Lebesgue perturbation of a quasi-definite Hermitian functional. The positive definite case

A. Cachafeiro [a,∗,1], F. Marcellán [b,2], C. Pérez [c,1]

[a]*Departamento de Matemática Aplicada, E.T.S. Ingenieros Industriales, Universidad de Vigo, 36280 Vigo, Spain*

[b]*Departamento de Matemáticas, Escuela Politécnica Superior, Universidad Carlos III de Madrid, 28911 Leganés, Spain*

[c]*Departamento de Matemática Aplicada, Universidad de Vigo, 36280 Vigo, Spain*

Abstract

In this work we study the problem of orthogonality with respect to a sum of measures or functionals. First we consider the case where one of the functionals is arbitrary and quasi-definite and the other one is the Lebesgue normalized functional. Next we study the sum of two positive measures. The first one is arbitrary and the second one is the Lebesgue normalized measure and we obtain some relevant properties concerning the new measure. Finally we consider the sum of a Bernstein–Szegő measure and the Lebesgue measure. In this case we obtain more simple explicit algebraic relations as well as the relation between the corresponding Szegő's functions.

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∗ Corresponding author.

E-mail addresses: acachafe@dma.uvigo.es, acachafe@uvigo.es (A. Cachafeiro), pacomarc@ing.uc3m.es (F. Marcellán), carlosp@uvigo.es (C. Pérez).

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1. Introduction

The study of the sequence of polynomials orthogonal with respect to a sum of measures or functionals in terms of the initial ones is an interesting question. It has been solved when some additional conditions on the measures or functionals (for instance one of them is a singular measure with finite support) are considered. Probably it is necessary to assume some additional conditions on the measures or functionals in order the problem is well-posed.

In this work we study the problem in a particular case where one of the functionals is arbitrary and quasi-definite and the other one is the Lebesgue normalized functional. In this case we deduce a necessary and sufficient condition for the quasi-definite character of the new functional and we also obtain the explicit expression for the corresponding sequence of orthogonal polynomials. This transformation in the linear functional appears in a very natural way in the study of Laguerre–Hahn affine functionals which are not semiclassical (see [1]).

Indeed we consider a quasi-definite hermitian linear functional $\mathscr{L}$ which is semiclassical. We denote by $\{c_n\}$ its moments and by $G(z)$ the formal series $G(z) = \sum_{-\infty}^{\infty}(c_n/z^n)$. We assume that G is not a rational function. Then there exist polynomials $A \neq 0$ and $B \neq 0$ such that $A(z)G'(z) + B(z)G(z) = 0$ with $z^2\overline{A}(1/z) + A(z)\overline{B}(1/z) = 0$. If we consider the functional $\widehat{\mathscr{L}} = \mathscr{L} + \mathscr{L}_0$, where $\mathscr{L}_0$ is the Lebesgue normalized functional and we denote by $\widehat{G}(z)$ the formal series associated with $\widehat{\mathscr{L}}$, then $\widehat{G}(z) = G(z) + 1$ and therefore the following equation holds:

$$A(z)\widehat{G}'(z) + B(z)\widehat{G}(z) - B(z) = 0.$$

Thus $\widehat{\mathscr{L}}$ is a Laguerre–Hahn affine functional which is not semiclassical.

On the other hand, it is easy to see that this transformation preserves the Laguerre–Hahn affine character of this functional.

In the second part of the work we will consider the sum of two positive measures, one is arbitrary and the second one is the Lebesgue measure and we obtain some properties of the new measure as well as the corresponding sequence of orthogonal polynomials. The addition of the Lebesgue normalized functional to a positive definite linear functional can be read in terms of Toeplitz matrices. Taking into account that the entries of an infinite Toeplitz matrix T with positive definite principal submatrices T_n are the moments of a positive Borel measure supported on the unit circle, the problem under consideration means that we modify the moment of order zero by the addition of a constant. This idea appears in the Pisarenko modelling problem [7].

Assume a finite positive definite hermitian Toeplitz matrix T_n is given. Consider the problem of designing a discrete stationary stochastic process $\{x(p) : p \in \mathbb{Z}\}$ of zero mean and having the entries $t_0, t_1, \ldots, t_n$, of T_n as its $n+1$ first covariances:

$$t_k = E[x(p)\overline{x}(p-k)]$$

for $k = 0, \ldots, n$. Stated in such general form the problem admits infinitely many solutions. Let us restrict our attention to the particular situation where the process

$\{x(p) : p \in \mathbb{Z}\}$ is further required to be made by adding two uncorrelated stationary stochastic processes $\{y(p) : p \in \mathbb{Z}\}$ and $\{z(p) : p \in \mathbb{Z}\}$ with $\{z(p) : p \in \mathbb{Z}\}$ the output of a discrete white noise generator of variance λ as large as possible. Thus one must have

$$t_k = c_k + \lambda \delta_{k,0}, \quad k = 0, 1, \ldots, n,$$

$c_k = E[y(p)\overline{y}(p-k)]$ and where λ assumes the largest possible nonnegative value.

Consider the covariance matrix $C_n = T_n - \lambda I_n$ associated with the process $\{y(p) : p \in \mathbb{Z}\}$. As C_n is required to be a covariance matrix it must be nonnegative definite. Thus the maximum value of λ is identified at once as the smallest eigenvalue of T_n. The positive function $f(z) = c_0 + 2c_1 z + \cdots$ associated with the matrix C_n is uniquely defined as a rational lossless function of degree n, i.e.,

$$f(z) = \sum_{m=1}^{n} h_m \frac{\mathrm{e}^{\mathrm{i}\theta_m} + z}{\mathrm{e}^{\mathrm{i}\theta_m} - z} + \mathrm{i}\alpha,$$

with $h_m > 0$, $m = 1, 2, \ldots, n$. As a straightforward consequence we get

$$c_k = \sum_{j=1}^{n} h_j \mathrm{e}^{\mathrm{i}k\theta_j}.$$

This shows that the process $\{y(p) : p \in \mathbb{Z}\}$ can be modelled by adding up the outputs of n sinusoidal wave generators of amplitude $\sqrt{h_j}$, $j = 1, 2, \ldots, n$, whose phases ϕ_j are uncorrelated random variables of zero mean.

Thus $\{y(p) : p \in \mathbb{Z}\}$ is modelled as $y(p) = \sum_{m=1}^{n} \sqrt{h_m} \mathrm{e}^{\mathrm{i}(p\theta_m + \phi_m)}$ and, in particular, $c_k = E[y(p)\overline{y}(p-k)]$. The values θ_m, $m = 1, 2, \ldots, n$, are called the Pisarenko model frequencies. They can be obtained by an efficient numerical procedure (the split Levinson algorithm) [2] when $t_k \in \mathbb{R}$.

Finally, as an example the sum of a Bernstein–Szegő measure and the Lebesgue measure is analyzed. We obtain explicit algebraic relations as well as the relation between the corresponding Szegő's functions.

2. Algebraic properties

Let Λ be the linear space of Laurent polynomials, that is, $\Lambda = \mathrm{span}\{z^k; k \in \mathbb{Z}\}$ and let $\mathscr{L} : \Lambda \longrightarrow \mathbb{C}$ be a linear functional which is quasi-definite and hermitian. If we denote the moments of the functional $\mathscr{L}$ by $c_n = \mathscr{L}(z^n)$ for every integer $n \in \mathbb{Z}$, we say that

(i) $\mathscr{L}$ is hermitian if $\forall n \geqslant 0$, $c_{-n} = \overline{c_n}$;
(ii) $\mathscr{L}$ is quasi-definite (positive definite) if the principal submatrices of the Toeplitz moment matrix associated with the sequence $\{c_n\}$ are nonsingular (resp. positive definite), i.e., $\forall n \geqslant 0$, $\Delta_n = \det(c_{i-j})_{i,j=0}^{n} \neq 0$ (resp. > 0).

We denote by $\langle , \rangle_{\mathscr{L}}$ the bilinear form defined by $\langle P(z), Q(z)\rangle_{\mathscr{L}} = \mathscr{L}(P(z)\overline{Q}(1/z))$.

When $\mathscr{L}$ is positive definite, it is well-known (see [6]) that there exists a finite positive Borel measure μ supported on $[0, 2\pi]$ such that

$$c_n = \int_0^{2\pi} z^n \, \mathrm{d}\mu(\theta), \quad z = \mathrm{e}^{\mathrm{i}\theta}.$$

Next we recall some definitions (see [6]).

A sequence of polynomials $\{P_n(z)\}$ is said to be a sequence of polynomials orthogonal with respect to a linear functional $\mathscr{L}$ if

(i) $\forall n \geqslant 0$, $\deg(P_n) = n$,
(ii) $\forall n, m \geqslant 0$, $\langle P_n(z), P_m(z)\rangle_{\mathscr{L}} = k_n \delta_{n,m}$ with $k_n \neq 0$.

If the leading coefficient of each P_n is 1, then $\{P_n(z)\}$ is said to be the monic orthogonal polynomial sequence with respect to $\mathscr{L}$, (MOPS($\mathscr{L}$)).

When $\mathscr{L}$ is positive definite $\{P_n(z)\}$ is the orthonormal polynomial sequence with respect to $\mathscr{L}$ if for every $n = 0, 1, 2, \ldots$

$$\langle P_n(z), P_n(z)\rangle_{\mathscr{L}} = \|P_n(z)\|_\mu^2 = 1.$$

The following result is also well-known.

Let $\mathscr{L}$ be a hermitian linear functional. Then the following conditions are equivalent:

(i) $\mathscr{L}$ is quasi-definite.
(ii) There exists a sequence of polynomials orthogonal with respect to $\mathscr{L}$.

If $\mathscr{L}$ is a quasi-definite and hermitian linear functional and $\{\Phi_n(z)\}$ is the corresponding MOPS($\mathscr{L}$), we define the nth kernel $K_n(z, y)$ by

$$K_n(z, y) = \sum_{k=0}^{n} \frac{1}{e_k} \Phi_k(z)\overline{\Phi_k(y)},$$

with $e_k = \langle \Phi_k(z), \Phi_k(z)\rangle_{\mathscr{L}}$.

If we denote by $\mathbb{P}_n$ the linear space of algebraic polynomials of degree at most n, then the reproducing property for the nth kernel is very well-known

$$\langle K_n(z, y), P(z)\rangle_{\mathscr{L}} = \overline{P(y)}, \quad \forall P \in \mathbb{P}_n.$$

An analog reproducing property holds for the derivatives. Indeed if we denote by

$$K_n^{(0,j)}(z, y) = \sum_{k=0}^{n} \frac{1}{e_k} \Phi_k(z)\overline{\Phi_k^{(j)}(y)},$$

then

$$\langle K_n^{(0,j)}(z, y), P(z)\rangle_{\mathscr{L}} = \overline{P^{(j)}(y)}, \quad \forall P \in \mathbb{P}_n.$$

Next we assume that $\mathscr{L}$ is a quasi-definite and hermitian linear functional and we denote by $\{\Phi_n(z)\}$ and $\{K_n(z, y)\}$ the corresponding MOPS($\mathscr{L}$) and the sequence of n-kernels respectively. Let $\mathscr{L}_0$ be the linear functional associated with the Lebesgue normalized measure, that is, $\mathscr{L}_0(1) = 1$ and $\mathscr{L}_0(z^n) = 0$ for $n \geqslant 1$. Consider the linear functional $\widehat{\mathscr{L}} = \mathscr{L} + \mathscr{L}_0$.

Next we study necessary and sufficient conditions in order to $\widehat{\mathscr{L}}$ be quasi-definite.

Theorem 1. *Let $\mathscr{L}$ be a quasi-definite and hermitian linear functional with* MOPS($\mathscr{L}$) $\{\Phi_n\}$ *and let $\widehat{\mathscr{L}} = \mathscr{L} + \mathscr{L}_0$ with $\mathscr{L}_0$ the Lebesgue normalized linear functional. Then the following statements are equivalent.*

(1) *$\widehat{\mathscr{L}}$ is quasi-definite.*
(2) *The matrix $\boldsymbol{A}_n$ is nonsingular as well as*

$$e_0 + 1 \neq 0, \text{ and}$$
$$e_n + 1 + \left(\overline{\Phi_n(0)}, \ldots, \overline{\Phi_n^{(n-1)}(0)} \right) \boldsymbol{A}_n^{-1} \left(\Phi_n(0), \ldots, \Phi_n^{(n-1)}(0) \right)^{\mathrm{T}} \neq 0,$$

for $n \geqslant 1$,

where

$$\boldsymbol{A}_n = \operatorname{diag}(1, (1!)^2, \ldots, ((n-1)!)^2) + \mathbb{K}_{n-1}$$

with

$$\mathbb{K}_{n-1} = (K_{n-1}^{(i,j)}(0, 0))_{i,j=0}^{n-1}.$$

Moreover, *the* MOPS $\{\widehat{\Phi}_n\}$ *with respect to $\widehat{\mathscr{L}}$ is given by*

$$\widehat{\Phi}_n(z) = \frac{1}{|\boldsymbol{A}_n|} \left| \begin{array}{c|cccc} \Phi_n(z) & K_{n-1}^{(0,0)}(z, 0) & K_{n-1}^{(0,1)}(z, 0) & \ldots & K_{n-1}^{(0,n-1)}(z, 0) \\ \hline \Phi_n(0) & & & & \\ \Phi_n^{(1)}(0) & & & & \\ \vdots & & & \boldsymbol{A}_n & \\ \Phi_n^{(n-1)}(0) & & & & \end{array} \right|, \tag{1}$$

for $n \geqslant 1$.

Proof. Let $\{\widehat{\Phi}_n(z)\}$ be the MOPS($\widehat{\mathscr{L}}$). If we write

$$\widehat{\Phi}_n(z) = \Phi_n(z) + \sum_{j=0}^{n-1} \alpha_{n,j} \Phi_j(z),$$

then for $j = 0, \ldots, n-1$, we have

$$0 = \langle \widehat{\Phi}_n(z), \Phi_j(z) \rangle_{\widehat{\mathscr{L}}} = \langle \widehat{\Phi}_n(z), \Phi_j(z) \rangle_{\mathscr{L}} + \langle \widehat{\Phi}_n(z), \Phi_j(z) \rangle_{\mathscr{L}_0}.$$

Since $\langle \widehat{\Phi}_n(z), \Phi_j(z)\rangle_{\mathscr{L}} = \alpha_{n,j} e_j$ and

$$\langle \widehat{\Phi}_n(z), \Phi_j(z)\rangle_{\mathscr{L}_0} = \sum_{k=0}^{j} \frac{\widehat{\Phi}_n^{(k)}(0)\overline{\Phi_j^{(k)}(0)}}{(k!)^2},$$

then for $j = 0, \ldots, n-1$, we get

$$\alpha_{n,j} = -\frac{1}{e_j}\sum_{k=0}^{j} \frac{\widehat{\Phi}_n^{(k)}(0)\overline{\Phi_j^{(k)}(0)}}{(k!)^2}.$$

Hence

$$\widehat{\Phi}_n(z) = \Phi_n(z) - \sum_{j=0}^{n-1} \frac{\widehat{\Phi}_n^{(j)}(0)}{(j!)^2} K_{n-1}^{(0,j)}(z,0). \tag{2}$$

Finally, if we evaluate at the point $z = 0$ the preceding equation as well as its derivatives up to the $(n-1)$th order derivative we get the following linear system

$$\sum_{j=0}^{n-1} [\delta_{i,j}(i!)^2 + K_{n-1}^{(i,j)}(0,0)] \frac{\widehat{\Phi}_n^{(j)}(0)}{(j!)^2} = \Phi_n^{(i)}(0), \quad i = 0, \ldots, n-1,$$

or equivalently with matrix notation

$$\mathbf{A}_n \begin{pmatrix} \widehat{\Phi}_n(0) \\ \vdots \\ \frac{\widehat{\Phi}_n^{(n-1)}(0)}{((n-1)!)^2} \end{pmatrix} = \begin{pmatrix} \Phi_n(0) \\ \vdots \\ \Phi_n^{(n-1)}(0) \end{pmatrix}.$$

Taking into account the existence and uniqueness of the family of monic orthogonal polynomials with respect to the linear functional $\widehat{\mathscr{L}}$, we deduce the matrix $\mathbf{A}_n$ is nonsingular in order to the existence and uniqueness of the solution of the above linear system. Notice that if $\mathscr{L}$ is positive definite then the matrix $\mathbf{A}_n$ is positive definite because it is the sum of two positive definite matrices (see [3]). As a consequence

$$\begin{pmatrix} \widehat{\Phi}_n(0) \\ \vdots \\ \frac{\widehat{\Phi}_n^{(n-1)}(0)}{((n-1)!)^2} \end{pmatrix} = \mathbf{A}_n^{-1} \begin{pmatrix} \Phi_n(0) \\ \vdots \\ \Phi_n^{(n-1)}(0) \end{pmatrix}.$$

Hence

$$\begin{aligned} \widehat{\Phi}_n(z) = \Phi_n(z) &- \left(K_{n-1}^{(0,0)}(z,0), K_{n-1}^{(0,1)}(z,0), \ldots, K_{n-1}^{(0,n-1)}(z,0)\right) \\ &\times \mathbf{A}_n^{-1} \begin{pmatrix} \Phi_n(0) \\ \vdots \\ \Phi_n^{(n-1)}(0) \end{pmatrix}, \end{aligned} \tag{3}$$

from which (1) follows.

Moreover, if $\widehat{\mathscr{L}}$ is quasi-definite and we denote by $\widehat{e}_k = \langle \widehat{\Phi}_k(z), \widehat{\Phi}_k(z)\rangle_{\widehat{\mathscr{L}}}$, then

$$\begin{aligned}
0 \neq \widehat{e}_n = \langle \widehat{\Phi}_n(z), \Phi_n(z)\rangle_{\widehat{\mathscr{L}}} &= \langle \widehat{\Phi}_n(z), \Phi_n(z)\rangle_{\mathscr{L}} + \langle \widehat{\Phi}_n(z), \Phi_n(z)\rangle_{\mathscr{L}_0} \\
&= \langle \widehat{\Phi}_n(z), \Phi_n(z)\rangle_{\mathscr{L}} + 1 + \sum_{k=0}^{n-1} \frac{\widehat{\Phi}_n^{(k)}(0)\overline{\Phi_n^{(k)}(0)}}{(k!)^2} \\
&= e_n + 1 + (\overline{\Phi_n(0)}, \ldots, \overline{\Phi_n^{(n-1)}(0)})\mathbf{A}_n^{-1} \\
&\quad \times (\Phi_n(0), \ldots, \Phi_n^{(n-1)}(0))^{\mathrm{T}}.
\end{aligned}$$

Conversely, since $\mathbf{A}_n$ is nonsingular, we define $\widehat{\Phi}_n(z)$ by (1). It is immediate to prove that $\{\widehat{\Phi}_n\}$ is the MOPS($\widehat{\mathscr{L}}$) and therefore $\widehat{\mathscr{L}}$ is quasi-definite. Indeed for $i = 0, \ldots, n-1$,

$$\begin{aligned}
\langle \widehat{\Phi}_n(z), \Phi_i(z)\rangle_{\widehat{\mathscr{L}}} &= \langle \widehat{\Phi}_n(z), \Phi_i(z)\rangle_{\mathscr{L}} + \langle \widehat{\Phi}_n(z), \Phi_i(z)\rangle_{\mathscr{L}_0} \\
&= \left\langle \Phi_n(z) - \sum_{j=0}^{n-1} \frac{\widehat{\Phi}_n^{(j)}(0)}{(j!)^2} K_{n-1}^{(0,j)}(z, 0)), \Phi_i(z) \right\rangle_{\mathscr{L}} \\
&\quad + \sum_{k=0}^{i} \frac{\widehat{\Phi}_n^{(k)}(0)\overline{\Phi_i^{(k)}(0)}}{(k!)^2} \\
&= -\sum_{k=0}^{n-1} \frac{\widehat{\Phi}_n^{(k)}(0)\overline{\Phi_i^{(k)}(0)}}{(k!)^2} + \sum_{k=0}^{i} \frac{\widehat{\Phi}_n^{(k)}(0)\overline{\Phi_i^{(k)}(0)}}{(k!)^2} = 0.
\end{aligned}$$

On the other hand, from the hypothesis $\langle \widehat{\Phi}_0(z), \Phi_0(z)\rangle_{\widehat{\mathscr{L}}} \neq 0$, and for $n \geqslant 1$

$$\begin{aligned}
\langle \widehat{\Phi}_n(z), \Phi_n(z)\rangle_{\widehat{\mathscr{L}}} &= \langle \widehat{\Phi}_n(z), \Phi_n(z)\rangle_{\mathscr{L}} + \langle \widehat{\Phi}_n(z), \Phi_n(z)\rangle_{\mathscr{L}_0} \\
&= e_n + 1 + \left(\overline{\Phi_n(0)}, \ldots, \overline{\Phi_n^{(n-1)}(0)}\right)\mathbf{A}_n^{-1} \\
&\quad \times (\Phi_n(0), \ldots, \Phi_n^{(n-1)}(0))^{\mathrm{T}} \neq 0. \qquad \square
\end{aligned}$$

Next we are going to see that relation (1) can be given explicitly in terms of $\Phi_n(z)$. Let us consider the Christoffel–Darboux formula (see [6])

$$e_{n+1}(1 - z\overline{y})K_n(z, y) = \Phi_{n+1}^*(z)\overline{\Phi_{n+1}^*(y)} - \Phi_{n+1}(z)\overline{\Phi_{n+1}(y)}.$$

(Recall that the $*$ operator is defined by $P^*(z) = z^n \overline{P}(1/z)$ if $\deg P = n$.)

If we take derivatives up to the jth order with respect to z and evaluate at $z = 0$ we have

$$\begin{aligned}
&e_{n+1}\left(K_n^{(j,0)}(0, y) - j\overline{y}K_n^{(j-1,0)}(0, y)\right) \\
&\quad = {\Phi_{n+1}^*}^{(j)}(0)\overline{\Phi_{n+1}^*(y)} - \Phi_{n+1}^{(j)}(0)\overline{\Phi_{n+1}(y)}.
\end{aligned}$$

Taking conjugates and interchanching y by z we can write

$$e_{n+1}\left(K_n^{(0,j)}(z,0) - jzK_n^{(0,j-1)}(z,0)\right) = \overline{{\Phi_{n+1}^*}^{(j)}(0)}\Phi_{n+1}^*(z) - \overline{\Phi_{n+1}^{(j)}(0)}\Phi_{n+1}(z). \tag{4}$$

Since

$$\Phi_{n+1}(z) = \sum_{k=0}^{n+1} \frac{\Phi_{n+1}^{(k)}(0)}{k!} z^k \text{ and } \Phi_{n+1}^*(z) = \sum_{k=0}^{n+1} \frac{\overline{\Phi_{n+1}^{(n+1-k)}(0)}}{(n+1-k)!} z^k$$

we deduce that

$$\frac{{\Phi_{n+1}^*}^{(k)}(0)}{k!} = \frac{\overline{\Phi_{n+1}^{(n+1-k)}(0)}}{(n+1-k)!}$$

and therefore (4) becomes

$$e_{n+1}\left(K_n^{(0,j)}(z,0) - jzK_n^{(0,j-1)}(z,0)\right) = j!\frac{\Phi_{n+1}^{(n+1-j)}(0)}{(n+1-j)!}\Phi_{n+1}^*(z) - \overline{\Phi_{n+1}^{(j)}(0)}\Phi_{n+1}(z).$$

Hence we have

$$e_{n+1}\left(\frac{K_n^{(0,j)}(z,0)}{j!} - z\frac{K_n^{(0,j-1)}(z,0)}{(j-1)!}\right) = \frac{\Phi_{n+1}^{(n+1-j)}(0)}{(n+1-j)!}\Phi_{n+1}^*(z) - \frac{\overline{\Phi_{n+1}^{(j)}(0)}}{j!}\Phi_{n+1}(z) \tag{5}$$

for $j \geqslant 1$ with the initial condition

$$K_n^{(0,0)}(z,0) = \frac{1}{e_{n+1}}\left(\Phi_{n+1}^*(z)\overline{\Phi_{n+1}^*(0)} - \Phi_{n+1}(z)\overline{\Phi_{n+1}(0)}\right) = \frac{1-|\Phi_{n+1}(0)|^2}{e_{n+1}}\Phi_n^*(z).$$

Taking into account the well-known recurrence relations for orthogonal polynomials on the unit circle (see [6]) one has

$$\frac{1-|\Phi_{n+1}(0)|^2}{e_{n+1}} = \frac{1}{e_n},$$

and therefore

$$K_n^{(0,0)}(z,0) = \frac{\Phi_n^*(z)}{e_n}. \tag{6}$$

Notice that (5) can be rewritten with matrix notation

$$e_{n+1}\left(\frac{K_n^{(0,0)}(z,0)}{0!},\ldots,\frac{K_n^{(0,n)}(z,0)}{n!}\right)\mathbf{M}(n)=\left(\Phi_{n+1}^*(z),-\Phi_{n+1}(z)\right)\mathbf{N}(n),$$

where

$$\mathbf{M}(n)=\begin{pmatrix}1 & -z & 0 & \cdots & 0\\ 0 & 1 & -z & \cdots & 0\\ \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & -z\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}\in\mathscr{M}_{n+1,n+1}$$

and

$$\mathbf{N}(n)=\begin{pmatrix}1 & \frac{\Phi_{n+1}^{(n)}(0)}{n!} & \frac{\Phi_{n+1}^{(n-1)}(0)}{(n-1)!} & \cdots & \frac{\Phi_{n+1}^{(1)}(0)}{1!}\\ \overline{\frac{\Phi_{n+1}(0)}{0!}} & \overline{\frac{\Phi_{n+1}^{(1)}(0)}{1!}} & \overline{\frac{\Phi_{n+1}^{(2)}(0)}{2!}} & \cdots & \overline{\frac{\Phi_{n+1}^{(n)}(0)}{n!}}\end{pmatrix}.$$

Hence

$$e_{n+1}\left(\frac{K_n^{(0,0)}(z,0)}{0!},\ldots,\frac{K_n^{(0,n)}(z,0)}{n!}\right)=\left(\Phi_{n+1}^*(z),-\Phi_{n+1}(z)\right)\mathbf{N}(n)(\mathbf{M}(n))^{-1}.$$

Since

$$(\mathbf{M}(n))^{-1}=\begin{pmatrix}1 & z & \cdots & z^n\\ 0 & 1 & \cdots & z^{n-1}\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix},$$

for $0\leqslant j\leqslant n$ we get

$$e_{n+1}\frac{K_n^{(0,j)}(z,0)}{j!}=p_j(z;n+1)\Phi_{n+1}^*(z)-q_j(z;n+1)\Phi_{n+1}(z),$$

where

$$p_j(z;n+1)=\sum_{k=0}^{j}\frac{\Phi_{n+1}^{(n+1-k)}(0)}{(n+1-k)!}z^{j-k}$$

and

$$q_j(z;n+1)=\sum_{k=0}^{j}\frac{\overline{\Phi_{n+1}^{(k)}(0)}}{(k)!}z^{j-k}.$$

On the other hand, if we take derivatives in (5) up to the pth order with respect to z and evaluate at the point $z = 0$ we get

$$e_{n+1}\left(\frac{K_n^{(p,j)}(0,0)}{j!} - p\frac{K_n^{(p-1,j-1)}(0,0)}{(j-1)!}\right)$$
$$= \frac{\Phi_{n+1}^{(n+1-j)}(0)}{(n+1-j)!}\frac{p!}{(n+1-p)!}\overline{\Phi_{n+1}^{(n+1-p)}(0)} - \frac{\overline{\Phi_{n+1}^{(j)}(0)}}{(j)!}\Phi_{n+1}^{(p)}(0).$$

Thus

$$e_{n+1}\left(\frac{K_n^{(p,j)}(0,0)}{p!j!} - \frac{K_n^{(p-1,j-1)}(0,0)}{(p-1)!(j-1)!}\right)$$
$$= \frac{\Phi_{n+1}^{(n+1-j)}(0)\overline{\Phi_{n+1}^{(n+1-p)}(0)}}{(n+1-j)!(n+1-p)!} - \frac{\overline{\Phi_{n+1}^{(j)}(0)}\Phi_{n+1}^{(p)}(0)}{j!p!}.$$

If $p < j$, taking into account the telescopic character of the first member in the above expression, we obtain that

$$e_{n+1}\frac{K_n^{(p,j)}(0,0)}{p!j!} = e_{n+1}\frac{K_n^{(0,j-p)}(0,0)}{0!(j-p)!}$$
$$+ \sum_{k=0}^{p}\left(\frac{\Phi_{n+1}^{(n+1-j+k)}(0)\overline{\Phi_{n+1}^{(n+1-p+k)}(0)}}{(n+1-j+k)!(n+1-p+k)!} - \frac{\overline{\Phi_{n+1}^{(j-k)}(0)}\Phi_{n+1}^{(p-k)}(0)}{(j-k)!(p-k)!}\right)$$

with the initial condition

$$\frac{K_n^{(0,l)}(0,0)}{0!l!} = \frac{1}{e_n}\frac{\overline{\Phi_{n+1}^{(n-l)}(0)}}{(n-l)!}.$$

This last result follows from (6) if we take derivatives with respect to z and evaluate at $z = 0$.

3. The positive definite case

Throughout this section we assume that $\mathscr{L}$ is a positive definite Hermitian functional and we denote by μ the finite Borel positive measure such that $\mathscr{L}(z^n) = \int_0^{2\pi} z^n \, d\mu(\theta)$, $z = e^{i\theta}$. Then $\widehat{\mathscr{L}} = \mathscr{L} + \mathscr{L}_0$ is also a positive definite linear functional and the associated measure $\widehat{\mu}$ is

$$\widehat{\mu}(\theta) = (\mu'(\theta) + 1)d\theta + \mu_s(\theta), \text{ where } \mu(\theta) = \mu'(\theta)d\theta + \mu_s(\theta).$$

Hence if $\mu'(\theta)$ is the Radon–Nikodym derivative of the measure μ (see [8]) then $\widehat{\mu}'(\theta) = \mu'(\theta) + 1$ is the Radon–Nikodym derivative of the measure $\widehat{\mu}$.

If we denote by $\| \ \|_{\mu}$, $\| \ \|_{\widehat{\mu}}$ and $\| \ \|_{\theta}$ the induced norms in the spaces $L^2_{\mu}[0, 2\pi]$, $L^2_{\widehat{\mu}}[0, 2\pi]$ and $L^2_{\theta}[0, 2\pi]$, then we obtain the following results.

Theorem 2. *Let* $\{\Phi_n\}$ *be the* MOPS(μ) *and let* $\{\widehat{\Phi}_n\}$ *be the* MOPS$(\widehat{\mu})$. *If*

$$ {}_0m = \lim_{n\to\infty} ||\Phi_n(z)||^2_{\mu} \quad \text{and} \quad {}_0\widehat{m} = \lim_{n\to\infty} \|\widehat{\Phi}_n(z)\|^2_{\widehat{\mu}}, $$

then

(i) $1 + \|\Phi_n\|^2_{\mu} \leqslant \|\widehat{\Phi}_n\|^2_{\widehat{\mu}} \leqslant 1 + c_0$, *with* $c_0 = \int_0^{2\pi} \mathrm{d}\mu(\theta)$.
(ii) $1 + {}_0m \leqslant {}_0\widehat{m} \leqslant 1 + c_0$.
(iii) $\|\widehat{\Phi}_n\|^2_{\mu} \leqslant c_0$ *and* $\|\widehat{\Phi}_n\|^2_{\theta} \leqslant c_0 + 1\ \forall n \geqslant 0$.
(iv) $\sum_{j=0}^{n-1} \frac{|\widehat{\Phi}_n^{(j)}(0)|^2}{(j!)^2} \leqslant c_0$.

Proof

(i) If we apply the extremal property of the norm of the orthogonal polynomials we get

$$ \begin{aligned} 1 + \|\Phi_n\|^2_{\mu} &\leqslant \|z^n\|^2_{\theta} + \|\widehat{\Phi}_n\|^2_{\mu} \\ &\leqslant \|\widehat{\Phi}_n\|^2_{\theta} + \|\widehat{\Phi}_n\|^2_{\mu} \\ &= \|\widehat{\Phi}_n\|^2_{\widehat{\mu}} \\ &\leqslant \|z^n\|^2_{\widehat{\mu}} \\ &= c_0 + 1. \end{aligned} $$

(ii) It is straightforward from (i), taking limits when n tends to ∞.
(iii), (iv) From $\|\widehat{\Phi}_n\|^2_{\theta} + \|\widehat{\Phi}_n\|^2_{\mu} \leqslant c_0 + 1$ it follows that

$$ \|\widehat{\Phi}_n\|^2_{\mu} \leqslant \sum_{j=0}^{n-1} \frac{|\widehat{\Phi}_n^{(j)}(0)|^2}{(j!)^2} \leqslant c_0, $$

which yields the result. □

Corollary 1

(i) $\widehat{\mu}$ *belongs to the Szegő class* S.
(ii) *The absolutely continuous part of* $\widehat{\mu}$, $\widehat{\mu}'(\theta) = \mu'(\theta) + 1$ *satisfies*

$$ \frac{1}{\widehat{\mu}'(\theta)} \leqslant 1. $$

Proof

(i) Applying Szegő's theorem (see [9,10]) we know that

$$\widehat{\mu} \in S \text{ if and only if } {}_0\widehat{m} > 0.$$

From Theorem 2 we get that ${}_0\widehat{m} > 0$, and our statement follows.

(ii) Since

$$\frac{1}{\widehat{\mu}'(\theta)} = \frac{1}{\mu'(\theta) + 1} \leqslant 1,$$

we get the result. Therefore

$$\frac{1}{\widehat{\mu}'(\theta)} \in L^1[0, 2\pi]. \qquad \square$$

Some other interesting relations between the norms are given in the next theorem.

Theorem 3. *Let* $\{\Phi_n\}$ *be the* MOPS(μ) *and let* $\{\widehat{\Phi}_n\}$ *be the* MOPS$(\widehat{\mu})$. *Then*

(i) $\|\widehat{\Phi}_n\|^2_{\widehat{\mu}} = \|\Phi_n\|^2_{\mu} + 1 + \sum_{j=0}^{n-1} \frac{\widehat{\Phi}_n^{(i)}(0)\overline{\Phi_n^{(i)}(0)}}{(i!)^2}.$

(ii) $0 \leqslant 1 + \sum_{j=0}^{n-1} \frac{\widehat{\Phi}_n^{(i)}(0)\overline{\Phi_n^{(i)}(0)}}{(i!)^2}.$

(iii) $\|\widehat{\Phi}_n\|^2_{\theta} \leqslant \|\Phi_n\|^2_{\theta}.$

Proof

(i) It is straightforward from $\|\widehat{\Phi}_n\|^2_{\widehat{\mu}} = \langle\widehat{\Phi}_n, \Phi_n\rangle_{\mu} + \langle\widehat{\Phi}_n, \Phi_n\rangle_{\theta} = \|\Phi_n\|^2_{\mu} + \langle\widehat{\Phi}_n, \Phi_n\rangle_{\theta}$.

(ii) Since $\|\Phi_n\|^2_{\mu} \leqslant \|\widehat{\Phi}_n\|^2_{\widehat{\mu}}$, then $0 \leqslant \|\widehat{\Phi}_n\|^2_{\widehat{\mu}} - \|\Phi_n\|^2_{\mu} = \langle\widehat{\Phi}_n, \Phi_n\rangle_{\theta}$, which implies (ii).

(iii) Since $\|\widehat{\Phi}_n\|^2_{\widehat{\mu}} \leqslant \|\Phi_n\|^2_{\widehat{\mu}}$, then $\|\widehat{\Phi}_n\|^2_{\mu} + \|\widehat{\Phi}_n\|^2_{\theta} \leqslant \|\Phi_n\|^2_{\mu} + \|\Phi_n\|^2_{\theta}$, which implies $0 \leqslant \|\widehat{\Phi}_n\|^2_{\mu} - \|\Phi_n\|^2_{\mu} \leqslant \|\Phi_n\|^2_{\theta} - \|\widehat{\Phi}_n\|^2_{\theta}$. Therefore $\|\widehat{\Phi}_n\|^2_{\theta} \leqslant \|\Phi_n\|^2_{\theta}$, that can be written

$$\sum_{j=0}^{n-1} \frac{|\widehat{\Phi}_n^{(j)}(0)|^2}{(j!)^2} \leqslant \sum_{j=0}^{n-1} \frac{|\Phi_n^{(j)}(0)|^2}{(j!)^2}. \qquad \square$$

Finally, in the next theorem we obtain some relations betweeen the coefficients of the polynomials Φ_n and $\widehat{\Phi}_n$.

Theorem 4. *If $\{c_n\}$ is the sequence of moments of the measure μ, then*

(i) $$\|\widehat{\Phi}_n\|^2_{\widehat{\mu}} = 1 + c_0 + \overline{c_1}\frac{\widehat{\Phi}_n^{(n-1)}(0)}{(n-1)!} + \cdots + \overline{c_n}\widehat{\Phi}_n(0), \text{ and}$$

$$\|\Phi_n\|^2_{\mu} = c_0 + \overline{c_1}\frac{\Phi_n^{(n-1)}(0)}{(n-1)!} + \cdots + \overline{c_n}\Phi_n(0).$$

(ii) $$\sum_{j=1}^{n}\overline{c_j}\frac{\widehat{\Phi}_n^{(n-j)}(0)}{(n-j)!} \leqslant 0, \text{ and } \sum_{j=1}^{n}\overline{c_j}\frac{\Phi_n^{(n-j)}(0)}{(n-j)!} \leqslant 0.$$

(iii) $$\sum_{j=1}^{n}\overline{c_j}\left(\frac{\widehat{\Phi}_n^{(n-j)}(0)}{(n-j)!} - \frac{\Phi_n^{(n-j)}(0)}{(n-j)!}\right) = \sum_{j=0}^{n-1}\frac{\widehat{\Phi}_n^{(j)}(0)\overline{\Phi_n^{(j)}(0)}}{(j!)^2}.$$

(iv) $$\sum_{j=0}^{n-1}\frac{\widehat{\Phi}_n^{(j)}(0)}{j!}\left(\overline{c_{n-j}} - \frac{\overline{\widehat{\Phi}_n^{(j)}(0)}}{j!}\right) \leqslant 0.$$

Proof

(i) From $\|\widehat{\Phi}_n\|^2_{\widehat{\mu}} = \langle\widehat{\Phi}_n, z^n\rangle_\mu + \langle\widehat{\Phi}_n, z^n\rangle_\theta = \langle\widehat{\Phi}_n, z^n\rangle_\mu + 1$ and $\|\Phi_n\|^2_\mu = \langle\Phi_n, z^n\rangle_\mu$, then (i) follows.

(ii) It is straightforward taking into account (i) and using (i) in Theorem 2.

(iii) We obtain the result using $\|\widehat{\Phi}_n\|^2_{\widehat{\mu}} - \|\Phi_n\|^2_\mu = \langle\widehat{\Phi_n}, \Phi_n\rangle_\theta$ and (i).

(iv) Since

$$\|\widehat{\Phi}_n\|^2_{\widehat{\mu}} = \|\Phi_n\|^2_\mu + 1 + \sum_{i=i}^{n-1}\frac{|\widehat{\Phi}_n^{(i)}(0)|^2}{(i!)^2}$$

and

$$\|\widehat{\Phi}_n\|^2_\mu \leqslant c_0,$$

from (i) the result follows. □

4. Lebesgue perturbation of a Bernstein–Szegő measure

When we choose as the first measure a Bernstein–Szegő measure we obtain some additional properties reflecting the meaning of the perturbation by the Lebesgue measure.

Let

$$\mathrm{d}\mu(\theta) = \frac{\mathrm{d}\theta}{2\pi|A_k(\mathrm{e}^{\mathrm{i}\theta})|^2}$$

be a Bernstein–Szegő measure where $A_k(z)$ a polynomial with $\deg A_k = k$, $A_k(0) > 0$ and $A_k(z) \neq 0, \forall z, |z| \leqslant 1$. Let $\{\varphi_n(z)\}$ be the corresponding sequence of orthonormal polynomials. It is well-known that $\varphi_n(z) = z^{n-k} A_k^*(z), \forall n \geqslant k$.

Theorem 5. *If $A_k^*(z) = \sum_{j=0}^{k} a_j z^j$ with $a_k > 0$ and $a_0 \neq 0$, then for each $n \geqslant k$:*

(i) *There exist $\beta_{n,n-j}\,(j = 1, \dots, k)$ such that*

$$\varphi_n(z) = a_k \widehat{\Phi}_n(z) + \beta_{n,n-1}\widehat{\Phi}_{n-1}(z) + \cdots + \beta_{n,n-k}\widehat{\Phi}_{n-k}(z). \tag{7}$$

(ii) *The sequences $\{\beta_{n,n-j}\}_{n\geqslant k}\,(j = 1, \dots, k)$ are convergent and*

$$|\beta_{n,n-j} - a_{k-j}| \leqslant \sqrt{c_0} \sum_{l=0}^{j-1} |a_{k-l}|. \tag{8}$$

(iii) $\|\widehat{\Phi}_n - z^n\|_{\widehat{\mu}}^2 \leqslant c_0 - \frac{1}{a_k^2}$ *for $n \geqslant k$.*

(iv) ${}_0\widehat{m} \in \left[1 + \frac{1}{a_k^2}, 1 + c_0\right]$.

Proof

(i) If we write

$$\varphi_n(z) = \sum_{k=0}^{n} \beta_{n,k} \widehat{\Phi}_k(z), \tag{9}$$

and take into account that $\langle \varphi_n(z), z^j \rangle_{\widehat{\mu}} = 0$ for $0 \leqslant j \leqslant n-k-1$, then we deduce that

$$\langle \varphi_n(z), \widehat{\Phi}_j(z)\rangle_{\widehat{\mu}} = \beta_{n,j} \|\widehat{\Phi}_j\|_{\widehat{\mu}}^2 = 0,$$

for $j = 0, \dots, n-k-1$. This means that $\beta_{n,j} = 0$ for $j = 0, \dots, n-k-1$. Finally, identifying the leading coefficients in (9) we get (7).

(ii) For $n \geqslant k$ and $j = 1, \dots, k$ from (7) we deduce

$$\begin{aligned}\beta_{n,n-j} &= \frac{\langle \varphi_n(z), \widehat{\Phi}_{n-j}(z)\rangle_{\theta}}{\|\widehat{\Phi}_{n-j}\|_{\widehat{\mu}}^2} = \frac{\langle \varphi_n(z), \widehat{\Phi}_{n-j}(z)\rangle_{\widehat{\mu}}}{\|\widehat{\Phi}_{n-j}\|_{\widehat{\mu}}^2} \\ &= a_{k-j} \frac{\langle z^{n-j}, \widehat{\Phi}_{n-j}(z)\rangle_{\widehat{\mu}}}{\|\widehat{\Phi}_{n-j}\|_{\widehat{\mu}}^2} + \frac{\langle \sum_{l=0}^{j-1} a_{k-l} z^{n-l}, \widehat{\Phi}_{n-j}(z)\rangle_{\widehat{\mu}}}{\|\widehat{\Phi}_{n-j}\|_{\widehat{\mu}}^2}.\end{aligned}$$

Hence

$$\beta_{n,n-j} - a_{k-j} = \sum_{l=0}^{j-1} a_{k-l} \frac{\langle z^{n-l}, \widehat{\Phi}_{n-j}(z)\rangle_{\widehat{\mu}}}{\|\widehat{\Phi}_{n-j}\|_{\widehat{\mu}}^2}.$$

For $n \geqslant k$, $\|\Phi_n\|_\mu^2 = (1/a_k^2)$ and by Theorem 2 we get $1 + (1/a_k^2) \leqslant \|\widehat{\Phi}_n\|_{\widehat{\mu}}^2$ for $n \geqslant k$.

Therefore

$$\begin{aligned}\frac{|\langle a_{k-l}z^{n-l}, \widehat{\Phi}_{n-j}(z)\rangle_\mu|^2}{\|\widehat{\Phi}_{n-j}\|_{\widehat{\mu}}^4} &\leqslant \frac{|a_{k-l}|^2\|z^{n-l}\|_\mu^2\|\widehat{\Phi}_{n-j}\|_\mu^2}{\|\widehat{\Phi}_{n-j}\|_{\widehat{\mu}}^4} \leqslant \frac{|a_{k-l}|^2\|z^{n-l}\|_\mu^2}{\|\widehat{\Phi}_{n-j}\|_{\widehat{\mu}}^2}\\ &\leqslant \frac{|a_{k-l}|^2 c_0}{1+\|\Phi_{n-j}\|_\mu^2} \leqslant \frac{|a_{k-l}|^2 c_0}{1+\|\Phi_n\|_\mu^2} = \frac{|a_{k-l}|^2 c_0}{1+\frac{1}{a_k^2}}\\ &\leqslant |a_{k-l}|c_0.\end{aligned}$$

Hence we get (8).

Again, from (7), if we identify coefficients we get

$$a_{k-1} = a_k \frac{\widehat{\Phi}_n^{(n-1)}(0)}{(n-1)!} + \beta_{n,n-1}.$$

If we take limits in the above expression and since $\widehat{\mu}$ belongs to the Szegö class, which implies that $\{(\widehat{\Phi}_n^{(n-1)}(0))/(n-1)!\}$ converges, then we obtain that $\{\beta_{n,n-1}\}$ converges. Proceeding in the same way, for each $j = 1, \ldots, k$ we deduce that $\{\beta_{n,n-j}\}$ converges.

(iii) $\|\widehat{\Phi}_n - z^n\|_{\widehat{\mu}}^2 = \|z^n\|_{\widehat{\mu}}^2 - \|\widehat{\Phi}_n\|_{\widehat{\mu}}^2 = c_0 + 1 - \|\widehat{\Phi}_n\|_{\widehat{\mu}}^2 \leqslant c_0 - \|\Phi_n\|_\mu^2$
$= c_0 - \dfrac{1}{a_k^2}$ for $n \geqslant k$.

(iv) Since ${}_0m = \lim_{n\to\infty}\|\Phi_n\|_\mu^2 = (1/a_k^2)$, from (ii) in Theorem 2 the result follows. □

Corollary 2. *For the measure $\widehat{\mu}$ the Szegö function is a rational transformation of the Szegö function for the measure μ.*

Proof. If we apply the $*^n$ operator (see [4]) in relation (7) we get

$$\overline{a_k} + \cdots + \overline{a_0}z^k = \overline{a_k}\widehat{\Phi}_n^*(z) + \beta_{n,n-1}z\widehat{\Phi}_{n-1}^*(z) + \cdots + \beta_{n,n-k}z^k\widehat{\Phi}_{n-k}^*(z). \tag{10}$$

Since $\Phi_n^*(z) = (\overline{a_0}z^k + \cdots + \overline{a_k})/\overline{a_k}$, the normalized Szegö function of measure μ (see [5]) is

$$\frac{\Pi(z)}{\Pi(0)} = \lim_{n\to\infty}\Phi_n^*(z) = \frac{\overline{a_0}z^k + \cdots + \overline{a_k}}{\overline{a_k}}$$

uniformly on compact subsets of $|z| < 1$.

If we denote $(\widehat{\Pi}(z)/\widehat{\Pi}(0))$ the normalized Szegő function of measure $\widehat{\mu}$, then

$$\frac{\widehat{\Pi}(z)}{\widehat{\Pi}(0)} = \lim_{n\to\infty} \widehat{\Phi}_n^*(z)$$

uniformly on compact subsets of $|z| < 1$. But if $B_k = \lim_{n\to\infty} \beta_{n,n-k}$ and taking limits in (10) we get

$$\frac{\widehat{\Pi}(z)}{\widehat{\Pi}(0)} = \frac{\overline{a_k}}{\overline{a_k} + B_1 z + \cdots + B_k z^k} \frac{\Pi(z)}{\Pi(0)} = \frac{\overline{a_0} z^k + \cdots + \overline{a_k}}{B_k z^k + \cdots + B_1 z + \overline{a_k}}. \qquad \square$$

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