3.11}$$

where

$$\mathrm{A}_3(x) = \phi_1(x-1), \qquad \mathrm{B}_2(x) = \mathrm{c}_1(x-1)\Delta\mathrm{c}_2(x-1) - \mathrm{c}_2(x-1)\Delta\mathrm{c}_1(x-1). \tag{3.12}$$

(iii) *The functional* u_0 *is a semiclassical discrete linear functional of class at most* 6 *since it verifies the distributional equation* $\Delta[\phi_0(x)u_0] = \psi_0(x)u_0$*, where* ϕ_0 *and* ψ_0 *are the polynomials of degree at most* 8 *and* 7 *given by*

$$\phi_0(x) = \phi_1(x)\phi_1(x-1)\mathrm{B}_2(x), \tag{3.13}$$

$$\psi_0(x) = \{\mathrm{B}_2(x-1)\psi_1(x) + \phi_1(x)\Delta\mathrm{B}_2(x-1)\}\mathrm{A}_3(x-1) + \phi_1(x)\mathrm{A}_3(x)\Delta\mathrm{B}_2(x). \tag{3.14}$$

Proof Let us write (3.6) using (2.7)

$$(n+1)\frac{\mathrm{P}_{n+1}(x)}{\langle u_0, \mathrm{P}_{n+1}^2\rangle}u_0 = \Delta[\mathrm{c}_{n+1}(x)u_1], \quad 0 \le n \le \min\{M_0-2, M_1-1\}. \tag{3.15}$$

For $n=0$ and $n=1$ we get in (3.15)

$$\begin{aligned} \frac{\mathrm{P}_1(x)}{\langle u_0, \mathrm{P}_1^2\rangle}u_0 &= \Delta[\mathrm{c}_1(x)u_1] = \mathrm{c}_1(x-1)\Delta u_1 + \Delta\mathrm{c}_1(x-1)u_1, \\ 2\frac{\mathrm{P}_2(x)}{\langle u_0, \mathrm{P}_2^2\rangle}u_0 &= \Delta[\mathrm{c}_2(x)u_1] = \mathrm{c}_2(x-1)\Delta u_1 + \Delta\mathrm{c}_2(x-1)u_1. \end{aligned} \tag{3.16}$$

(i) From (3.16) it follows that

$$\begin{aligned} &\left(2\frac{\mathrm{P}_2(x)}{\langle u_0, \mathrm{P}_2^2\rangle}\mathrm{c}_1(x-1) - \frac{\mathrm{P}_1(x)}{\langle u_0, \mathrm{P}_1^2\rangle}\mathrm{c}_2(x-1)\right)\Delta u_1 \\ &\qquad + \left(2\frac{\mathrm{P}_2(x)}{\langle u_0, \mathrm{P}_2^2\rangle}\Delta\mathrm{c}_1(x-1) - \frac{\mathrm{P}_1(x)}{\langle u_0, \mathrm{P}_1^2\rangle}\Delta\mathrm{c}_2(x-1)\right)u_1 = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\Delta\left[\left(2\frac{\mathrm{P}_2(x+1)}{\langle u_0, \mathrm{P}_2^2\rangle}\mathrm{c}_1(x) - \frac{\mathrm{P}_1(x+1)}{\langle u_0, \mathrm{P}_1^2\rangle}\mathrm{c}_2(x)\right)u_1\right] \\ &\quad = \left(\frac{\mathrm{P}_1(x)}{\langle u_0, \mathrm{P}_1^2\rangle}\Delta\mathrm{c}_2(x-1) - 2\frac{\mathrm{P}_2(x)}{\langle u_0, \mathrm{P}_2^2\rangle}\Delta\mathrm{c}_1(x-1) + \Delta\phi_1(x-1)\right)u_1. \end{aligned}$$

Hence, $\Delta[\phi_1(x)u_1] = \psi_1(x)u_1$, where ϕ_1 and ψ_1 are given in (3.8) and (3.9), respectively. Thus (i) follows.

(ii) Eliminating Δu_1 in the system (3.16) we obtain $\mathrm{A}_3(x)u_0 = \mathrm{B}_2(x)u_1$, where polynomials A_3 and B_2 are given in (3.12).

(iii) Finally, by using Proposition 2.7 appropriately we have

$$\begin{aligned} &\Delta[\phi_1(x)\phi_1(x-1)\mathrm{B}_2(x)u_0] \\ &\quad = \Delta[\phi_1(x)\mathrm{B}_2(x)\phi_1(x-1)u_0] = \Delta[\phi_1(x)\mathrm{B}_2^2(x)u_1] \\ &\quad = \mathrm{B}_2^2(x-1)\Delta[\phi_1(x)u_1] + \Delta\mathrm{B}_2^2(x-1)\phi_1(x)u_1 \\ &\quad = \mathrm{B}_2^2(x-1)\psi_1(x)u_1 + (\mathrm{B}_2(x-1)\Delta\mathrm{B}_2(x-1) + \mathrm{B}_2(x)\Delta\mathrm{B}_2(x-1))\phi_1(x)u_1 \\ &\quad = \phi_1(x)\Delta\mathrm{B}_2(x-1)\mathrm{A}_3(x)u_0 + \{\mathrm{B}_2(x-1)\psi_1(x) + \phi_1(x)\Delta\mathrm{B}_2(x-1)\}\mathrm{A}_3(x-1)u_0. \end{aligned}$$

■

4 GENERAL PROBLEM OF Δ-COHERENCE

In Theorem 3.4 we have proved that if (u_0, u_1) is a Δ-coherent pair of linear functionals, then both u_0, u_1 are semiclassical discrete functionals of class at most 6 and 1, respectively. The main goal of this section is to prove that if (u_0, u_1) is a Δ-coherent pair of linear functionals then at least one of the functionals u_0, u_1 must be a classical discrete functional under certain conditions on the order of u_0 and u_1. The proof of this statement will consist in 3 steps. Let us denote by ξ and η the zeros of the polynomial $\mathrm{B}_2(x)$ defined in (3.12). In the first one, we prove that if $\eta = \xi + 1$, u_0 must be a classical discrete functional (Theorem 4.2). In the second step we prove that if $\eta \neq \xi$ and $\eta \neq \xi+1$ then u_1 must be a classical discrete functional (Theorem 4.6). Finally, as a remark, we prove that the case $\eta = \xi$ can not be hold.

PROPOSITION 4.1 *Let (u_0, u_1) be a Δ-coherent pair of linear functionals, and let $\{\mathrm{P}_n(x)\}_{n=0}^{M_0}$ and $\{\mathrm{T}_n(x)\}_{n=0}^{M_1}$ the corresponding MOPS associated with u_0 and u_1, respectively with $M_1 > 3$. Let $\mathrm{c}_n(x)$ be the polynomials defined in* (3.10). *For each $1 \leq n \leq \min\{M_0 - 1, M_1\}$, we have*

$$n \frac{\mathrm{P}_n(x)}{\langle u_0, \mathrm{P}_n^2 \rangle} \mathrm{B}_2(x) = \mathrm{A}_3(x) \Delta \mathrm{c}_n(x-1) + \mathrm{c}_n(x-1)\pi(x), \tag{4.1}$$

where the polynomials $\mathrm{A}_3(x)$ and $\mathrm{B}_2(x)$ are defined in (3.12) *and $\pi(x) := \psi_1(x) - \Delta \mathrm{A}_3(x)$, with $\psi_1(x)$ given in* (3.9).

Proof Using (3.11), (3.15) and Proposition 2.7 we obtain

$$\begin{aligned} n \frac{\mathrm{P}_n(x)}{\langle u_0, \mathrm{P}_n^2 \rangle} \mathrm{B}_2(x) u_1 &= n \frac{\mathrm{P}_n(x)}{\langle u_0, \mathrm{P}_n^2 \rangle} \mathrm{A}_3(x) u_0 = \mathrm{A}_3(x) \Delta[\mathrm{c}_n(x) u_1] \\ &= \mathrm{A}_3(x)\{\mathrm{c}_n(x-1)\Delta u_1 + \Delta \mathrm{c}_n(x-1) u_1\}, \quad 1 \leq n \leq \min\{M_0 - 1, M_1\}. \end{aligned}$$

From Proposition 2.7 and (3.7) we get

$$\phi_1(x-1)\Delta u_1 = \pi(x) u_1 \tag{4.2}$$

and then (4.1) holds, since $M_1 > 3$. ■

THEOREM 4.2 *Let (u_0, u_1) be a Δ-coherent pair of linear functionals. Let ξ and η be the zeros of the polynomial $\mathrm{B}_2(x)$ defined in* (3.12) *and suppose that $\eta = \xi + 1$, i.e., let*

$$\mathrm{B}_2(x) = \frac{\sigma_1 \sigma_2}{\langle u_1, \mathrm{T}_1^2 \rangle \langle u_1, \mathrm{T}_2^2 \rangle} (x - \xi)(x - (\xi + 1)). \tag{4.3}$$

Then,

(i) *If the order of the linear functional u_1 is greater than 3, then the functional $\tilde{u} = (x - \xi) u_1$ is a classical discrete linear functional verifying*

$$\Delta[\tilde{\phi}(x)\tilde{u}] = \tilde{\psi}(x)\tilde{u}, \tag{4.4}$$

for some polynomials $\tilde{\phi}$ and $\tilde{\psi}$, with $\deg(\tilde{\phi}) \leq 2$ *and* $\deg(\tilde{\psi}) = 1$. *Moreover,*

$$\tilde{\phi}(x-1) u_0 = \frac{\sigma_1 \sigma_2}{\langle u_1, \mathrm{T}_1^2 \rangle \langle u_1, \mathrm{T}_2^2 \rangle} \tilde{u}. \tag{4.5}$$

(ii) *The functional u_0 is a classical discrete linear functional satisfying*

$$\Delta[\tilde{\phi}(x-1) u_0] = \tilde{\psi}_0(x) u_0, \tag{4.6}$$

for some polynomial $\tilde{\psi}_0$ with $\deg(\tilde{\psi}_0) = 1$.

Proof We have proved in Theorem 3.4 that u_1 is a semiclassical discrete linear functional satisfying the distributional Eq. (3.7).

(i) From the definition of $\mathrm{B}_2(x)$ in (4.3) it follows

$$\Delta \mathrm{B}_2(x) = 2\frac{\sigma_1\sigma_2}{\langle u_1, \mathrm{T}_1^2\rangle\langle u_1, \mathrm{T}_2^2\rangle}(x-\xi).$$

Moreover, from (3.12) we get

$$\Delta \mathrm{B}_2(x) = 2\frac{\sigma_2}{\langle u_1, \mathrm{T}_2^2\rangle}\mathrm{c}_1(x),$$

using (3.10). Thus we obtain that $\mathrm{c}_1(\xi)=0$. From (3.12) with $x=\xi+1$ it follows that

$$0 = \mathrm{B}_2(\xi+1) = \mathrm{c}_1(\xi)\Delta\mathrm{c}_2(\xi) - \mathrm{c}_2(\xi)\Delta\mathrm{c}_1(\xi) = -\mathrm{c}_2(\xi)\frac{\sigma_1}{\langle u_1, \mathrm{T}_1^2\rangle}.$$

Thus $\mathrm{c}_2(\xi)=0$. Since $\mathrm{c}_1(\xi)=\mathrm{c}_2(\xi)=0$ from (3.8) we obtain $\phi_1(\xi)=0$ as well as $\psi_1(\xi)=0$ using (3.8) and (3.9).
Hence, we can write

$$\phi_1(x) = (x-\xi)\tilde{\phi}(x), \qquad \psi_1(x) = (x-\xi)\tilde{\psi}(x).$$

Let us define $\tilde{u} = (x-\xi)u_1$. From (3.7) and the definition of polynomials $\tilde{\phi}$ and $\tilde{\psi}$, it follows that $\tilde{u}$ satisfies (4.4). In Theorem 3.4 we have proved that $\deg(\phi_1)$ is at most 3, so $\deg(\tilde{\phi})\le 2$. Since $\deg(\psi_1)$ is at most 2, we deduce $\deg(\tilde{\psi})\le 1$. If we prove that $\tilde{\psi}$ can not be a constant polynomial then we deduce part (i) of the Theorem. Indeed, we shall distinguish two situations:

(1) If $\tilde{\psi}$ is a non-zero constant v, then $\langle u_1, v(x-\xi)\rangle = \langle v(x-\xi)u_1, 1\rangle = \langle v\tilde{u}, 1\rangle = \langle \Delta[\tilde{\phi}(x)\tilde{u}], 1\rangle = 0$. Hence $\mathrm{T}_1(x) = x-\xi$. From (3.10) then $\mathrm{c}_1(\xi)\neq 0$ and this contradicts that $\mathrm{c}_1(\xi)=0$.
(2) Suppose that $\tilde{\psi}\equiv 0$ and let us denote by $x^{[n]} = x(x-1)\cdots(x-n+1)$, $x^{[0]}=1$. Since $\Delta x^{[n+1]} = x^{[n]}(n+1)$ for each $n\ge 0$, it should be

$$\langle \phi_1(x)u_1, x^{[n]}\rangle = -\left\langle \Delta[\phi_1(x)u_1], \frac{x^{[n+1]}}{n+1}\right\rangle = -\left\langle \Delta[\tilde{\phi}(x)\tilde{u}], \frac{x^{[n+1]}}{n+1}\right\rangle = \langle \tilde{\psi}(x)\tilde{u}, x^{[n]}\rangle = 0.$$

So, $\langle \phi_1(x)u_1, p(x)\rangle = 0$ for every $p\in\mathbb{P}$, and then u_1 should not be a weakly quasi-definite linear functional of order greater than 3. Hence $\tilde{\psi}\not\equiv 0$.

From the above situations we conclude that $\deg(\tilde{\psi})=1$.

Moreover, since $\mathrm{c}_1(\xi)=\mathrm{c}_2(\xi)=0$ then $\mathrm{c}_1(x)$ divides $\mathrm{c}_2(x)$. From (3.11) and (3.12) we can write

$$(x-(\xi+1))\tilde{\phi}(x-1)u_0 = \mathrm{B}_2(x)u_1.$$

Multiplying both sides of this equality by $(x-(\xi+1))^{-1}$, and using (2.1) we obtain

$$\tilde{\phi}(x-1)u_0 = \frac{\sigma_1\sigma_2}{\langle u_1, \mathrm{T}_1^2\rangle\langle u_1, \mathrm{T}_2^2\rangle}(x-\xi)u_1 + \delta_{\xi+1}[(\tilde{\phi}(x-1)u_0)_0 - ((x-\xi)u_1)_0].$$

With an appropriate choice of the first moments of the functionals u_0 and u_1, it yields

$$\tilde{\phi}(x-1)u_0 = \frac{\sigma_1\sigma_2}{\langle u_1, \mathrm{T}_1^2\rangle\langle u_1, \mathrm{T}_2^2\rangle}(x-\xi)u_1,$$

Thus we get (4.5).

(ii) From the above equality and (3.15)

$$\Delta[\tilde{\phi}(x-1)u_0] = \Delta\left[\frac{\sigma_1\sigma_2}{\langle u_1, \mathrm{T}_1^2\rangle\langle u_1, \mathrm{T}_2^2\rangle}(x-\xi)u_1\right] = \frac{\sigma_2}{\langle u_1, \mathrm{T}_2^2\rangle}\Delta[\mathrm{c}_1(x)u_1] = \frac{\sigma_2}{\langle u_1, \mathrm{T}_2^2\rangle}\frac{\mathrm{P}_1(x)}{\langle u_0, \mathrm{P}_1^2\rangle}u_0.$$

If we define $\tilde{\psi}_0(x) = (\sigma_2/\langle u_1, \mathrm{T}_2^2\rangle)(\mathrm{P}_1(x)/\langle u_0, \mathrm{P}_1^2\rangle)$, then $\deg(\tilde{\psi}_0) = 1$ and (4.6) holds. ■

For the remaining steps described in the introduction of this section, some previous lemmas are needed.

LEMMA 4.3 *Let (u_0, u_1) be a Δ-coherent pair of linear functionals and $\mathrm{A}_3(x)$ and $\mathrm{B}_2(x)$ the polynomials defined in* (3.12). *Let ξ be a zero of $\mathrm{B}_2(x)$ such that $\mathrm{A}_3(\xi) \neq 0$. Then, there exists a non-zero parameter k independent of n, such that*

$$\mathrm{c}_n(\xi-1) + k\Delta\mathrm{c}_n(\xi-1) = 0, \quad 1 \le n \le \min\{M_0-1, M_1\},$$

where $M_0 \ge 2$ and $M_1 \ge 1$ are the order of u_0 and u_1, respectively.

Proof Since $\mathrm{B}_2(\xi) = 0$, (4.1) for $n = 1$ and $x = \xi$ reads as

$$0 = \mathrm{A}_3(\xi)\Delta\mathrm{c}_1(\xi-1) + \mathrm{c}_1(\xi-1)\pi(\xi).$$

Since $\Delta\mathrm{c}_1(\xi-1) = \sigma_1/\langle u_1, \mathrm{T}_1^2\rangle$ is a non-zero constant then $\pi(\xi) \neq 0$ and

$$\mathrm{c}_1(\xi-1) = -\frac{\mathrm{A}_3(\xi)\sigma_1}{\pi(\xi)\langle u_1, \mathrm{T}_1^2\rangle} \neq 0.$$

If we define

$$k = \frac{\mathrm{A}_3(\xi)}{\pi(\xi)} = -\frac{\mathrm{c}_1(\xi-1)}{\Delta\mathrm{c}_1(\xi-1)}$$

our result follows from (4.1) for $1 \le n \le \min\{M_0-1, M_1\}$. ■

LEMMA 4.4 *Let (u_0, u_1) be a Δ-coherent pair of linear functionals. Suppose that there exist parameters ξ_1, ξ_2, $k_1 \neq 0$ and $k_2 \neq 0$ such that*

$$\mathrm{c}_n(\xi_1) + k_1\Delta\mathrm{c}_n(\xi_1) = 0, \quad \text{and } \mathrm{c}_n(\xi_2) + k_2\Delta\mathrm{c}_n(\xi_2) = 0, \quad 1 \le n \le \min\{M_0-1, M_1\}, \tag{4.7}$$

where M_0 *and* M_1 *are the order of* u_0 *and* u_1, *respectively. If* $M_1 \geq 2$, $M_1 \geq 3$, *and* $|\xi_1 - \xi_2| \neq 1$ *then* $\xi_1 = \xi_2$ *and* $k_1 = k_2$.

Proof From (3.10), Eqs. (4.7) can be written

$$\sigma_n\left\{\frac{\mathrm{T}_n(\xi_j)}{\langle u_1, \mathrm{T}_n^2\rangle} + k_j\frac{\Delta\mathrm{T}_n(\xi_j)}{\langle u_1, \mathrm{T}_n^2\rangle}\right\}$$
$$= \frac{\mathrm{T}_{n-1}(\xi_j)}{\langle u_1, \mathrm{T}_{n-1}^2\rangle} + k_j\frac{\Delta\mathrm{T}_{n-1}(\xi_j)}{\langle u_1, \mathrm{T}_{n-1}^2\rangle}, \quad j = 1, 2, \quad 1 \leq n \leq \min\{M_0 - 1, M_1\}.$$

Let us denote

$$\mathrm{h}_n^{(j)}(\xi_j) = \frac{\mathrm{T}_n(\xi_j)}{\langle u_1, \mathrm{T}_n^2\rangle} + k_j\frac{\Delta\mathrm{T}_n(\xi_j)}{\langle u_1, \mathrm{T}_n^2\rangle}, \quad j = 1, 2, \quad 1 \leq n \leq \min\{M_0 - 1, M_1\}$$

and observe that $\mathrm{h}_0^{(j)}(\xi_j) = 1/\langle u_1, 1\rangle$. For each $1 \leq n \leq \min\{M_0 - 1, M_1\}$ and for $j = 1, 2$ we write $\sigma_n \mathrm{h}_n^{(j)}(\xi_j) = \mathrm{h}_{n-1}^{(j)}(\xi_j)$, so $\mathrm{h}_n^{(j)}(\xi_j) \neq 0$, for every $1 \leq n \leq \min\{M_0 - 1, M_1\}$, and we get

$$\frac{\mathrm{h}_n^{(1)}(\xi_1)}{\mathrm{h}_n^{(2)}(\xi_2)} = \frac{\mathrm{h}_{n-1}^{(1)}(\xi_1)}{\mathrm{h}_{n-1}^{(2)}(\xi_2)}, \quad 1 \leq n \leq \min\{M_0 - 1, M_1\}.$$

Repeating this process

$$\frac{\mathrm{h}_n^{(1)}(\xi_1)}{\mathrm{h}_n^{(2)}(\xi_2)} = \frac{\mathrm{h}_0^{(1)}(\xi_1)}{\mathrm{h}_0^{(2)}(\xi_2)} = \frac{1/\langle u_1, 1\rangle}{1/\langle u_1, 1\rangle} = 1, \quad 1 \leq n \leq \min\{M_0 - 1, M_1\},$$

i.e., $\mathrm{h}_n^{(1)}(\xi_1) = \mathrm{h}_n^{(2)}(\xi_2)$ for $1 \leq n \leq \min\{M_0 - 1, M_1\}$ or, equivalently,

$$\mathrm{T}_n(\xi_1) + k_1\Delta\mathrm{T}_n(\xi_1) = \mathrm{T}_n(\xi_2) + k_2\Delta\mathrm{T}_n(\xi_2), \quad 1 \leq n \leq \min\{M_0 - 1, M_1\}. \tag{4.8}$$

So from the initial problem of characterizing ξ_1, ξ_2, k_1 and k_2 such that (4.7) holds, we propose a new problem: determine ξ_1, ξ_2, k_1 and k_2 such that (4.8) holds.

In order to solve it, we study a more general one: Find all μ, v, δ and η such that

$$\mu\mathrm{T}_n(\xi_1) + v\Delta\mathrm{T}_n(\xi_1) = \delta\mathrm{T}_n(\xi_2) + \eta\Delta\mathrm{T}_n(\xi_2), \quad 1 \leq n \leq \min\{M_0 - 1, M_1\}. \tag{4.9}$$

Since $\{\mathrm{T}_n(x)\}_{n=0}^{M_1}$ is a MOPS, it satisfies a three-term recurrence relation [6] which can be written $\mathrm{T}_{n+1}(x) = (x - \beta_n^{\mathrm{T}})\mathrm{T}_n(x) - \gamma_n^{\mathrm{T}}\mathrm{T}_{n-1}(x)$, $0 \leq n \leq M_1 - 1$ with the initial conditions $\mathrm{T}_{-1}(x) = 0$ and $\mathrm{T}_0(x) = 1$. If we apply the Δ operator to this relation

$$\Delta\mathrm{T}_{n+1}(x) = \mathrm{T}_n(x) + (x - \beta_n^{\mathrm{T}} + 1)\Delta\mathrm{T}_n(x) - \gamma_n^{\mathrm{T}}\Delta\mathrm{T}_{n-1}(x), \quad 0 \leq n \leq M_1 - 1. \tag{4.10}$$

Using the three-term recurrence relation for $\{\mathrm{T}_n(x)\}_n$ and (4.10) a new equation is obtained:

$$\begin{aligned}(v + \mu\xi_1)\mathrm{T}_n(\xi_1) + v(1 + \xi_1)\Delta\mathrm{T}_n(\xi_1) =& (\eta + \delta\xi_2)\mathrm{T}_n(\xi_2) + \eta(1 + \xi_2)\Delta\mathrm{T}_n(\xi_2), \\ & 0 \leq n \leq M_1.\end{aligned} \tag{4.11}$$

Let us repeat the process from (4.9) to (4.11), but starting with (4.11) instead of (4.9). Finally, mimicking the process starting with this new last equation we find an homogeneous system of four linear equations with variables $\mathrm{T}_n(\xi_1)$, $\mathrm{T}_n(\xi_2)$, $\Delta\mathrm{T}_n(\xi_1)$ and $\Delta\mathrm{T}_n(\xi_2)$. The determinant of the matrix of coefficients is, after replacing $\mu = 1$, $v = k_1$, $\delta = 1$ and $\eta = k_2$,

$$k_1 k_2 (k_1 - 1)(k_2 - 1)(\xi_1 - \xi_2)^2 (\xi_2 - \xi_1 - 1)(\xi_2 - \xi_1 + 1). \tag{4.12}$$

Then, we need to study how are the solutions of this linear system, depending on the value of (4.12).

If the determinant (4.12) is different of zero, then the solution of the linear system is

$$\mathrm{T}_n(\xi_i) = \Delta\mathrm{T}_n(\xi_i) = 0, \quad 3 \le n \le M_1, \quad i = 1, 2.$$

Hence it should be $\mathrm{T}_n(\xi_1) = \mathrm{T}_n(\xi_2) = \mathrm{T}_n(\xi_1 + 1) = \mathrm{T}_n(\xi_2 + 1) = 0$, for every $3 \le n \le M_1$, but this contradicts that $\{\mathrm{T}_n(x)\}_{n=0}^{M_1}$ is a MOPS.

Now we discuss what happens when (4.12) vanishes.

(1) If $k_i = 1$ $(i = 1, 2)$ then the solution of the system is $\mathrm{T}_n(\xi_i) = -\Delta\mathrm{T}_n(\xi_i)$, $\mathrm{T}_n(\xi_j) = \Delta\mathrm{T}_n(\xi_j) = 0$, for every $3 \le n \le M_1$. Hence $\mathrm{T}_n(\xi_i + 1) = \mathrm{T}_n(\xi_j) = \mathrm{T}_n(\xi_j + 1) = 0$ for every $3 \le n \le M_1$. But since $\{\mathrm{T}_n(x)\}_{n=0}^{M_1}$ is a MOPS these equations can not be held.

(2) If $k_i \ne 1$ $(i = 1, 2)$,

   (a) Suppose $\xi_1 = \xi_2$. Then, it is trivial to check that $k_1 = k_2$.

   (b) The case $|\xi_1 - \xi_2| = 1$ can not be held because of the hypothesis of the Lemma.

■

LEMMA 4.5 *Let (u_0, u_1) be a Δ-coherent pair of linear functionals, where $M_0 \ge 2$ and $M_1 \ge 4$ are the order of u_0 and u_1, respectively. Let $\mathrm{A}_3(x)$ and $\mathrm{B}_2(x)$ be the polynomials defined in* (3.12) *and $\mathrm{c}_n(x)$ the polynomials defined in* (3.10). *If $\mathrm{B}_2(x)$ has not a double zero and that no zero of $\mathrm{B}_2(x)$ is a root of $\Delta\mathrm{B}_2(x) = 0$, then there exists a parameter ξ such that $\mathrm{B}_2(\xi) = \mathrm{A}_3(\xi) = 0$. Furthermore, we have $\mathrm{c}_1(\xi) \ne 0$, $\mathrm{c}_1(\xi - 1) \ne 0$ and $\pi(\xi) = 0$.*

Proof Let us denote ξ_1 and ξ_2 the zeros of $\mathrm{B}_2(x)$. If both ξ_i $(i = 1, 2)$ are not zeros of $\mathrm{A}_3(x)$, we can apply Lemma 4.3 to obtain two constants $k_1 \ne 0$ and $k_2 \ne 0$ such that $\mathrm{c}_n(\xi_1 - 1) + k_1 \Delta\mathrm{c}_n(\xi_1 - 1) = 0$ and $\mathrm{c}_n(\xi_2 - 1) + k_2 \Delta\mathrm{c}_n(\xi_2 - 1) = 0$, for every $1 \le n \le \min\{M_0 - 1, M_1\}$. Using Lemma 4.4 we obtain that $k_1 = k_2$ and also that $\xi_1 = \xi_2$ in contradiction with the hypothesis of this Lemma.

Let us denote ξ the common zero of $\mathrm{B}_2(x)$ and $\mathrm{A}_3(x)$. If $\mathrm{B}_2(\xi) = 0$ we have $\Delta\mathrm{B}_2(\xi) \ne 0$, and hence $\mathrm{c}_1(\xi) \ne 0$. But we also obtain that $\mathrm{c}_1(\xi - 1) \ne 0$, because if $\mathrm{c}_1(\xi - 1) = 0$ then it should be $\mathrm{B}_2(\xi - 1) = \Delta\mathrm{B}_2(\xi - 1) = 0$ which is not possible.

From (4.1), setting $n = 1$ and $x = \xi$, we obtain that $\pi(\xi) = 0$. ■

THEOREM 4.6 *Let (u_0, u_1) be a Δ-coherent pair of linear functionals, where $M_0 \ge 2$ and $M_1 \ge 4$ are the order of u_0 and u_1, respectively. Suppose that $\mathrm{B}_2(x)$ has not a double zero and also that no zero of $\mathrm{B}_2(x)$ is a root of $\Delta\mathrm{B}_2(x) = 0$. Then*

(i) *There exist a parameter* ξ *and two polynomials* $\tilde{A}(x)$ *and* $\pi_1(x)$ *with* $\deg(\tilde{A}) \leq 2$ *and* $\deg(\pi_1) \leq 1$ *such that*

$$\tilde{A}(x)u_0 = \frac{\sigma_1\sigma_2}{\langle u_1, \mathrm{T}_1^2\rangle\langle u_1, \mathrm{T}_2^2\rangle}(x-\xi)u_1, \tag{4.13}$$

$$\pi_1(x)u_0 = \frac{\sigma_1\sigma_2}{\langle u_1, \mathrm{T}_1^2\rangle\langle u_1, \mathrm{T}_2^2\rangle}(x-\xi)\Delta u_1. \tag{4.14}$$

(ii) *If* $\tilde{A}(\xi) = 0$ *then* $\pi_1(\xi) = 0$.

(iii) *The functional* u_1 *is a classical discrete linear functional verifying* $\Delta[\tilde{A}(x+1)u_1] = \tilde{\psi}_1(x)u_1$, *where* $\deg(\tilde{\psi}_1) = 1$.

Proof

(i) Let us denote ξ_1 and ξ_2 the zeros of $\mathrm{B}_2(x)$. Using Lemma 4.5, at least one of them is also a zero of $\mathrm{A}_3(x)$. Suppose that $\mathrm{A}_3(\xi_1) = 0$. Using again Lemma 4.5 we obtain that $\pi(\xi_1) = 0$. Let us define

$$\mathrm{B}_2(x) = (x-\xi_1)\tilde{B}(x), \quad \mathrm{A}_3(x) = (x-\xi_1)\tilde{A}(x), \quad \pi(x) = (x-\xi_1)\pi_1(x). \tag{4.15}$$

Then, we can divide both members of (4.1) by $x - \xi_1$ and we obtain

$$n\frac{\mathrm{P}_n(x)}{\langle u_0, \mathrm{P}_n^2\rangle}\tilde{B}(x) = \tilde{A}(x)\Delta \mathrm{c}_n(x-1) + \mathrm{c}_n(x-1)\pi_1(x), \quad 1 \leq n \leq \min\{M_0 - 1, M_1\}. \tag{4.16}$$

If we eliminate u_1 in (3.16) we get $\pi(x)u_0 = \mathrm{B}_2(x)\Delta u_1$. From (4.15) we obtain

$$\tilde{A}(x)u_0 = \tilde{B}(x)u_1 \tag{4.17}$$

with an appropriate choice of the first moments of the functionals u_0 and u_1, which was to be proved. Furthermore, from (3.11) and (4.2) we get

$$\pi_1(x)u_0 = \tilde{B}(x)\Delta u_1 + R\delta_{\xi_1}, \qquad \tilde{A}(x)\Delta u_1 = \pi_1(x)u_1 + K\delta_{\xi_1}. \tag{4.18}$$

Hence from (4.16) and (3.15) we obtain for $1 \leq n \leq \min\{M_0 - 1, M_1\}$

$$\begin{aligned}(\tilde{A}(x)\Delta \mathrm{c}_n(x-1) + \mathrm{c}_n(x-1)\pi_1(x))u_0 &= \left(n\tilde{B}(x)\frac{\mathrm{P}_n(x)}{\langle u_0, \mathrm{P}_n^2\rangle}\right)u_0 \\ &= \tilde{B}(x)(\Delta \mathrm{c}_n(x-1)u_1 + \mathrm{c}_n(x-1)\Delta u_1),\end{aligned}$$

i.e.,

$$\Delta \mathrm{c}_n(x-1)(\tilde{A}(x)u_0 - \tilde{B}(x)u_1) = \mathrm{c}_n(x-1)(\tilde{B}(x)\Delta u_1 - \pi_1(x)u_0), \quad 1 \leq n \leq \min\{M_0 - 1, M_1\}.$$

Moreover, using (4.17) the above identity becomes $\mathrm{c}_n(x-1)R\delta_{\xi_1} = 0$ for every $1 \leq n \leq \min\{M_0 - 1, M_1\}$. Since $\mathrm{c}_1(\xi_1 - 1) \neq 0$ we obtain $R = 0$ and this proves (4.14).

(ii) From the definition of $\tilde{A}(x)$ we have $\tilde{A}(\xi_2) = 0$ and then, using Lemma 4.5 it follows that $\pi_1(\xi_2) = 0$, so part (ii) of the Theorem is proved.

(iii) Finally, using (4.16) with $n = 1$,

$$\tilde{A}(x)\Delta c_1(x-1) = \frac{P_1(x)}{\langle u_0, P_1^2\rangle}\tilde{B}(x) - c_1(x-1)\pi_1(x)$$

holds. If we take the first equation of (3.16) and the last equation it follows

$$\begin{aligned}\tilde{A}(x)\frac{P_1(x)}{\langle u_0, P_1^2\rangle}u_0 &= \tilde{A}(x)\Delta c_1(x-1)u_1 + \tilde{A}(x)c_1(x-1)\Delta u_1\\ &= \left(\frac{P_1(x)}{\langle u_0, P_1^2\rangle}\tilde{B}(x) - c_1(x-1)\pi_1(x)\right)u_1 + \tilde{A}(x)c_1(x-1)\Delta u_1,\end{aligned}$$

whence

$$\frac{P_1(x)}{\langle u_0, P_1^2\rangle}(\tilde{A}(x)u_0 - \tilde{B}(x)u_1) = c_1(x-1)(\tilde{A}(x)\Delta u_1 - \pi_1(x)u_1).$$

From the previous equation we obtain $Kc_1(\xi_1 - 1) = 0$, by using (4.17) and (4.18). Since $c_1(\xi_1 - 1) \neq 0$, then $K = 0$. Thus, the second equation in (4.18) reads as $\tilde{A}(x)\Delta u_1 = \pi_1(x)u_1$. Therefore, by using Proposition 2.7 and this last equality, we get

$$\Delta[\tilde{A}(x+1)u_1] = \Delta\tilde{A}(x)u_1 + \tilde{A}(x)\Delta u_1 = (\Delta\tilde{A}(x) + \pi_1(x))u_1 = \tilde{\psi}_1(x)u_1,$$

where $\deg(\tilde{\psi}_1) \leq 1$. As in Theorem 4.2, we use that u_1 is weakly quasi-definite of order $M_1 \geq 4$ to conclude $\deg(\tilde{\psi}_1) = 1$, *i.e.*, u_1 is a classical discrete linear functional. ■

Remark Note that if u_1 is a linear functional of order $M_1 > 7$, the polynomial $B_2(x)$ can not have a double zero. For the proof, if ξ is a double zero of $B_2(x)$, then

$$c_1(x) = \frac{\sigma_1}{\langle u_1, T_1^2\rangle}\left(x - \left(\xi - \frac{1}{2}\right)\right) \tag{4.19}$$

applying the Δ operator in the definition of $B_2(x)$ given in (3.12). From (3.5) for $n = 0$, it follows

$$\frac{\phi_0(x)}{\langle u_0, \phi_0\rangle}u_0 = -c_1(x)u_1,$$

using (2.7) and the definition of $c_1(x)$ in (3.10). We can now use the definition of $\phi_0(x)$ given in (3.13) as well as (3.11) in order to obtain

$$\left(\frac{\phi_1(x)B_2^2(x)}{\langle u_0, \phi_0\rangle} + c_1(x)\right)u_1 = 0.$$

Since u_1 is a weakly quasi-definite linear functional of order greater than 7, then

$$\frac{\phi_1(x)\mathrm{B}_2^2(x)}{\langle u_0, \phi_0 \rangle} + \mathrm{c}_1(x) = 0.$$

By using (4.19) and since $\mathrm{B}_2(\xi) = 0$, the above expression for $x = \xi$ gives $\sigma_1 = 0$ which is not possible.

We can summarize the results obtained in this section in the following theorem.

THEOREM 4.7 *Let (u_0, u_1) be a Δ-coherent pair of linear functionals and let $\{\mathrm{P}_n(x)\}_{n=0}^{M_0}$ and $\{\mathrm{T}_n(x)\}_{n=0}^{M_1}$ be the corresponding MOPS associated with u_0 and u_1, respectively, with $M_0 \geq 2$ and $M_1 \geq 8$. Let*

$$\mathrm{B}_2(x) := \begin{vmatrix} \mathrm{c}_1(x-1) & \mathrm{c}_2(x-1) \\ \Delta\mathrm{c}_1(x-1) & \Delta\mathrm{c}_2(x-1) \end{vmatrix} = \frac{\sigma_1\sigma_2}{t_1 t_2}(x-\xi)(x-\eta), \tag{4.20}$$

where

$$\mathrm{c}_n(x) := \sigma_n \frac{\mathrm{T}_n(x)}{t_n} - \frac{\mathrm{T}_{n\;1}(x)}{t_{n\;1}}, \quad t_{n\;1} := \langle u_1, \mathrm{T}_{n\;1}^2 \rangle, \quad 1 \leq n \leq \min\{M_0 - 1, M_1\}. \tag{4.21}$$

One of the following situations hold

(1) *If $|\xi - \eta| = 1$, then u_0 is a classical discrete linear functional satisfying*

$$\Delta[\tilde{\phi}(x-1)u_0] = \tilde{\psi}_0(x)u_0.$$

Moreover,

$$\tilde{\phi}(x-1)u_0 = \frac{\sigma_1\sigma_2}{t_1 t_2}(x-\xi)u_1. \tag{4.22}$$

(2) *If $|\xi - \eta| \neq 1$, then u_1 is a classical discrete linear functional satisfying*

$$\Delta[\tilde{A}(x+1)u_1] = \tilde{\psi}_1(x)u_1.$$

Furthermore,

$$\tilde{A}(x)u_0 = \frac{\sigma_1\sigma_2}{t_1 t_2}(x-\xi)u_1, \qquad \pi_1(x)u_0 = \frac{\sigma_1\sigma_2}{t_1 t_2}(x-\xi)\Delta u_1, \tag{4.23}$$

where

$$\pi_1(x) := \tilde{\psi}_1(x) - \Delta\tilde{A}(x). \tag{4.24}$$

Finally, if $\tilde{A}(\xi) = 0$ then $\pi_1(\xi) = 0$.

From the above theorem, we have obtained in Ref. [5] the classification of all Δ-coherent pairs for Hahn, Kravchuk, Meixner and Charlier linear functionals, which allowed us to recover the classification given by Meijer in Ref. [20], using a limit process.

5 EXAMPLES

In this section we present examples of Δ-coherent pairs (u_0, u_1) of linear functionals. In the first example, we deal with quasi-definite linear functionals where u_0 or u_1 is the Meixner linear functional $u^{(\gamma,\mu)}$. On the other hand, in the second example we present Δ-coherent pairs of linear functionals where u_0 or u_1 is the Hahn linear functional $u^{(\alpha,\beta,N)}$, which is a weakly quasi-definite linear functional of order $N-1$.

5.1 Meixner Case

Let (u_0, u_1) be a Δ-coherent pair of linear functionals and assume that $u_1 \equiv u^{(\gamma,\mu)}$ is the Meixner linear functional defined by

$$\langle u^{(\gamma,\mu)}, p\rangle = \sum_{s=0}^{\infty} p(s)\, \frac{\mu^s\, \Gamma(\gamma+s)\, (1-\mu)^\gamma}{\Gamma(s+1)\, \Gamma(\gamma)}, \quad 0<\mu<1, \ \ \gamma>0, \ \ \text{for every } p\in\mathbb{P}, \tag{5.1}$$

[see Ref. 6 and references therein], which satisfies the distributional equation

$$\Delta[\mu(x+\gamma)u^{(\gamma,\mu)}] = (\gamma\mu - x(1-\mu))u^{(\gamma,\mu)}.$$

For this quasi-definite linear functional the moments are given by

$$(u^{(\gamma,\mu)})_n = \sum_{m=0}^{n} \mathcal{S}_m(n)\, (\gamma)_m \left(\frac{\mu}{1-\mu}\right)^m, \quad n\ge 0, \tag{5.2}$$

where $\mathcal{S}_m(n)$ denotes the Stirling numbers of second kind [2]

$$\mathcal{S}_m(n) = \sum_{j=0}^{m} \frac{(-1)^{m-j} j^n}{(m-j)!\,j!}. \tag{5.3}$$

Let us denote by

$$\mathrm{M}_n^{(\gamma,\mu)}(x) = \left(\frac{\mu}{\mu-1}\right)^n (\gamma)_n \; {}_2\mathrm{F}_1\left(\begin{matrix}-n, & -x\\ & \gamma\end{matrix}\;\middle|\; 1-\frac{1}{\mu}\right), \quad n\ge 0,$$

the polynomials orthogonal with respect to $u^{(\gamma,\mu)}$, which are called Meixner polynomials [see Ref. 10, p. 45, Ref. 21, p. 51]. The following relation between two families of Meixner polynomials holds [21, (2.4.16)]

$$\mathrm{M}_n^{(\gamma+1,\mu)}(x) = \frac{\Delta \mathrm{M}_{n+1}^{(\gamma,\mu)}(x)}{n+1}, \quad n\ge 0. \tag{5.4}$$

5.1.1 Case $u_0 \equiv u^{(\gamma,\mu)}$

Let (u_0, u_1) be a Δ-coherent pair of linear functionals and assume that $u_0 \equiv u^{(\gamma,\mu)}$ is the Meixner linear functional. In this situation, the linear functional u_1 can be computed from (4.22)

$$\mu(x+\gamma)u^{(\gamma,\mu)} = (x-\xi)u_1.$$

By using Proposition 2.9 and (5.4), the above equation can be written as

$$u^{(\gamma+1,\mu)} = (x-\xi)u_1.$$

Then, from (2.1) we get

$$u_1 = (x-\xi)^{\;1}u^{(\gamma+1,\mu)} + L\delta_{\xi}.$$

As an example, for $L=0$ and $\xi=0$ we shall obtain a recurrence relation for the sequences $\{\sigma_n\}$ and $\{\mathrm{T}_n(x)\}_{n=0}^{M_1}$, the MOPS associated with u_1 which can be computed from Ref. [8]

$$\mathrm{T}_n(0)\ x\ \mathrm{M}_n^{(\gamma+1,\mu)}(x) = \mathrm{T}_n(0)\ \mathrm{T}_{n+1}(x) - \mathrm{T}_{n+1}(0)\mathrm{T}_n(x). \tag{5.5}$$

By using the above equation, (5.4), the three-term recurrence relation

$$(x-\beta_n^{(\gamma+1,\mu)})\ \mathrm{M}_n^{(\gamma+1,\mu)}(x) = \mathrm{M}_{n+1}^{(\gamma+1,\mu)}(x) + \gamma_n^{(\gamma+1,\mu)}\ \mathrm{M}_{n\;1}^{(\gamma+1,\mu)}(x), \tag{5.6}$$

satisfied by monic Meixner polynomials $\mathrm{M}_n^{(\gamma+1,\mu)}(x)$, where

$$\beta_n^{(\gamma+1,\mu)} = \frac{(1+\gamma)\ \mu + (1+\mu)\ n}{1-\mu},$$
$$\gamma_n^{(\gamma+1,\mu)} = \frac{\mu n\ (\gamma+n)}{(1-\mu)^2},$$

and the relation between $\{\mathrm{M}_n^{(\gamma+1,\mu)}(x)\}_n$ and $\{\mathrm{T}_n(x)\}_n$ since (u_0, u_1) is a Δ-coherent pair of linear functionals,

$$\mathrm{T}_n(x) = \frac{\Delta\mathrm{M}_{n+1}^{(\gamma,\mu)}(x)}{n+1} - \sigma_n\frac{\Delta\mathrm{M}_n^{(\gamma,\mu)}(x)}{n} = \mathrm{M}_n^{(\gamma+1,\mu)}(x) - \sigma_n\mathrm{M}_{n\;1}^{(\gamma+1,\mu)}(x), \quad n \geq 1, \tag{5.7}$$

we obtain

$$(\beta_n^{(\gamma+1,\mu)} + \sigma_{n+1} + \varpi_n)\ \mathrm{M}_n^{(\gamma+1,\mu)}(x) + (\gamma_n^{(\gamma+1,\mu)} - \sigma_n\varpi_n)\ \mathrm{M}_{n\;1}^{(\gamma+1,\mu)}(x) = 0,$$

where

$$\varpi_n = \frac{\mathrm{M}_{n+1}^{(\gamma+1,\mu)}(0) - \sigma_{n+1}\mathrm{M}_n^{(\gamma+1,\mu)}(0)}{\mathrm{M}_n^{(\gamma+1,\mu)}(0) - \sigma_n\mathrm{M}_{n\;1}^{(\gamma+1,\mu)}(0)},$$

and [21]

$$\mathrm{M}_n^{(\gamma+1,\mu)}(0) = \left(\frac{\mu}{\mu-1}\right)^n (\gamma+1)_n,$$

being $(A)_s$ the Pochhammer symbol. Since $\{\mathrm{M}_n^{(\gamma+1,\mu)}(x)\}_{n=0}^{N-2}$ is a set of linearly independent vectors in $\mathbb{P}$, it yields the following recurrence relation for the coefficients σ_n

$$\sigma_{n+1} = -\beta_n^{(\gamma+1,\mu)} - \frac{\gamma_n^{(\gamma+1,\mu)}}{\sigma_n}, \quad n \geq 1, \tag{5.8}$$

by using again the three-term recurrence relation (5.6).

If we choose $(u_1)_0 := \langle u_1, 1\rangle = 1$, then $\mathrm{T}_1(x) = x - 1$. From the Δ-coherence relation,

$$\sigma_1 = 1 - \beta_0^{(\gamma+1,\mu)}.$$

Moreover, by using (5.6) and (5.8), from (5.7) we have

$$\begin{aligned} \mathrm{T}_n(x) &= \mathrm{M}_n^{(\gamma+1,\mu)}(x) + \left(\beta_{n-1}^{(\gamma+1,\mu)} + \frac{\gamma_{n-1}^{(\gamma+1,\mu)}}{\sigma_{n-1}}\right)\mathrm{M}_{n-1}^{(\gamma+1,\mu)}(x) \\ &= x\mathrm{M}_{n-1}^{(\gamma+1,\mu)}(x) + \gamma_{n-1}^{(\gamma+1,\mu)}\left(\frac{\mathrm{M}_{n-1}^{(\gamma+1,\mu)}(x)}{\sigma_{n-1}} - \mathrm{M}_{n-2}^{(\gamma+1,\mu)}(x)\right) \\ &= x\mathrm{M}_{n-1}^{(\gamma+1,\mu)}(x) + \frac{\gamma_{n-1}^{(\gamma+1,\mu)}}{\sigma_{n-1}}\mathrm{T}_{n-1}(x), \quad n \geq 2. \end{aligned}$$

a recurrence relation for the sequence $\{\mathrm{T}_n(x)\}_n$.

5.1.2 Case $u_1 \equiv u^{(\gamma,\mu)}$

Let (u_0, u_1) be a Δ-coherent pair of linear functionals and assume that $u_1 \equiv u^{(\gamma,\mu)}$ is the Meixner linear functional. We shall consider the following three situations:
Case $\gamma > 1$ In this situation, from (4.23) we get

$$u_0 = (x-\xi)u^{(\gamma-1,\mu)}.$$

We assume that $\xi \leq 0$ in order to obtain positive-definite linear functionals. We shall obtain a recurrence relation for the MOPS $\{\mathrm{P}_n(x)\}_n$ associated to u_0, as well as a recurrence relation for the coherence parameters σ_n.

The sequence $\{\mathrm{P}_n(x)\}_n$ can be computed from Ref. [8]

$$(x-\xi)\mathrm{P}_n(x) = \mathrm{M}_{n+1}^{(\gamma-1,\mu)}(x) - \frac{\mathrm{M}_{n+1}^{(\gamma-1,\mu)}(\xi)}{\mathrm{M}_n^{(\gamma-1,\mu)}(\xi)}\mathrm{M}_n^{(\gamma-1,\mu)}(x). \tag{5.9}$$

Moreover, they satisfy the following three-term recurrence relation

$$\mathrm{P}_{n+1}(x) = (x-\tilde{B}_n)\mathrm{P}_n(x) - \tilde{C}_n\mathrm{P}_{n-1}(x), \;\; n \geq 1, \quad \mathrm{P}_0(x) := 1, \quad \mathrm{P}_1(x) := x - \tilde{B}_0, \tag{5.10}$$

where

$$\tilde{B}_n := B_{n+1}^{(\gamma-1,\mu)} + \frac{\mathrm{M}_{n+2}^{(\gamma-1,\mu)}(\xi)}{\mathrm{M}_{n+1}^{(\gamma-1,\mu)}(\xi)} - \frac{\mathrm{M}_{n+1}^{(\gamma-1,\mu)}(\xi)}{\mathrm{M}_{n}^{(\gamma-1,\mu)}(\xi)}, \quad n \geq 0,$$

$$\tilde{C}_n := \frac{\mathrm{M}_{n-1}^{(\gamma-1,\mu)}(\xi)\ \mathrm{M}_{n+1}^{(\gamma-1,\mu)}(\xi)}{(\mathrm{M}_{n}^{(\gamma-1,\mu)}(\xi))^2} C_n^{(\gamma-1,\mu)}, \quad n \geq 1,$$

and

$$B_n^{(\gamma-1,\mu)} = \frac{(\gamma-1)\mu + n(1+\mu)}{1-\mu}, \quad C_n^{(\gamma-1,\mu)} = \frac{\mu n(\gamma+n-2)}{(1-\mu)^2},$$

are the coefficients of the three-term recurrence relation satisfied by monic Meixner polynomials $\mathrm{M}_n^{(\gamma-1,\mu)}(x)$

$$\mathrm{M}_{n+1}^{(\gamma-1,\mu)}(x) = \left(x - B_n^{(\gamma-1,\mu)}\right) \mathrm{M}_{n}^{(\gamma-1,\mu)}(x) - C_n^{(\gamma-1,\mu)}\ \mathrm{M}_{n-1}^{(\gamma-1,\mu)}(x). \tag{5.11}$$

By using

$$\mathrm{M}_n^{(\gamma,\mu)}(x) = \frac{\Delta \mathrm{P}_{n+1}(x)}{n+1} - \sigma_n \frac{\Delta \mathrm{P}_n(x)}{n}, \quad n \geq 1,$$

as well as (5.9), (5.10) and (5.11), we obtain

$$\sigma_n = \xi - \tilde{B}_n - \frac{\tilde{C}_n}{\sigma_{n-1}}, \quad n \geq 2,$$

with

$$\sigma_1 := \frac{\gamma\, \mu\ (-\xi + \mu\ (-1+\gamma+\xi))}{(-1+\mu)\ (\gamma^2\ \mu^2 + (-1+\mu)^2\ (-1+\xi)\ \xi + \gamma\mu\ (-\mu + 2\ (-1+\mu)\ \xi))}.$$

Case $\gamma = 1$ Then, up to numerical factors, u_0 can be deduced from (4.23)

$$x u_0 = (x - \xi) u^{(1,\mu)}.$$

From Lemma 4.5, $\xi = 0$ and

$$u_0 = u^{(1,\mu)} + K\delta_0$$

using (2.1). Let $K > 0$ and let us denote by $\{\mathrm{P}_n(x)\}_{n=0}^{\infty}$ the sequence of polynomials orthogonal with respect to this quasi-definite linear functional u_0 (perturbation of $u^{(1,\mu)}$ by a Dirac functional). These polynomials satisfy the following three-term recurrence relation [13]

$$\mathrm{P}_{n+1}(x) = (x - \tilde{B}_n)\mathrm{P}_n(x) - \tilde{C}_n \mathrm{P}_{n-1}(x), \quad n \geq 1,$$
$$\mathrm{P}_0(x) = 1, \qquad \mathrm{P}_1(x) = x - \tilde{B}_0,$$

where

$$\tilde{B}_n = -n - \frac{2n+1}{\mu - 1} + \frac{n(K+1)}{1 + K(1-\mu^n)} + \frac{(n+1)(K+1)}{K(\mu^{n+1}-1) - 1}, \quad n \geq 0,$$
$$\tilde{C}_n = \frac{n^2 \mu \left(K\left(\mu^{n\;\;1} - 1\right) - 1\right)\left(K(\mu^{n+1}-1) - 1\right)}{(1-\mu)^2 \left(K\left(\mu^n - 1\right) - 1\right)^2}, \quad n \geq 1.$$

Furthermore, they can be written in terms of hypergeometric series [4]

$$\mathrm{P}_n(x) = n! \left(\frac{\mu}{\mu - 1}\right)^n {}_3\mathrm{F}_2\left(\begin{matrix} -n, & -x, & 1 + \frac{x}{\tau_n} \\ 1, & \frac{x}{\tau_n} & \end{matrix} \,\middle|\, 1 - \frac{1}{\mu}\right),$$

where

$$\tau_n = \frac{K\mu^n}{1 + K(1-\mu^n)}, \quad K > 0.$$

It can be checked that the Δ-coherence relation reads

$$\mathrm{M}_n^{(1,\mu)}(x) = \frac{\Delta \mathrm{P}_{n+1}(x)}{n+1} - \sigma_n \frac{\Delta \mathrm{P}_n(x)}{n}, \quad n \geq 1,$$

where

$$\sigma_n = n\left(\frac{1}{\mu - 1} + \frac{K+1}{1 + K(1-\mu^{n+1})}\right), \quad n \geq 1,$$

are non-zero real numbers, since $K > 0$.

Case $0 < \gamma < 1$ In this situation $u_0 = u_1 = u^{(\gamma,\mu)}$. The coherence relation reads

$$\mathrm{M}_n^{(\gamma,\mu)}(x) = \frac{\Delta \mathrm{M}_{n+1}^{(\gamma,\mu)}(x)}{n+1} - \sigma_n \frac{\Delta \mathrm{M}_n^{(\gamma,\mu)}(x)}{n}, \quad n \geq 1,$$

where

$$\sigma_n = n \frac{\mu}{\mu - 1}.$$

5.2 Hahn Case

Let $u^{(\alpha,\beta,N)}$ be the Hahn linear functional given by

$$\langle u^{(\alpha,\beta,N)}, r\rangle = \sum_{s=0}^{N\;\;1} \frac{\Gamma(N)\Gamma(\alpha+\beta+2)\Gamma(\alpha+N-s)\Gamma(\beta+s+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+N+1)\Gamma(N-s)\Gamma(s+1)} r(s),$$

for every $r \in \mathbb{P}$,

where $\alpha > -1$, $\beta > -1$, and $N \in \mathbb{N}$, which satisfies the distributional equation

$$\Delta[(N-x-1)(x+\beta+1)u^{(\alpha,\beta,N)}] = ((N-1)(\beta+1) - x(\alpha+\beta+2))u^{(\alpha,\beta,N)}.$$

For this weakly quasi-definite linear functional of order $N-1$ the moments are given by

$$(u^{(\alpha,\beta,N)})_n = \sum_{m=0}^{n} \mathcal{S}_m(n)(-1)^m \frac{(1-N)_m\ (\beta+1)_m}{(\alpha+\beta+2)_m}, \quad n \geq 0,$$

where $(A)_n$ denotes the Pochhammer symbol and $\mathcal{S}_m(n)$ are the Stirling numbers of second kind given in (5.3).

Let us denote by

$$\mathrm{h}_n^{(\alpha,\beta)}(x;N) = \frac{(-1)^n(N-n)_n(\beta+1)_n}{(\alpha+\beta+n+1)_n}\ {}_3\mathrm{F}_2\left(\begin{matrix}-n, & -x, & n+\alpha+\beta+1 \\ & \beta+1, & 1-N\end{matrix}\ \middle|\ 1\right),$$
$$0 \leq n \leq N-1,$$

the polynomials orthogonal with respect to $u^{(\alpha,\beta,N)}$, which are called the Hahn polynomials [see Ref. 10, p. 33, Ref. 21, p. 52]. The following relation between two families of Hahn polynomials holds [21, (2.4.13)]

$$\mathrm{H}_n^{(\alpha+1,\beta+1)}(x;N-1) = \frac{\Delta\mathrm{H}_{n+1}^{(\alpha,\beta)}(x;\ N)}{n+1}, \quad 0 \leq n \leq N-2. \tag{5.12}$$

5.2.1 *Case* $u_0 \equiv u^{(\alpha,\beta,N)}$

Let (u_0, u_1) be a Δ-coherent pair of linear functionals and assume that $u_0 \equiv u^{(\alpha,\beta,N)}$ is the Hahn linear functional. In this situation, the linear functional u_1 can be computed from (4.22)

$$(N-x-1)(x+\beta+1)u^{(\alpha,\beta,N)} = (x-\xi)u_1.$$

By using Proposition 2.9 and (5.12), the above equation can be written as

$$u^{(\alpha+1,\beta+1,N-1)} = (x-\xi)u_1.$$

Then, from (2.1) we get

$$u_1 = (x-\xi)^{-1}u^{(\alpha+1,\beta+1,N-1)} + L\delta_\xi.$$

If $L=0$ and $\xi=0$, we shall obtain a recurrence relation for the sequences $\{\sigma_n\}$ and $\{\mathrm{T}_n(x)\}_{n=0}^{M_1}$, the MOPS associated with u_1 which can be computed from [8]

$$\mathrm{T}_n(0)\ x\ \mathrm{H}_n^{(\alpha+1,\beta+1)}(x;\ N-1) = \mathrm{T}_n(0)\ \mathrm{T}_{n+1}(x) - \mathrm{T}_{n+1}(0)\mathrm{T}_n(x). \tag{5.13}$$

By using the above equation, (5.12), the three-term recurrence relation

$$\begin{aligned}(x-\beta_n^{(\alpha+1,\beta+1,N-1)})\ \mathrm{H}_n^{(\alpha+1,\beta+1)}(x;\ N-1) = \mathrm{H}_{n+1}^{(\alpha+1,\beta+1)}(x;\ N-1) + \gamma_n^{(\alpha+1,\beta+1,N-1)} \\ \times \mathrm{H}_{n-1}^{(\alpha+1,\beta+1)}(x;\ N-1),\end{aligned} \tag{5.14}$$

satisfied by monic Hahn polynomials $\mathrm{H}_n^{(\alpha+1,\beta+1)}(x; N-1)$, where

$$\beta_n^{(\alpha+1,\beta+1,N-1)} = \frac{n^2(2N+\alpha-\beta-4)+(N-2)(\beta+2)(\alpha+\beta+2)+n(2N+\alpha-\beta-4)(\alpha+\beta+3)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)},$$

$$\gamma_n^{(\alpha+1,\beta+1,N-1)} = \frac{n(N-n-1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+2)(n+N+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2\,(2n+\alpha+\beta+3)},$$

and the relation between $\{\mathrm{H}_n^{(\alpha+1,\beta+1)}(x;\ N-1)\}_{n=0}^{N-2}$ and $\{\mathrm{T}_n(x)\}_{n=0}^{M_1}$ since (u_0, u_1) is a Δ-coherent pair of linear functionals,

$$\mathrm{T}_n(x) = \frac{\Delta \mathrm{H}_{n+1}^{(\alpha,\beta)}(x;\ N)}{n+1} - \sigma_n \frac{\Delta \mathrm{H}_n^{(\alpha,\beta)}(x;\ N)}{n} = \mathrm{H}_n^{(\alpha+1,\beta+1)}(x;\ N-1) - \sigma_n \mathrm{H}_{n-1}^{(\alpha+1,\beta+1)}(x;\ N-1),\ \ 1 \le n \le \min\{N-2, M_1\}, \tag{5.15}$$

we obtain

$$\left(\beta_n^{(\alpha+1,\beta+1,N-1)} + \sigma_{n+1} + \varpi_n\right) \mathrm{H}_n^{(\alpha+1,\beta+1)}(x;\ N-1) + \left(\gamma_n^{(\alpha+1,\beta+1,N-1)} - \sigma_n \varpi_n\right) \mathrm{H}_{n-1}^{(\alpha+1,\beta+1)}(x;\ N-1) = 0,$$

where

$$\varpi_n = \frac{\mathrm{H}_{n+1}^{(\alpha+1,\beta+1)}(0;\ N-1) - \sigma_{n+1}\mathrm{H}_n^{(\alpha+1,\beta+1)}(0;\ N-1)}{\mathrm{H}_n^{(\alpha+1,\beta+1)}(0;\ N-1) - \sigma_n \mathrm{H}_{n-1}^{(\alpha+1,\beta+1)}(0;\ N-1)},$$

and [21]

$$\mathrm{H}_n^{(\alpha+1,\beta+1)}(0;\ N-1) = (-1)^n \frac{(\beta+2)_n\ (N-n-1)_n}{(n+\alpha+\beta+3)_n},$$

being $(A)_s$ the Pochhammer symbol. Since $\{\mathrm{H}_n^{(\alpha+1,\beta+1)}(x;\ N-1)\}_{n=0}^{N-2}$ is a set of linearly independent vectors in $\mathbb{P}_{N-2}$, it yields the following recurrence relation for the coefficients σ_n

$$\sigma_{n+1} = -\beta_n^{(\alpha+1,\beta+1,N-1)} - \frac{\gamma_n^{(\alpha+1,\beta+1,N-1)}}{\sigma_n}, \quad n \ge 1, \tag{5.16}$$

by using again the three-term recurrence relation (5.14).

If we choose $(u_1)_0 := \langle u_1, 1\rangle = 1$, then $\mathrm{T}_1(x) = x-1$. From the Δ-coherence relation,

$$\sigma_1 = 1 - \beta_0^{(\alpha+1,\beta+1,N-1)}.$$

Moreover, by using (5.14) and (5.16), from (5.15) we have

$$
\begin{aligned}
\mathrm{T}_n(x) &= \mathrm{H}_n^{(\alpha+1,\beta+1)}(x;\ N-1) + \left(\beta_{n-1}^{(\alpha+1,\beta+1,N-1)} + \frac{\gamma_{n-1}^{(\alpha+1,\beta+1,N-1)}}{\sigma_{n-1}}\right)\mathrm{H}_{n-1}^{(\alpha+1,\beta+1)}(x;\ N-1)\\
&= x\mathrm{H}_{n-1}^{(\alpha+1,\beta+1)}(x;\ N-1) + \gamma_{n-1}^{(\alpha+1,\beta+1,N-1)}\\
&\qquad\qquad \times\left(\frac{\mathrm{H}_{n-1}^{(\alpha+1,\beta+1)}(x;\ N-1)}{\sigma_{n-1}} - \mathrm{H}_{n-2}^{(\alpha+1,\beta+1)}(x;\ N-1)\right)\\
&= x\mathrm{H}_{n-1}^{(\alpha+1,\beta+1)}(x;\ N-1) + \frac{\gamma_{n-1}^{(\alpha+1,\beta+1,N-1)}}{\sigma_{n-1}}\mathrm{T}_{n-1}(x), \quad n\geq 2.
\end{aligned}
$$

a recurrence relation for the sequence $\{\mathrm{T}_n(x)\}_{n=0}^{M_1}$.

5.2.2 *Case* $u_1 \equiv u^{(\alpha,\beta,N)}$

Let (u_0, u_1) be a Δ-coherent pair of linear functionals and assume that $u_1 \equiv u^{(\alpha,\beta,N)}$ is the Hahn linear functional. We must consider the following situations:

Case $\alpha, \beta > 0$ In this situation, from (4.23) we get

$$u_0 = (x-\xi)u^{(\alpha-1,\beta-1,N+1)}.$$

We assume that $\xi \leq 0$ or $\xi \geq N$ in order to obtain nonnegative-definite linear functionals. We shall obtain a recurrence relation for the MOPS $\{\mathrm{P}_n(x)\}_{n=0}^{M_0}$ associated to u_0, as well as a recurrence relation for the coherence parameters σ_n.

The monic sequence $\{\mathrm{P}_n(x)\}_{n=0}^{M_0}$ can be computed from [8]

$$(x-\xi)\mathrm{P}_n(x) = \mathrm{H}_{n+1}^{(\alpha-1,\beta-1)}(x;\ N+1) - \frac{\mathrm{H}_{n+1}^{(\alpha-1,\beta-1)}(\xi;\ N+1)}{\mathrm{H}_n^{(\alpha-1,\beta-1)}(\xi;\ N+1)}\mathrm{H}_n^{(\alpha-1,\beta-1)}(x;\ N+1). \quad (5.17)$$

Moreover, they satisfy the following three-term recurrence relation

$$\mathrm{P}_{n+1}(x) = (x-\tilde{B}_n)\mathrm{P}_n(x) - \tilde{C}_n\mathrm{P}_{n-1}(x), \ \ 1\leq n\leq M_0-1, \quad \mathrm{P}_0(x) := 1, \quad \mathrm{P}_1(x) := x-\tilde{B}_0, \quad (5.18)$$

where

$$
\begin{aligned}
\tilde{B}_n &:= B_{n+1}^{(\alpha-1,\beta-1,N+1)} + \frac{\mathrm{H}_{n+2}^{(\alpha-1,\beta-1)}(\xi;\ N+1)}{\mathrm{H}_{n+1}^{(\alpha-1,\beta-1)}(\xi;\ N+1)} - \frac{\mathrm{H}_{n+1}^{(\alpha-1,\beta-1)}(\xi;\ N+1)}{\mathrm{H}_n^{(\alpha-1,\beta-1)}(\xi;\ N+1)}, \quad n\geq 0,\\
\tilde{C}_n &:= \frac{\mathrm{H}_{n-1}^{(\alpha-1,\beta-1)}(\xi;\ N+1)\ \mathrm{H}_{n+1}^{(\alpha-1,\beta-1)}(\xi;\ N+1)}{(\mathrm{H}_n^{(\alpha-1,\beta-1)}(\xi;\ N+1))^2}C_n^{(\alpha-1,\beta-1,N+1)}, \quad n\geq 1,
\end{aligned}
$$

and

$$B_n^{(\alpha-1,\beta-1,N+1)} = \frac{\beta\,(-2+\alpha+\beta)\,N + n\,(-1+\alpha+\beta+n)\,(\alpha-\beta+2N)}{(-2+\alpha+\beta+2n)\,(\alpha+\beta+2n)}$$

$$C_n^{(\alpha-1,\beta-1,N+1)} = -\frac{n\,(\alpha+n-1)\,(\beta+n-1)\,(\alpha+\beta+n-2)\,(n-N-1)\,(\alpha+\beta+n+N-1)}{(\alpha+\beta+2n-3)\,(-2+\alpha+\beta+2n)^2\,(-1+\alpha+\beta+2n)},$$

are the coefficients of the three-term recurrence relation

$$\begin{aligned}\mathrm{H}_{n+1}^{(\alpha-1,\beta-1)}(x;\,N+1) = (x - B_n^{(\alpha-1,\beta-1,N+1)})\,\mathrm{H}_n^{(\alpha-1,\beta-1)}(x;\,N+1)\\ - C_n^{(\alpha-1,\beta-1,N+1)}\,\mathrm{H}_{n-1}^{(\alpha-1,\beta-1)}(x;\,N+1).\end{aligned} \tag{5.19}$$

satisfied by monic Hahn polynomials $\mathrm{H}_n^{(\alpha-1,\beta-1)}(x;\,N+1)$. By using

$$\mathrm{H}_n^{(\alpha,\beta)}(x;\,N) = \frac{\Delta \mathrm{P}_{n+1}(x)}{n+1} - \sigma_n \frac{\Delta \mathrm{P}_n(x)}{n}, \quad n \geq 1,$$

as well as (5.17), (5.18) and (5.19), we obtain

$$\sigma_n = \xi - \tilde{B}_n - \frac{\tilde{C}_n}{\sigma_{n-1}}, \quad n \geq 2,$$

with

$$\sigma_1 := \frac{(1+\alpha)(1+\beta)(N-1)(\vartheta+N)\,((\alpha+\beta)-\beta N\xi)}{(1+\vartheta)(2+\vartheta)(\beta(1+\beta)(N-1)N - \vartheta(\alpha-\beta+2(1+\beta)N)\xi+\vartheta)\,(1+\vartheta)\xi^2)},$$

and $\vartheta := 1+\alpha+\beta$.

Case $\beta = 0$ From Lemma 4.5 then $\xi = 0$, and from (4.23)

$$u_0 = u^{(\alpha-1,0,N+1)} + L\delta_0, \quad \alpha > 0.$$

Let us assume that $L \geq 0$ and let us denote by $\{\mathrm{P}_n(x)\}_{n=0}^N$ the sequence of polynomials orthogonal with respect to this weakly quasi-definite linear functional u_0 (perturbation of $u^{(\alpha-1,0,N+1)}$ by a Dirac functional). These polynomials can be written in terms of hypergeometric series [3]

$$\mathrm{P}_n(x) = \frac{(-1)^n\,(N)!n!\,\Gamma(\alpha+n)}{(N-n)!\,\Gamma(\alpha+2n)}\,{}_4\mathrm{F}_3\left(\begin{matrix}-n,\ -x,\ \alpha+n-1,\ \eta_0+1\\ -N,\ 1,\ \eta_0\end{matrix}\,\middle|\,1\right), \quad 0 \leq n \leq N,$$

where

$$
\begin{aligned}
\eta_0 &:= \frac{x(\alpha+n-1)}{x+(\alpha+2n-1)\tau_n}, \\
\tau_n &:= \frac{L}{(1+L\ \mathrm{Ker}_{n-1}^{(\alpha-1,0)}(0,0))}\ \frac{(1+\alpha)_{2(n-1)}(N-n+1)_n}{(\alpha)_{n-1}(\alpha+n)_n\ (1+\alpha+N)_{n-1}}, \\
\mathrm{Ker}_{n-1}^{(\alpha-1,0)}(0,0) &:= \sum_{m=0}^{n-1} \frac{(-1)^m\ (1+\alpha)_{2m}\ ((N-m+1)_m)^2}{(\alpha)_m\ (\alpha+m)_m\ (-N)_m\ (1+\alpha+N)_m}.
\end{aligned}
$$

It can be checked that the Δ-coherence relation reads

$$
\mathrm{H}_n^{(\alpha,0)}(x;\ N) = \frac{\Delta \mathrm{P}_{n+1}(x)}{n+1} - \sigma_n \frac{\Delta \mathrm{P}_n(x)}{n}, \quad 1 \le n \le N-1,
$$

where,

$$
\sigma_n = \frac{-n\ (\alpha+n)\ (N-n)\ (\alpha+N+n)\ ((\alpha+N+1)_{n-1}\ (LN+\alpha(L+1)) - L(N-n+1)_n)}{(\alpha+2n)\ (\alpha+2n+1)\ ((\alpha+N+1)_n\ (LN+\alpha(L+1)) - L(N-n)_{n+1})},
$$

are non-zero real numbers.

Remark 3 Note that the classification of Δ-coherent pairs, assuming that one of the linear functionals u_0 or u_1 is the Charlier or the Kravchuk linear functional, can be done by using the same arguments as in the previous examples [5].

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