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Matrix orthogonal Laurent polynomials on the unit circle and Toda type integrable systems

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Abstract

Matrix orthogonal Laurent polynomials in the unit circle and the theory of Toda-like integrable systems are connected us-ing the Gauss–Borel factorization of two, left and a right, Cantero–Morales–Velázquez block moment matrices, which are constructed using a quasi-definite matrix measure. A b l o c k Gauss–Borel factorization problem of these moment matrices leads to two sets of biorthogonal matrix orthogonal Laurent polynomials and matrix Szegő polynomials, which can be expressed in terms of Schur complements of bordered trun-cations of the block moment matrix. The corresponding block extension of the Christoffel–Darboux theory is derived. De-formations of the quasi-definite matrix measure leading to integrable systems of Toda type are studied. The integrable theory is given in this matrix scenario; wave and adjoint wave functions, Lax and Zakharov–Shabat equations, bilinear equa-tions and discrete flows — connected with Darboux transfor-mations. We generalize the integrable flows of the Cafasso's matrix extension of the Toeplitz lattice for the Verblunsky coefficients of Szegő polynomials. An analysis of the Miwa shifts allows for the finding of interesting connections between Christoffel–Darboux kernels and Miwa shifts of the matrix or-thogonal Laurent polynomials.

Keywords	Matrix orthogonal Laurent polynomials
	Borel–Gauss factorization
	Christoffel–Darboux kernels
	Toda type integrable hierarchies

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1. Introduction

In this paper we extend previous results on orthogonal Laurent polynomials in the unit circle (OLPUC) [13] to the matrix realm (MOLPUC). To explain better our aims and results we need a brief account on orthogonal polynomials, Laurent orthogonal polynomials and their matrix extensions, and also some facts about integrable systems.

1.1. Historical background

1.1.1. Szegő polynomials

We will denote the unit circle by $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ stands for the unit disk; when $z \in \mathbb{T}$ we will use the parametrization $z = \mathrm{e}^{\mathrm{i}\theta}$ with $\theta \in [0, 2\pi)$. In the scalar case, one deals with a complex Borel measure μ supported in $\mathbb{T}$ that is said to be positive definite if it maps measurable sets onto non-negative numbers, that in the absolutely continuous situation (with respect to the Lebesgue measure $\mathrm{d}\theta$) has the form $w(\theta)\mathrm{d}\theta$. For the positive definite situation the orthogonal polynomials in the unit circle (OPUC) or Szegő polynomials are defined as those monic polynomials P_n of degree n that satisfy the following system of equations, called orthogonality relations: $\int_{\mathbb{T}} P_n(z) z^{-k} \mathrm{d}\mu(z) = 0$, for $k = 0, 1, \ldots, n-1$ [90]. The connections between orthogonal polynomials on the real line (OPRL) supported in the interval $[-1, 1]$ and OPUC have been explored in the literature, see for example [53,22]. Let us observe that

for this analysis the use of spectral theory techniques requires the study of the operator of multiplication by z. Recursion relations for OPRL and OPUC are well known; however, in the real case, the three term recurrence laws provide a tridiagonal matrix, the so-called Jacobi operator, while in the unit circle support case, the problem leads to a Hessenberg matrix [61], being a more involved scenario that the Jacobi one (as it is not a sparse matrix with a finite number of non-vanishing diagonals). In fact, OPUCs recursion relation requires the introduction of reciprocal or reverse Szegő polynomials $P_l^*(z) := z^l \overline{P_l(\bar{z}^{-1})}$ and the reflection or Verblunsky (Schur parameters is another usual name) coefficients $\alpha_l := P_l(0)$. The recursion relations for the Szegő polynomials can be written as $\begin{pmatrix} P_l \\ P_l^* \end{pmatrix} = \begin{pmatrix} z & \alpha_l \\ z\bar{\alpha}_l & 1 \end{pmatrix} \begin{pmatrix} P_{l-1} \\ P_{l-1}^* \end{pmatrix}$. There exist numerous studies on the zeroes of the OPUC [10,16,19,54,58,60,74,80] with interesting applications to signal analysis theory [63,65,82,83]. Despite the mentioned advances for the OPUC theory, the corresponding state of the art in the OPRL context is still much more developed. An issue to stress here is that Szegő polynomials are, in general, not a dense set in the Hilbert space $L^2(\mathbb{T}, \mu)$; Szegő's theorem implies for a non-trivial probability measure $d\mu$ on $\mathbb{T}$ with Verblunsky coefficients $\{\alpha_n\}_{n=0}^\infty$ that the corresponding Szegő's polynomials are dense in $L^2(\mathbb{T}, \mu)$ if and only if $\prod_{n=0}^\infty (1 - |\alpha_n|^2) = 0$. For an absolutely continuous probability measure Kolmogorov's density theorem ensures that density in $L^2(\mathbb{T}, \mu)$ of the OPUC holds iff the so-called Szegő's condition $\int_\mathbb{T} \log(w(\theta)) d\theta = -\infty$ is fulfilled [89]. We refer the reader to Barry Simon's books [85] and [86] for a very detailed studied of OPUC.

1.1.2. Orthogonal Laurent polynomials

Orthogonal Laurent polynomials on the real line (OLPRL) were introduced in [66, 67] in the context of the strong Stieltjes moment problem. When this moment problem has a solution, there exist polynomials $\{Q_n\}$, known as Laurent polynomials, such that $\int_\mathbb{R} x^{-n+j} Q_n(x) d\mu(x) = 0$ for $j = 0, \ldots, n-1$. The theory of Laurent polynomials on the real line was developed in parallel with the theory of orthogonal polynomials, see [33,43,64] and [81]. The theory of orthogonal Laurent polynomials was carried from the real line to the circle [91] and subsequent works broadened the matter (e.g. [37,29,35,36]), treating subjects like recursion relations, Favard's theorem, quadrature problems, and Christoffel–Darboux formulae. The Cantero–Moral–Velázquez (CMV) [29] representation is a hallmark in the study of certain aspects of Szegő polynomials, as we mentioned already while the OLPUC are always dense in $L^2(\mathbb{T}, \mu)$, this is not true in general for the OPUC [25,37]. The bijection between OLPUC in the CMV representation and the ordinary Szegő polynomials implies the replacement of complicated recursion relations with five term relations similar to the OPRL situation. Other papers have reviewed and broadened the study of CMV matrices, see for example [87,68]; in particular alternative or generic orders in the base used to span the space of OLPUC can be found in [36]. In particular, the reading of Simon's account of the CMV theory [87] is illuminating. In fact, the discovery of the advantages of the CMV ordering goes back to the previous work [93].

1.1.3. Matrix orthogonal polynomials

Orthogonal polynomials with matrix coefficients on the real line were considered in detail by Krein [69,70] in 1949, and thereafter were studied sporadically until the last decade of the XX-th century. Some relevant papers on this subject are [20,56,17]; in particular, in [17] the scattering problem is solved for a kind of discrete Sturm–Liouville operators that are equivalent to the recursion equation for scalar orthogonal polynomials. They found that polynomials that satisfy a relation of the form

$$xP_k(x) = A_k P_{k+1}(x) + B_k P_k(x) + A_{k-1}^* P_{k-1}(x), \quad k = 0, 1, \dots,$$

are orthogonal with respect to a positive definite measure. This is a matrix version of Favard's theorem for scalar orthogonal polynomials. Then, in the 1990s and the 2000s some authors found that matrix orthogonal polynomials (MOPs) satisfy in certain cases some properties that satisfy scalar-valued orthogonal polynomials; for example, Laguerre, Hermite and Jacobi polynomials, i.e., the scalar-type Rodrigues' formula [47,48,34] and a second order differential equation [44,46,24]. Later on, it has been proven [45] that operators of the form $D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0$ have as eigen-functions different infinite families of MOPs. Moreover, in [24] a new family of MOPs satisfying second order differential equations, whose coefficients do not behave asymptotically as the identity matrix, was found; see also [30]. In [31] the Riemann–Hilbert problem for this matrix situation and the appearance of non-Abelian discrete versions of Painlevé I were explored, showing singularity confinement — see [32]; for Riemann–Hilbert problems see also [62]. Let us mention that in [75,76] and [27] the MOPs are expressed in terms of Schur complements that play the role of determinants in the standard scalar case. For a survey on matrix orthogonal polynomials, we refer the reader to [39].

1.1.4. Integrable hierarchies and the Gauss–Borel factorization

The seminal paper of M. Sato [84] and further developments performed by the Kyoto school [40–42] settled the Lie-group theoretical description of the integrable hierarchies. It was Mulase [78] the one who made the connection between factorization problems, dressing procedures and integrability. In this context, Ueno and Takasaki [92] performed an analysis of the Toda type hierarchies and their soliton-like solutions. Adler and van Moerbeke [4–8,3,9] have clarified the connection between the Lie-group factorization, applied to Toda type hierarchies — what they call discrete Kadomtsev–Petviashvilii (KP) — and the Gauss–Borel factorization applied to a moment matrix that comes from orthogonality problems; thus, the corresponding orthogonal polynomials are closely related to specific solutions of the integrable hierarchy. See [21,52,71,11] for further developments in relation with the factorization problem, multicomponent Toda lattices and generalized orthogonality. In [3] a profound study of the OPUC and the Toda type associated lattice, called the Toeplitz lattice (TL), was performed. A relevant reduction of the equations of the TL has been found by Golinskii [59] in the context of Schur flows when the measure is invariant under conjugation (also studied in [88] and [49]), another interesting paper

on this subject is [77]. The Toeplitz lattice was proven to be equivalent to the Ablowitz–Ladik lattice (ALL) [1,2], and that work has been generalized to the link between matrix orthogonal polynomials and the non-Abelian ALL in [27]. Both of them have to deal with the Hessenberg operator for the multiplication by z. Research about the integrable structure of Schur flows and its connection with ALL has been done (in recent and not so recent works) from a Hamiltonian point of view in [79], and other works also introduce connections with Laurent polynomials and τ-functions, like [50,51,23].

1.2. Preliminary material

1.2.1. Semi-infinite block matrices

For the matrix extension considered in the present work we need to deal with block matrices and block Gauss–Borel factorizations. For each $m \in \mathbb{N}$, the directed set of natural numbers, we consider ring of the complex $m \times m$ matrices $\mathbb{M}_m := \mathbb{C}^{m \times m}$, and its direct limit $\mathbb{M}_\infty := \varinjlim \mathbb{M}_m$, the ring of semi-infinite complex matrices. We will denote by $\mathrm{diag}_m \subset \mathbb{M}_m$ the set of diagonal matrices. For any $A \in \mathbb{M}_\infty$, $A_{ij} \in \mathbb{C}$ denotes the (i,j)-th element of A, while $(A)_{ij} \in \mathbb{M}_m$ denotes the (i,j)-th block of it when subdivided into $m \times m$ blocks. We will denote by G_∞ the group of invertible semi-infinite matrices of $\mathbb{M}_\infty$. In this paper two important subgroups are $\mathscr{U}$, the invertible upper triangular — by blocks — matrices, and $\mathscr{L}$, the lower triangular — by blocks — matrices with the identity matrix along their block diagonal. The corresponding restriction on invertible upper triangular block matrices is denoted by $\widehat{\mathscr{U}}$. Block diagonal matrices will be denoted by $\mathscr{D} = \{D \in \mathbb{M}_\infty : (D)_{i,j} = d_i \cdot \delta_{i,j} \text{ with } d_i \in \mathbb{M}_m\}$. Given a semi-infinite matrix $A \in \mathbb{M}_\infty$, we consider its l-th block leading submatrix

$$A^{[l]} = \begin{pmatrix} (A)_{0,0} & (A)_{0,1} & \ldots & (A)_{0,l-1} \\ (A)_{1,0} & (A)_{1,1} & \ldots & (A)_{1,l-1} \\ \vdots & & & \vdots \\ (A)_{l-1,0} & (A)_{l-1,1} & \ldots & (A)_{l-1,l-1} \end{pmatrix} \in \mathbb{M}_{ml}, \quad (A)_{i,j} \in \mathbb{M}_m,$$

and we write

$$A = \left(\begin{array}{c|c} A^{[l]} & A^{[l,\geq l]} \\ \hline A^{[\geq l,l]} & A^{[\geq l]} \end{array} \right), \tag{1}$$

for the corresponding block partition of a matrix A where, for example, $A^{[l,\geq l]}$ denotes all the $(A)_{i,j}$-th blocks of the matrix A with $i < l, j \geq l$. Very much related to the block partition of a matrix M are the Schur complements. The Schur complement with respect to the upper left block of the block partition

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{M}_{p+q}, \quad A = (a_{i,j}) \in \mathbb{M}_p, \ D \in \mathbb{M}_q,$$

is

$$M \diagup A := \mathrm{SC} \left(\begin{array}{c|c} (a_{i,j}) & B \\ \hline C & D \end{array} \right) := D - CA^{-1}D,$$

where we have assumed that A is an invertible matrix.

1.2.2. Quasi-definiteness

Let us recall the reader that measures and linear functionals are closely connected; given a linear functional $\mathcal{L}$ on $\varLambda_{[\infty]}$, the set of Laurent polynomials on the circle — or polynomial loops $L_{\mathrm{pol}}\mathbb{C}$, we define the corresponding moments of $\mathcal{L}$ as $c_n := \mathcal{L}[z^n]$ for all the possible integer values of $n \in \mathbb{Z}$. The functional $\mathcal{L}$ is said to be Hermitian whenever $c_{-n} = \overline{c_n}$, $\forall n \in \mathbb{Z}$. Moreover, the functional $\mathcal{L}$ is defined as quasi-definite (positive definite) when the principal submatrices of the Toeplitz moment matrix $(\Delta_{i,j})$, $\Delta_{i,j} := c_{i-j}$, associated to the sequence c_n, are non-singular (positive definite), i.e., $\forall n \in \mathbb{Z}$, $\Delta_n := \det(c_{i-j})_{i,j=0}^n \neq 0 \ (> 0)$. Some aspects on quasi-definite functionals and their perturbations are studied in [15,26]. It is known [57] that when the linear functional $\mathcal{L}$ is Hermitian and positive definite, there exists a finite positive Borel measure with a support lying on $\mathbb{T}$ such that $\mathcal{L}[f] = \int_{\mathbb{T}} f \mathrm{d}\mu$, $\forall f \in \varLambda_{[\infty]}$. In addition, a Hermitian positive definite linear functional $\mathcal{L}$ defines a sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{L}} : \varLambda_{[\infty]} \times \varLambda_{[\infty]} \mapsto \mathbb{C}$ as $\langle f, g \rangle_{\mathcal{L}} = \mathcal{L}[f\bar{g}]$, $\forall f, g \in \varLambda_{[\infty]}$. Two Laurent polynomials $\{f, g\} \subset \varLambda_{[\infty]}$ are said to be orthogonal with respect to $\mathcal{L}$ if $\langle f, g \rangle_{\mathcal{L}} = 0$. From the properties of $\mathcal{L}$ it is easy to see that $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is a scalar product and if μ is the positive finite Borel measure associated to $\mathscr{L}$ we are led to the corresponding Hilbert space $L^2(\mathbb{T}, \mu)$, the closure of $\varLambda_{[\infty]}$. The more general setting, when $\mathcal{L}$ is just quasi-definite is associated to a corresponding quasi-definite complex measure μ, see [55]. As before, a sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is defined for any such linear functional $\mathcal{L}$; thus, we just have the linearity (in the first entry) and skew-linearity (in the second entry) properties. However, we have no symmetry allowing the interchange of the two arguments. We formally broaden the notion of orthogonality and say that f is orthogonal to g if $\langle f, g \rangle_{\mathcal{L}} = 0$, but we must be careful as in this general situation it could happen that $\langle f, g \rangle_{\mathcal{L}} = 0$ but $\langle g, f \rangle_{\mathcal{L}} \neq 0$.

1.2.3. Matrix Laurent polynomials and orthogonality

A matrix-valued measure $\mu = (\mu_{i,j})$ supported on $\mathbb{T}$ is said to be Hermitian and/or positive definite, if for every Borel subset $\mathscr{B}$ of $\mathbb{T}$, the matrix $\mu(\mathscr{B})$ is a Hermitian and/or positive definite matrix. When the scalar measures $\mu_{i,j}$, $i, j = 1, \ldots, m$, are absolutely continuous with respect to the Lebesgue measure on the circle $\mathrm{d}\theta$, according to the Radon–Nikodym theorem, it can be always expressed using complex weight (density or Radon–Nikodym derivative of the measure) functions $w_{i,j}$, $i, j = 1, \ldots, m$, so that $\mathrm{d}\mu_{i,j}(\theta) = w_{i,j}(\theta)\mathrm{d}\theta$, $\theta \in [0, 2\pi)$. If, in addition, the matrix measure μ is Hermitian and positive definite, then the matrix $(w_{i,j}(\theta))$ is a positive definite Hermitian matrix. For the sake of notational simplicity we will use, whenever it is convenient, the complex notation $\mathrm{d}\mu(z) = \mathrm{i}e^{\mathrm{i}\theta}\mathrm{d}\mu(\theta)$.

The moments of the matrix measure μ are

$$c_n := \frac{1}{2\pi} \oint_{\mathbb{T}} z^{-n} \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\theta} \mathrm{d}\mu(\theta) \in \mathbb{M}_m,$$

while the Fourier series of the measure is

$$F_\mu(u) := \sum_{n=-\infty}^{\infty} c_n u^n, \tag{2}$$

that for absolutely continuous measures, $\mathrm{d}\mu(\theta) = w(\theta)\mathrm{d}\theta$ satisfies $F_\mu(\theta) = w(\theta)$. Let $D(0; r, R) = \{z \in \mathbb{C} : r < |z| < R\}$ denote the annulus around $z = 0$ with interior and exterior radii r and R, $R_{ij,\pm} := (\limsup_{n\to\infty} \sqrt[n]{|c_{ij,\pm n}|})^{\mp 1}$ and $R_+ = \min_{i,j=1,\dots,m} R_{ij,+}$ and $R_{ij,-} = \max_{i,j=1,\dots,m} R_{ij,-}$. Then, according to the Cauchy–Hadamard theorem, the series $F_\mu(z)$ converges uniformly in any compact set K, $K \subset D(0; R_-, R_+)$.

The space $\Lambda_{m,[p,q]} := \mathbb{M}_m\{\mathbb{I}z^{-p}, \mathbb{I}z^{-p+1}, \dots, \mathbb{I}z^q\}$ (where $\mathbb{I} \in \mathbb{M}_m$ is the identity matrix) of complex Laurent polynomials with $m \times m$ matrix coefficients and the corresponding restrictions on their degrees is an $\mathbb{M}_m$ free module of rank $p + q + 1$. We denote by $L_{\mathrm{pol}}\mathbb{M}_m$ the infinite set of Laurent matrix polynomials or polynomial loops in $\mathbb{M}_m$.

Given a matrix measure μ, we introduce the following left and right matrix-valued sesquilinear forms in the loop space $L\mathbb{M}_m$ considered as left and right modules for the ring $\mathbb{M}_m$, respectively,

$$\langle\!\langle f, g \rangle\!\rangle_L := \oint_{\mathbb{T}} g(z) \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} f(z)^\dagger \in \mathbb{M}_m, \tag{3}$$

$$\langle\!\langle f, g \rangle\!\rangle_R := \oint_{\mathbb{T}} f(z)^\dagger \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} g(z) \in \mathbb{M}_m. \tag{4}$$

The sesquilinearity of these forms means that the following two properties hold:

(1) $\langle\!\langle f_1 + f_2, g \rangle\!\rangle_H = \langle\!\langle f_1, g \rangle\!\rangle_H + \langle\!\langle f_2, g \rangle\!\rangle_H$ and $\langle\!\langle f, g_1 + g_2 \rangle\!\rangle_H = \langle\!\langle f, g_1 \rangle\!\rangle_H + \langle\!\langle f, g_2 \rangle\!\rangle_H$ for all $f, f_1, f_2, g, g_1, g_2 \in L\mathbb{M}_m$ and $H = L, R$.
(2) $\langle\!\langle mf, g \rangle\!\rangle_L = \langle\!\langle f, g \rangle\!\rangle_L m^\dagger$, $\langle\!\langle f, mg \rangle\!\rangle_L = m\langle\!\langle f, g \rangle\!\rangle_L$, $\langle\!\langle fm, g \rangle\!\rangle_R = m^\dagger\langle\!\langle f, g \rangle\!\rangle_R$ and $\langle\!\langle f, gm \rangle\!\rangle_R = \langle\!\langle f, g \rangle\!\rangle_R m$, for all $f, g \in L\mathbb{M}_m$ and $m \in \mathbb{M}_m$.

Moreover, if the matrix measure is Hermitian, then so are these forms; i.e.,

$$\langle\!\langle f, g \rangle\!\rangle_H^\dagger = \langle\!\langle g, f \rangle\!\rangle_H, \quad H = L, R.$$

Actually, from these sesquilinear forms, for a positive definite Hermitian measure, we can derive the corresponding scalar products

$$\langle f, g\rangle_H^\dagger = \langle f, g\rangle_H := \mathrm{Tr}\big[\langle\!\langle f, g\rangle\!\rangle_H\big], \qquad \|f\|_H^2 = \langle f, f\rangle_H, \quad H = L, R,$$

and corresponding Hilbert spaces $\mathcal{H}^H$ with a norm — of Frobenius type — given by

$$\|f\|_H = +\sqrt{\langle f, f\rangle_H}, \quad H = L, R.$$

A set $\{p_l\}_{l=0}^N \subset \mathcal{H}^H$, $H = L, R$, is an orthogonal set if and only if

$$\langle\!\langle p_l^H, p_j^H\rangle\!\rangle_K = \delta_{ij} h_j, \quad h_j \in \mathbb{M}_m.$$

1.3. On the content of the paper

In previous papers we have approached the study of the link between orthogonality and integrability within an algebraic/group theoretical point of view. Our keystone relies on the fact that a number of facets of orthogonality and integrability can be described with the aid of the Gauss–Borel factorization of an infinite matrix. This approach was applied in [12] for the analysis of multiple orthogonal polynomials of mixed type, allowing for an algebraic proof of the Christoffel–Darboux formula, alternative to the analytic one, based on the Riemann–Hilbert problem (and constrained therefore by convenient analytic conditions) given in [38]. This approach was also used successfully in [13], where a CMV ordering of the Fourier basis gave, for a given measure on the unitary circle, a moment matrix whose Gauss–Borel factorization leads to OLPUC. Recursion relations and Christoffel–Darboux formula appeared also in a straightforward manner. Also continuous and discrete deformations and τ-function theory were extended to the circular case under the suitable choice of moment matrices and shift operators. In this last paper we only requested to the measure to be quasi-definite, condition that implies the existence of the Gauss–Borel factorization. Let us mention that we have applied this method in the finding of Christoffel–Darboux type formulae in other situations, see [18,14].

In this paper we consider two semi-infinite block matrices, whose coefficients (matrices in $\mathbb{C}^{m\times m}$) are left and right matrix moments, ordered in a Cantero–Morales–Velázquez style, of a matrix measure on the circle. The corresponding block Gauss–Borel factorization of these CMV block moment matrices leads to MOLPUC. To be more precise, we get the right and left versions of two biorthogonal families of matrix Laurent polynomials and corresponding Szegő polynomials. When the matrix measure is Hermitian, these two families happen to be proportional resulting in two families of MOLPUC. Following [75,27] we express them as Schur complements of bordered truncated moment matrices. We also prove, in an algebraic manner using the Gauss–Borel factorization, the five term recursion relations and the Christoffel–Darboux formula. Let us stress that in this paper we introduce an intertwining operator η not used in [13] that clarifies the appearance of reciprocal polynomials and simplifies the algebraic proofs. The recursion relations indicate which deformations of the quasi-definite matrix measure lead to integrable systems of Toda type. Thus, we discuss the following elements: wave and adjoint wave functions,

Lax and Zakharov–Shabat equations, bilinear equations and discrete flows — connected with Darboux transformations. In this context we find a generalization of the matrix Cafasso's extension of the Toeplitz lattice for the Verblunsky coefficients of Szegő polynomials. The Cafasso flows correspond to what we call total flows, which are only a part of the integrable flows associated to MOLPUC. We unsuccessfully tried to get a matrix τ theory, but despite this failure, we get interesting byproducts. We analyze the role of Miwa shifts in this context and, as a collateral effect, nicely connect them with the Christoffel–Darboux kernels. These formulae suggest a link of these kernels with the Cauchy propagators that in the Grassmannian $\bar{\partial}$ approach to multicomponent KP hierarchy was used in [72,73]. This identification allows us to give in Theorem 6 expressions of the MOLPUC in terms of products of their Miwa shifted and non-shifted quasi-norms. Despite that these expressions lead to the τ-function representation in the scalar case, this is not the case within the matrix context.

Let us mention that the submodules of matrix Laurent polynomials considered in this paper have the higher and lower powers constrained to be of some particular form, implied by the chosen CMV ordering. In [13] this limitation was overcome by the introduction of extended CMV orderings of the Fourier basis, which allowed for general subspaces of Laurent polynomials. A similar procedure can be performed in this matrix situation; but, as its development follows very closely the ideas of [13], we prefer to avoid its inclusion here.

The layout of this paper is as follows. Section 2 is devoted to orthogonality theory, in particular in Section 2.1 we consider the left and right block CMV moment matrices and perform corresponding block Gaussian factorizations in Section 2.2, getting the associated families of right and left MOLPUC and matrix Szegő polynomials and their biorthogonality relations. We also get the recursion relations and Schur complement expressions of them in terms of bordered truncations of the moment matrices. Then, in Section 2.3 we introduce the matrix second kind functions that are connected with the Fourier series of the measure and that will be relevant later on for the adjoint Baker functions. The reconstruction of the recursion relations from the Gauss–Borel factorizations is performed in Section 2.4; the Christoffel–Darboux formulae for this non-Abelian scenario are given in Section 2.5. Observe that in this case, the projection operators are projectors in a module over the ring $\mathbb{C}^{m \times m}$, that in the Hermitian definite positive situation lead to orthogonal projections in the standard geometrical sense. The integrability aspects are treated in Section 3. Given adequate deformations of the moment matrices, we find wave functions, Lax equations and Zakharov–Shabat equations in Section 3.1; here we also consider a generalization of the Cafasso's Toeplitz lattice and the bilinear equations formulation of the hierarchy. Finally, we extend to this matrix context the discrete flows for the Toeplitz lattice, intimately related to Darboux transformations in Section 3.2 and also derive the bilinear equations fulfilled by the MOLPUC in Section 3.1.3. Finally, in Section 3.3 we consider the action of Miwa transformations and get the previously mentioned results. We conclude the paper with a series of appendices that serve as support of certain sections.

Finally, let us stress that this paper is not just an extension of the results of [13] to the matrix realm but we also have introduced important elements not discussed there, which also hold in that scalar case, as the η operator, a different proof of the Christoffel–Darboux formula with no need of associated polynomials and new relations between Christoffel–Darboux kernels and Miwa shifted MOLPUC.

2. Matrix orthogonality and block Gauss–Borel factorization

In this section, inspired by the CMV construction [29] and the previous work [13], for a given matrix measure, we introduce an appropriate block moment matrix that, when factorized as a product of lower and upper block matrices, gives a set of biorthogonal matrix Laurent polynomials on the unit circle. This Borel–Gauss factorization problem also allows us to derive the recursion relations and the Christoffel–Darboux theory.

2.1. The CMV right and left moment matrices for quasi-definite matrix measures

The following $m \times m$ matrix-valued vectors will be relevant in the construction of biorthogonal families of MOLPUC

Definition 1. The CMV vectors are given by

$$
\chi_1(z) := \big(\mathbb{I}, 0, \mathbb{I}z, 0, \mathbb{I}z^2, \ldots\big)^\top,
$$

$$
\chi_2(z) := \big(0, \mathbb{I}, 0, \mathbb{I}z, 0, \mathbb{I}z^2, \ldots\big)^\top,
$$

$$
\chi_a^*(z) := z^{-1}\chi_a\big(z^{-1}\big), \quad a = 1, 2,
$$

$$
\chi(z) := \chi_1(z) + \chi_2^*(z) = \big(\mathbb{I}, \mathbb{I}z^{-1}, \mathbb{I}z, \mathbb{I}z^{-2}, \mathbb{I}z^2, \ldots\big)^\top.
$$

In the sequel, the matrix $\chi^{(l)}$ will denote the l-th component of the matrix vector χ

$$
\chi = \big(\chi^{(0)}, \chi^{(1)}, \chi^{(2)}, \ldots\big)^\top.
$$

Definition 2. The CMV left and right moment matrices of the measure μ are

$$
g^L := \oint_{\mathbb{T}} \chi(z) \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} \big(\chi(z)\big)^\dagger = 2\pi \begin{pmatrix} c_0 & c_{-1} & c_1 & c_{-2} & \ldots \\ c_1 & c_0 & c_2 & c_{-1} & \ldots \\ c_{-1} & c_{-2} & c_0 & c_{-3} & \ldots \\ c_2 & c_1 & c_3 & c_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5}
$$

$$g^R := \oint_{\mathbb{T}} \left(\chi(z)^\top \right)^\dagger \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} \chi(z)^\top = 2\pi \begin{pmatrix} c_0 & c_1 & c_{-1} & c_2 & \cdots \\ c_{-1} & c_0 & c_{-2} & c_1 & \cdots \\ c_1 & c_2 & c_0 & c_3 & \\ c_{-2} & c_{-1} & c_{-3} & c_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6}$$

Notice that when $\mathrm{d}\mu(\theta)$ is Hermitian, so are the moment matrices g^L and g^R.

In the scalar case [13], the only requirement that the moment matrix needs to meet is to be Gaussian factorable; i.e., all the principal minors of the matrix are requested to be not degenerated. The measure from which this moment matrix is constructed receives the name of quasi-definite measure. This condition is related to the existence of *biorthogonal* polynomials of all degrees — also called non-triviality of the measure. In the matrix case, the requirement is a bit more relaxed.

Definition 3. The matrix measure μ is said to be *quasi-definite* if its truncated moment matrices satisfy

$$\det\left(\left(g^H \right)^{[l]} \right) \neq 0 \quad \text{for } H = R, L \text{ and } l = 1, 2, 3, \ldots.$$

Notice that $(g^H)^{[l]} \in \mathbb{M}_{ml}$; a quite different situation from the scalar case in which all the principal minors had to be non-degenerate, while in the matrix case only the ml-order principal minors should meet this requirement. Actually, this is the only restriction — besides having compact support on $\mathbb{T}$ — that from hereon the matrix measures must satisfy, since when this condition holds

Proposition 1. *The moment matrices g^H, $H = L, R$, of a matrix quasi-definite measure μ admit a block Gauss–Borel factorization.*

Proof. See Appendix A. □

2.1.1. The generalized matrix Szegő polynomials
Definition 4. Given a matrix quasi-definite measure μ, the set of monic matrix polynomials $\{P_{i,l}^L\}_{l=0}^\infty$, $\{P_{i,l}^R\}_{l=0}^\infty$, $i = 1, 2$, with $\deg P_{i,l}^H = l$, $H = L, R$, satisfying

$$\left\langle\!\left\langle z^j \mathbb{I}, P_{1,l}^L(z) \right\rangle\!\right\rangle_L = \oint_{\mathbb{T}} P_{1,l}^L(z) \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} z^{-j} = 0, \quad j = 0, \ldots, l-1,$$

$$\left\langle\!\left\langle P_{2,l}^L(z), z^j \mathbb{I} \right\rangle\!\right\rangle_L = \oint_{\mathbb{T}} z^j \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} \left[P_{2,l}^L(z) \right]^\dagger = 0, \quad j = 0, \ldots, l-1,$$

$$\left\langle\!\left\langle P_{2,l}^R(z), z^j \mathbb{I} \right\rangle\!\right\rangle_R = \oint_{\mathbb{T}} \left[P_{2,l}^R(z) \right]^\dagger \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} z^j = 0, \quad j = 0, \ldots, l-1,$$

$$\left\langle\!\left\langle z^j \mathbb{I}, P_{1,l}^R(z) \right\rangle\!\right\rangle_R = \oint_{\mathbb{T}} z^{-j} \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} P_{1,l}^R(z) = 0, \quad j = 0, \ldots, l-1,$$

are said to be Szegő polynomials.

Proposition 2. *The matrix Szegő polynomials introduced in* Definition 4 *for the quasi-definite situation exist and are unique. Moreover, there exist matrices $h_r^H \in \mathbb{M}_m$, $H = L, R$, such that the biorthogonality conditions are fulfilled*

$$\delta_{r,j} h_r^H := \left\langle\!\left\langle P_{2,r}^H, P_{1,j}^H \right\rangle\!\right\rangle_H, \quad H = R, L.$$

Now we introduce the matrix extension of the Verblunsky coefficients.

Definition 5. The Verblunsky matrices of a matrix quasi-definite measure are

$$\alpha_{i,l}^H := P_{i,l}^H(0), \quad i = 1, 2, \; l = 1, 2, 3, \ldots, \; H = L, R,$$

and the reciprocal or reversed Szegő matrix polynomials are given by

$$\left(P_l^H\right)^*(z) := z^l \left(P_l^H\left(\bar{z}^{-1}\right)\right)^\dagger, \quad H = L, R.$$

Notice that in the Hermitian positive definite case, the matrices h_l^H, $H = L, R$, $l = 0, 1, 2, \ldots$, can be interpreted as a kind of "matrix-valued norms" for the matrix Szegő polynomials, as the square-root of their traces is a norm indeed.

2.2. The CMV matrix Laurent polynomials

We consider now the $m \times m$ block LU factorization of the moment matrices (5) and (6); in fact, there are two block Gauss–Borel factorizations, for both the right and left moment matrices, to consider

$$g^L := S_1^{-1} D^L \widehat{S_2} = S_1^{-1} S_2, \quad S_1 \in \mathscr{L}, \; S_2 \in \mathscr{U}, \; \widehat{S_2} \in \widehat{\mathscr{U}}, \; D_L \in \mathscr{D}, \tag{7}$$

$$g^R := Z_2 D^R \widehat{Z_1}^{-1} = Z_2 Z_1^{-1}, \quad Z_2 \in \mathscr{L}, \; Z_1 \in \mathscr{U}, \; \widehat{Z_1} \in \widehat{\mathscr{U}}, \; D_R \in \mathscr{D}. \tag{8}$$

For the entries of the block diagonal matrices, we use the notation

$$D^H = \mathrm{diag}\left(D_0^H, D_1^H, \ldots\right), \quad H = L, R. \tag{9}$$

The reader should notice that in the Hermitian case, the two normalized matrices of the factorization are related

$$S_1^\dagger = \widehat{S_2}^{-1}, \qquad Z_2^\dagger = \widehat{Z_1}^{-1}, \tag{10}$$

and the block diagonal matrices are Hermitian; $(D^H)^\dagger = D^H$, $H = L, R$.

Definition 6. We introduce the following partial CMV matrix Laurent polynomials

$$\phi_{1,1}^L := S_1 \chi_1(z), \qquad \phi_{1,2}^L := S_1 \chi_2^*(z),$$

$$\phi_{2,1}^L := \big(S_2^{-1}\big)^\dagger \chi_1(z), \qquad \phi_{2,2}^L := \big(S_2^{-1}\big)^\dagger \chi_2^*(z),$$

$$\phi_{1,1}^R := \chi_1^\top(z) Z_1, \qquad \phi_{1,2}^R := \big[\chi_2^*\big]^\top(z) Z_1,$$

$$\phi_{2,1}^R := \chi_1^\top(z)\big(Z_2^{-1}\big)^\dagger, \qquad \phi_{2,2}^R := \big[\chi_2^*\big]^\top(z)\big(Z_2^{-1}\big)^\dagger,$$

and CMV matrix Laurent polynomials

$$\phi_1^L := \phi_{1,1}^L + \phi_{1,2}^L = S_1 \chi(z), \qquad \phi_2^L := \phi_{2,1}^L + \phi_{2,2}^L = \big(S_2^{-1}\big)^\dagger \chi(z), \tag{11}$$

$$\phi_1^R := \phi_{1,1}^R + \phi_{1,2}^R = \chi^\top(z) Z_1, \qquad \phi_2^R := \phi_{2,1}^R + \phi_{2,2}^R = \chi^\top(z)\big(Z_2^{-1}\big)^\dagger. \tag{12}$$

Notice that these semi-infinite vectors with matrix coefficients $(\varphi_j^H)^{(l)}(z), l = 0, 1, \ldots,$ can be written as

$$\phi_j^L =: \begin{pmatrix} (\varphi_j^L)^{(0)}(z) \\ (\varphi_j^L)^{(1)}(z) \\ \vdots \end{pmatrix}, \qquad \phi_j^R =: \big((\varphi_j^R)^{(0)}(z), (\varphi_j^R)^{(1)}(z), \ldots\big), \quad j = 1, 2.$$

For the Hermitian case, we have

$$\big(\varphi_2^L\big)^{(l)}(z) = \big(D_l^L\big)^{-1}\big(\varphi_1^L\big)^{(l)}(z), \qquad \big(\varphi_2^R\big)^{(l)}(z) = \big(\varphi_1^R\big)^{(l)}(z) D_l^R,$$

$$l = 0, 1, \ldots. \tag{13}$$

2.2.1. Biorthogonality

From the Gaussian factorization, whose existence is ensured for quasi-definite matrix measures, we infer that these matrix Laurent polynomials satisfy biorthogonal type relations.

Theorem 1. *The matrix Laurent polynomials* $\{(\varphi^H)_1^{(l)}\}_{l=0}^\infty$ *and* $\{(\varphi^H)_2^{(l)}\}_{l=0}^\infty$, $H = L, R$, *introduced in* (11) *and* (12), *are biorthogonal on the unit circle*

$$\big\langle\!\big\langle \big(\varphi_2^H\big)^{(j)}, \big(\varphi_1^H\big)^{(k)} \big\rangle\!\big\rangle_H = \mathbb{I}\delta_{j,k}, \quad H = L, R, \; j, k = 0, 1, \ldots. \tag{14}$$

Proof. It is straightforward to check that

$$\oint_{\mathbb{T}} \phi_1^L(z) \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} \big(\phi_2^L(z)\big)^\dagger = S_1\bigg(\oint_{\mathbb{T}} \chi(z) \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} \chi(z)^\dagger\bigg) S_2^{-1} = S_1 g^L S_2^{-1} = \mathbb{I},$$

$$\oint_{\mathbb{T}} \big(\phi_2^R(z)\big)^\dagger \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} \phi_1^R(z) = Z_2^{-1}\bigg(\oint_{\mathbb{T}} \overline{\chi(z)} \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} \chi(z)^\top\bigg) Z_1 = Z_2^{-1} g^R Z_1 = \mathbb{I}. \qquad \square$$

In order to relate the CMV matrix Laurent polynomials to the Szegő polynomials, we rewrite the *quasi-orthogonality* conditions from Theorem 1

$$\oint_{\mathbb{T}} \big(\varphi_1^L\big)^{(2l)}(z)\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}z^{-k} = 0, \quad k = -l,\ldots,l-1,$$

$$\oint_{\mathbb{T}} \big(\varphi_1^L\big)^{(2l+1)}(z)\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}z^{-k} = 0, \quad k = -l,\ldots,l,$$

$$\oint_{\mathbb{T}} z^k\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}\big[\big(\varphi_2^L\big)^{(2l)}(z)\big]^{\dagger} = 0, \quad k = -l,\ldots,l-1,$$

$$\oint_{\mathbb{T}} z^k\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}\big[\big(\varphi_2^L\big)^{(2l+1)}(z)\big]^{\dagger} = 0, \quad k = -l,\ldots,l, \tag{15}$$

$$\oint_{\mathbb{T}} \big[\big(\varphi_2^R\big)^{(2l)}(z)\big]^{\dagger}\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}z^{k} = 0, \quad k = -l,\ldots,l-1,$$

$$\oint_{\mathbb{T}} \big[\big(\varphi_2^R\big)^{(2l+1)}(z)\big]^{\dagger}\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}z^{k} = 0, \quad k = -l,\ldots,l,$$

$$\oint_{\mathbb{T}} z^{-k}\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}\big(\varphi_1^R\big)^{(2l)}(z) = 0, \quad k = -l,\ldots,l-1,$$

$$\oint_{\mathbb{T}} z^{-k}\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}\big(\varphi_1^R\big)^{(2l+1)}(z) = 0, \quad k = -l,\ldots,l. \tag{16}$$

Proposition 3. *For a quasi-definite matrix measure μ, the matrix Szegő polynomials and the CMV matrix Laurent polynomials are related in the following way for the left case*

$$z^{l}\big(\varphi_1^L\big)^{(2l)}(z) = P_{1,2l}^L(z),$$

$$z^{l+1}\big(\varphi_1^L\big)^{(2l+1)}(z) = \big(P_{2,2l+1}^R\big)^{*}(z),$$

$$z^{l}\big(D_{2l}^L\big)^{\dagger}\big(\varphi_2^L\big)^{(2l)}(z) = P_{2,2l}^L(z),$$

$$z^{l+1}\big(D_{2l+1}^L\big)^{\dagger}\big(\varphi_2^L\big)^{(2l+1)}(z) = \big(P_{1,2l+1}^R\big)^{*}(z), \tag{17}$$

and

$$z^{l}\big(\varphi_2^R\big)^{(2l)}(z) = P_{2,2l}^R(z),$$

$$z^{l+1}\big(\varphi_2^R\big)^{(2l+1)}(z) = \big(P_{1,2l+1}^L\big)^{*}(z),$$

$$z^{l}\big(\varphi_1^R\big)^{(2l)}(z)D_{2l}^R = P_{1,2l}^R(z),$$

$$z^{l+1}\big(\varphi_1^R\big)^{(2l+1)}(z)D_{2l+1}^R = \big(P_{2,2l+1}^L\big)^{*}(z) \tag{18}$$

for the right case.

Proof. Taking the differences between the RHS and LHS of the equalities, we get matrix polynomials, of degree $d = 2l - 1, 2l$, that when paired via $\langle\!\langle \cdot, \cdot \rangle\!\rangle_H$, $H = L, R$, to all the powers z^j, $j = 0, \ldots, q$ cancels. Therefore, as we have a quasi-definite matrix measure, with moment matrices having non-null principal block minors, the only possibility for the difference is to be 0. $\square$

The last identifications together with (4) define some of the entries of the Gaussian factorization matrices.

Proposition 4. *The* matrix quasi-norms h_k^H *introduced in Definition 4 and the coefficients* D_k^H *given in* (9) *satisfy*

$$
\begin{aligned}
h_{2l}^L &= D_{2l}^L, & h_{2l+1}^L &= D_{2l+1}^R, \\
h_{2l}^R &= D_{2l}^R, & h_{2l+1}^R &= D_{2l+1}^L.
\end{aligned}
$$

For the first non-trivial block diagonal of the factors in the Gauss–Borel factorization, we get

Proposition 5. *The matrices of the block LU factorization can be written more explicitly in terms of the Verblunsky coefficients as follows*

$$
S_1 = \begin{pmatrix}
\mathbb{I} & 0 & 0 & 0 & 0 & \ldots \\
[\alpha_{2,1}^R]^\dagger & \mathbb{I} & 0 & 0 & 0 & \ldots \\
* & \alpha_{1,2}^L & \mathbb{I} & 0 & 0 & \ldots \\
* & * & [\alpha_{2,3}^R]^\dagger & \mathbb{I} & 0 & \ldots \\
* & * & * & \alpha_{1,4}^L & \mathbb{I} & \\
\vdots & \vdots & \vdots & & \ddots & \ddots
\end{pmatrix},
$$

$$
\widehat{S_2}^{-1} = \begin{pmatrix}
\mathbb{I} & \alpha_{1,1}^R & * & * & * & \ldots \\
0 & \mathbb{I} & [\alpha_{2,2}^L]^\dagger & * & * & \ldots \\
0 & 0 & \mathbb{I} & \alpha_{1,3}^R & * & \ldots \\
0 & 0 & 0 & \mathbb{I} & [\alpha_{2,4}^L]^\dagger & \\
0 & 0 & 0 & 0 & \mathbb{I} & \ddots \\
\vdots & \vdots & \vdots & \vdots & & \ddots
\end{pmatrix},
$$

$$
Z_2^{-1} = \begin{pmatrix}
\mathbb{I} & 0 & 0 & 0 & 0 & \ldots \\
\alpha_{1,1}^L & \mathbb{I} & 0 & 0 & 0 & \ldots \\
* & [\alpha_{2,2}^R]^\dagger & \mathbb{I} & 0 & 0 & \ldots \\
* & * & \alpha_{1,3}^L & \mathbb{I} & 0 & \ldots \\
* & * & * & [\alpha_{2,4}^R]^\dagger & \mathbb{I} & \\
\vdots & \vdots & \vdots & & \ddots & \ddots
\end{pmatrix},
$$

$$\widehat{Z_1} = \begin{pmatrix} \mathbb{I} & [\alpha_{2,1}^L]^\dagger & * & * & * & \cdots \\ 0 & \mathbb{I} & \alpha_{1,2}^R & * & * & \cdots \\ 0 & 0 & \mathbb{I} & [\alpha_{2,3}^L]^\dagger & * & \cdots \\ 0 & 0 & 0 & \mathbb{I} & \alpha_{1,4}^R & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

This gives the following structure for the MOLPUC

Proposition 6.

(1) *The MOLPUC are of the form*

$$\left(\phi_1^L\right)^{(2l)} = \alpha_{1,2l}^L z^{-l} + \cdots + z^l,$$

$$\left(\phi_1^L\right)^{(2l+1)} = z^{-l-1} + \cdots + \left(\alpha_{2,2l+1}^R\right)^\dagger z^l,$$

$$\left(\phi_2^L\right)^{(2l)} = \left(\left(h_{2l}^L\right)^\dagger\right)^{-1}\left(\alpha_{2,2l}^L z^{-l} + \cdots + z^l\right),$$

$$\left(\phi_2^L\right)^{(2l+1)} = \left(\left(h_{2l+1}^R\right)^\dagger\right)^{-1}\left(z^{-l-1} + \cdots + \left(\alpha_{1,2l+1}^R\right)^\dagger z^l\right),$$

$$\left(\phi_1^R\right)^{(2l)} = \left(\alpha_{1,2l}^R z^{-l} + \cdots + z^l\right)\left(h_{2l}^R\right)^{-1},$$

$$\left(\phi_1^R\right)^{(2l+1)} = \left(z^{-l-1} + \cdots + \left(\alpha_{2,2l+1}^L\right)^\dagger z^l\right)\left(h_{2l+1}^L\right)^{-1},$$

$$\left(\phi_2^R\right)^{(2l)} = \alpha_{2,2l}^R z^{-l} + \cdots + z^l,$$

$$\left(\phi_2^R\right)^{(2l+1)} = z^{-l-1} + \cdots + \alpha_{1,2l+1}^L z^l.$$

(2) *The "quasi-norms" and the MOLPUC fulfill*

$$h_{2l+1}^R = \oint_{\mathbb{T}} \left(\phi_1^L\right)^{(2l+1)}(z)\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}z^{l+1}, \qquad h_{2l}^L = \oint_{\mathbb{T}} \left(\phi_1^L\right)^{(2l)}(z)\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}z^{-l},$$

$$h_{2l}^R = \oint_{\mathbb{T}} \left(\left(\phi_2^R\right)^{(2l)}(z)\right)^\dagger\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}z^l, \qquad h_{2l+1}^L = \oint_{\mathbb{T}} \left(\left(\phi_2^R\right)^{(2l+1)}(z)\right)^\dagger\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}z^{-l-1}.$$

(19)

Proof.

(1) Use (7), (8) and Propositions 4 and 5.
(2) Consider the biorthogonality (14) together with the explicit expressions of the first item in this proposition and orthogonality relations (15) and (16). $\square$

Recalling (10), we conclude from Proposition 5 that in the Hermitian context, we have

$$\alpha_{1,l}^H = \alpha_{2,l}^H, \quad H = L, R, \ l = 0, 1, \ldots,$$

$$\left(D_l^H\right)^\dagger = D_l^H, \quad H = L, R, \ l = 0, 1, \ldots.$$

It is not difficult to see comparing the previous result with the proof of the Gaussian factorization (A.1) that in terms of Schur complements, we have

Proposition 7.

(1) *The matrices $D_l^H \in \mathbb{C}^{m \times m}$, $H = L, R$, $l = 0, 1, \ldots$, from the diagonal block of the bock LU factorization can be written as the following Schur complements*

$$D_l^H = \left(g^H\right)^{[l+1]} \Big/ \left(g^H\right)^{[l]}, \quad H = L, R, \ l = 0, 1, \ldots. \tag{20}$$

(2) *The Verblunsky matrices can be expressed as*

$$\alpha_{1,2k}^L = -\sum_{i=0}^{2k-1} \left(g^L\right)_{2k,i} \left(\left(\left(g^L\right)^{[2k]}\right)^{-1}\right)_{i,2k-1},$$

$$\left[\alpha_{2,2k+1}^R\right]^\dagger = -\sum_{i=0}^{2k} \left(g^L\right)_{2k+1,i} \left(\left(\left(g^L\right)^{[2k+1]}\right)^{-1}\right)_{i,2k},$$

$$\left[\alpha_{2,2k}^R\right]^\dagger = -\sum_{i=0}^{2k-1} \left(g^R\right)_{2k,i} \left(\left(\left(g^R\right)^{[2k]}\right)^{-1}\right)_{i,2k-1},$$

$$\alpha_{1,2k+1}^L = -\sum_{i=0}^{2k} \left(g^R\right)_{2k+1,i} \left(\left(\left(g^R\right)^{[2k+1]}\right)^{-1}\right)_{i,2k},$$

$$\left[\alpha_{2,2k}^L\right]^\dagger = -\sum_{i=0}^{2k-1} \left(\left(\left(g^L\right)^{[2k]}\right)^{-1}\right)_{2k-1,i} \left(g^L\right)_{i,2k},$$

$$\alpha_{1,2k+1}^R = -\sum_{i=0}^{2k} \left(\left(\left(g^L\right)^{[2k+1]}\right)^{-1}\right)_{2k,i} \left(g^L\right)_{i,2k+1},$$

$$\alpha_{1,2k}^R = -\sum_{i=0}^{2k-1} \left(\left(\left(g^R\right)^{[2k]}\right)^{-1}\right)_{2k-1,i} \left(g^R\right)_{i,2k},$$

$$\left[\alpha_{2,2k+1}^L\right]^\dagger = -\sum_{i=0}^{2k} \left(\left(\left(g^R\right)^{[2k+1]}\right)^{-1}\right)_{2k,i} \left(g^R\right)_{i,2k+1}.$$

2.2.2. Alternative ways to express the CMV matrix Laurent polynomials

For later use, we now present some alternative expressions for the MOLPUC $(\varphi_i^H)^{(l)}(z)$, $H = L, R$, $l = 0, 1, \ldots$ in terms of Schur complements of bordered truncated matrices

Lemma 1. *The next expressions hold true*

$$
\left(\varphi_1^L\right)^{(l)}(z) = (S_2)_{ll} \begin{pmatrix} 0 & 0 & \cdots & 0 & \mathbb{I} \end{pmatrix} \left(\left(g^L\right)^{[l+1]}\right)^{-1} \chi^{[l+1]}
$$

$$
= \chi^{(l)} - \left(\begin{matrix} (g^L)_{l,0} & (g^L)_{l,1} & \cdots & (g^L)_{l,l-1} \end{matrix}\right) \left(\left(g^L\right)^{[l]}\right)^{-1} \chi^{[l]}
$$

$$
= \mathrm{SC} \left(\begin{array}{cccc|c}
(g^L)_{0,0} & (g^L)_{0,1} & \cdots & (g^L)_{0,l-1} & \chi(z)^{(0)} \\
(g^L)_{1,0} & (g^L)_{1,1} & \cdots & (g^L)_{1,l-1} & \chi(z)^{(1)} \\
& & \vdots & & \vdots \\
(g^L)_{l-1,0} & (g^L)_{l-1,1} & \cdots & (g^L)_{l-1,l-1} & \chi(z)^{(l-1)} \\
\hline
(g^L)_{l,0} & (g^L)_{l,1} & \cdots & (g^L)_{l,l-1} & \chi(z)^{(l)}
\end{array}\right), \tag{21}
$$

$$
\left[\left(\varphi_2^L\right)^{(l)}(z)\right]^\dagger = \left(\chi^{[l+1]}\right)^\dagger \left(\left(g^L\right)^{[l+1]}\right)^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \mathbb{I} \end{pmatrix}
$$

$$
= \left(\left(\chi^{(l)}\right)^\dagger - \left(\chi^{[l]}\right)^\dagger \left(\left(g_L\right)^{[l]}\right)^{-1} \begin{pmatrix} (g^L)_{0,l} \\ (g^L)_{1,l} \\ \vdots \\ (g^L)_{l-1,l} \end{pmatrix} \right) \left(D^L\right)_l
$$

$$
= \mathrm{SC} \left(\begin{array}{cccc|c}
(g^L)_{0,0} & (g^L)_{0,1} & \cdots & (g^L)_{0,l-1} & (g^L)_{0,l} \\
(g^L)_{1,0} & (g^L)_{1,1} & \cdots & (g^L)_{1,l-1} & (g^L)_{1,l} \\
& & \vdots & & \vdots \\
(g^L)_{l-1,0} & (g^L)_{l-1,1} & \cdots & (g^L)_{l-1,l-1} & (g^L)_{l-1,l} \\
\hline
[\chi(z)^\dagger]^{(0)} & [\chi(z)^\dagger]^{(1)} & \cdots & [\chi(z)^\dagger]^{(l-1)} & [\chi(z)^\dagger]^{(l)}
\end{array}\right) \left(D^L\right)_l \tag{22}
$$

and

$$
\left(\varphi_1^R\right)^{(l)}(z) = \left[\chi^{[l+1]}\right]^\top \left(\left(g^R\right)^{[l+1]}\right)^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \mathbb{I} \end{pmatrix}
$$

$$
= \left[\chi^{(l)}\right]^\top - \left[\chi^{[l]}\right]^\top \left(\left(g^R\right)^{[l]}\right)^{-1} \begin{pmatrix} (g^R)_{0,l} \\ (g^R)_{1,l} \\ \vdots \\ (g^R)_{l-1,l} \end{pmatrix} \left(\left(D^R\right)^{-1}\right)_l
$$

$$= \mathrm{SC} \begin{pmatrix} (g^R)_{0,0} & (g^R)_{0,1} & \cdots & (g^R)_{0,l-1} & (g^R)_{0,l} \\ (g^R)_{1,0} & (g^R)_{1,1} & \cdots & (g^R)_{1,l-1} & (g^R)_{1,l} \\ \vdots & \vdots & & \vdots & \vdots \\ (g^R)_{l-1,0} & (g^R)_{l-1,1} & \cdots & (g^R)_{l-1,l-1} & (g^R)_{l-1,l} \\ \hline [\chi(z)^\top]^{(0)} & [\chi(z)^\top]^{(1)} & \cdots & [\chi(z)^\top]^{(l-1)} & [\chi(z)^\top]^{(l)} \end{pmatrix} \left((D^R)^{-1} \right)_l,$$

$$\left[(\varphi_2^R)^{(l)}(z) \right]^\dagger = (D^R)_l \begin{pmatrix} 0 & \cdots & 0 & \mathbb{I} \end{pmatrix} \left((g^R)^{[l+1]} \right)^{-1} \left[(\chi^{[l+1]})^\top \right]^\dagger$$

$$= \left[(\chi^{(l)})^\top \right]^\dagger - \begin{pmatrix} (g^R)_{l,0} & \cdots & (g^R)_{l,l-1} \end{pmatrix} \left((g^R)^{[l]} \right)^{-1} \left[(\chi^{[l]})^\top \right]^\dagger$$

$$= \mathrm{SC} \begin{pmatrix} (g^R)_{0,0} & (g^R)_{0,1} & \cdots & (g^R)_{0,l-1} & [(\chi(z)^\top)^\dagger]^{(0)} \\ (g^R)_{1,0} & (g^R)_{1,1} & \cdots & (g^R)_{1,l-1} & [(\chi(z)^\top)^\dagger]^{(1)} \\ & & \vdots & & \vdots \\ (g^R)_{l-1,0} & (g^R)_{l-1,1} & \cdots & (g^R)_{l-1,l-1} & [(\chi(z)^\top)^\dagger]^{(l-1)} \\ \hline (g^R)_{l,0} & (g^R)_{l,1} & \cdots & (g^R)_{l,l-1} & [(\chi(z)^\top)^\dagger]^{(l)} \end{pmatrix}.$$

Proof. See Appendix A. $\square$

Following [27] we give expressions in terms of Schur complements for the matrix Szegő polynomials, in terms of bordered truncated matrices of the right and left block CMV moment matrices, extending though similar expressions given in [27] in terms of standard block moment matrices.

Corollary 1. *The left matrix Szegő polynomials can be rewritten as the following Schur complements of bordered truncated CMV moment matrices*

$$P_{1,2l}^L(z) = z^l \, \mathrm{SC} \left(\begin{array}{ccc|c} & (g^L)^{[2l]} & & \chi(z)^{[2l]} \\ \hline (g^L)_{2l,0} & \cdots & (g^L)_{2l,2l-1} & \chi(z)^{(2l)} \end{array} \right),$$

$$P_{1,2l+1}^L(z) = z^{l+1} \, \mathrm{SC} \left(\begin{array}{ccc|c} & (g^R)^{[2l+1]} & & \chi^*(z)^{[2l+1]} \\ \hline (g^R)_{2l+1,0} & \cdots & (g^R)_{2l+1,2l-1} & \chi^*(z)^{(2l+1)} \end{array} \right),$$

$$\left[P_{2,2l}^L(z) \right]^\dagger = \bar{z}^l \, \mathrm{SC} \left(\begin{array}{c|c} & (g^L)_{0,2l} \\ (g^L)^{[2l]} & \vdots \\ & (g^L)_{2l-1,2l} \\ \hline (\chi(z)^\dagger)^{[2l]} & (\chi(z)^\dagger)^{(2l)} \end{array} \right),$$

$$\left[P_{2,2l+1}^L(z) \right]^\dagger = \bar{z}^{l+1} \, \mathrm{SC} \left(\begin{array}{c|c} & (g^R)_{0,2l+1} \\ (g^R)^{[2l+1]} & \vdots \\ & (g^R)_{2l,2l+1} \\ \hline (\chi^*(z)^\dagger)^{[2l+1]} & (\chi^*(z)^\dagger)^{(2l+1)} \end{array} \right),$$

while for the right polynomials we have

$$P_{1,2l}^R(z) = z^l \, \mathrm{SC} \left(\begin{array}{c|c} (g^R)^{[2l]} & \begin{array}{c} (g^R)_{0,2l} \\ \vdots \\ (g^R)_{2l-1,2l} \end{array} \\ \hline (\chi(z)^\top)^{[2l]} & (\chi(z)^\top)^{(2l)} \end{array} \right),$$

$$P_{1,2l+1}^R(z) = z^{l+1} \, \mathrm{SC} \left(\begin{array}{c|c} (g^L)^{[2l+1]} & \begin{array}{c} (g^L)_{0,2l+1} \\ \vdots \\ (g^L)_{2l,2l+1} \end{array} \\ \hline (\chi^*(z)^\top)^{[2l+1]} & (\chi^*(z)^\top)^{(2l+1)} \end{array} \right),$$

$$\left[P_{2,2l}^R(z) \right]^\dagger = \bar{z}^l \, \mathrm{SC} \left(\begin{array}{ccc|c} & (g^R)^{[2l]} & & (\chi(z)^\dagger)^{[2l]} \\ \hline (g^R)_{2l,0} & \cdots & (g^R)_{2l,2l-1} & (\chi(z)^\dagger)^{(2l)} \end{array} \right),$$

$$\left[P_{2,2l+1}^R(z) \right]^\dagger = \bar{z}^{l+1} \, \mathrm{SC} \left(\begin{array}{ccc|c} & (g^L)^{[2l+1]} & & (\chi^*(z)^\dagger)^{[2l+1]} \\ \hline (g^R)_{2l+1,0} & \cdots & (g^R)_{2l+1,2l-1} & (\chi^*(z)^\dagger)^{(2l+1)} \end{array} \right).$$

Proof. These relations appear when one introduces in (17) and (18) the expressions of the CMV polynomials in terms of Schur complements. $\square$

2.3. Matrix second kind functions

The following matrix fashion of rewriting previous left objects

$$\begin{pmatrix} \phi_{1,1}^L & \phi_{1,2}^L \\ \phi_{2,1}^L & \phi_{2,2}^L \end{pmatrix} = \begin{bmatrix} S_1 & 0 \\ 0 & [S_2^{-1}]^\dagger \end{bmatrix} \begin{pmatrix} \chi_1 & \chi_2^* \\ \chi_1 & \chi_2^* \end{pmatrix} = \begin{pmatrix} S_1 \chi_1 & S_1 \chi_2^* \\ [S_2^{-1}]^\dagger \chi_1 & [S_2^{-1}]^\dagger \chi_2^* \end{pmatrix},$$

$$\begin{pmatrix} \phi_1^L \\ \phi_2^L \end{pmatrix} = \begin{pmatrix} \phi_{1,1}^L & \phi_{1,2}^L \\ \phi_{2,1}^L & \phi_{2,2}^L \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\begin{bmatrix} [g^L]^\dagger & 0 \\ 0 & g^L \end{bmatrix} = \begin{bmatrix} [S_2]^\dagger [S_1^{-1}]^\dagger & 0 \\ 0 & S_1^{-1} S_2 \end{bmatrix} = \begin{bmatrix} [S_2]^\dagger & 0 \\ 0 & S_1^{-1} \end{bmatrix} \begin{bmatrix} [S_1^{-1}]^\dagger & 0 \\ 0 & S_2 \end{bmatrix}$$

and the right ones

$$\begin{pmatrix} \phi_{1,1}^R & \phi_{2,1}^R \\ \phi_{1,2}^R & \phi_{2,2}^R \end{pmatrix} = \begin{pmatrix} \chi_1^\top & \chi_1^\top \\ [\chi_2^*]^\top & [\chi_2^*]^\top \end{pmatrix} \begin{bmatrix} Z_1 & 0 \\ 0 & [Z_2^{-1}]^\dagger \end{bmatrix},$$

$$\begin{pmatrix} \phi_1^R & \phi_2^R \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \phi_{1,1}^R & \phi_{2,1}^R \\ \phi_{1,2}^R & \phi_{2,2}^R \end{pmatrix},$$

$$\begin{bmatrix} [g^R]^\dagger & 0 \\ 0 & g^R \end{bmatrix} = \begin{bmatrix} [Z_1^{-1}]^\dagger & 0 \\ 0 & Z_2 \end{bmatrix} \begin{bmatrix} [Z_2]^\dagger & 0 \\ 0 & Z_1^{-1} \end{bmatrix}$$

inspires the next

Definition 7. The partial matrix CMV second kind sequences are given by

$$
\begin{pmatrix} C_{1,1}^L & C_{1,2}^L \\ C_{2,1}^L & C_{2,2}^L \end{pmatrix} := \begin{bmatrix} [S_1^{-1}]^\dagger & 0 \\ 0 & S_2 \end{bmatrix} \begin{pmatrix} \chi_1^* & \chi_2 \\ \chi_1^* & \chi_2 \end{pmatrix} = \begin{pmatrix} [S_1^{-1}]^\dagger \chi_1^* & [S_1^{-1}]^\dagger \chi_2 \\ S_2 \chi_1^* & S_2 \chi_2 \end{pmatrix},
$$

$$
\begin{pmatrix} C_{1,1}^R & C_{2,1}^R \\ C_{1,2}^R & C_{2,2}^R \end{pmatrix} := \begin{pmatrix} [\chi_1^*]^\top & [\chi_1^*]^\top \\ \chi_2^\top & \chi_2^\top \end{pmatrix} \begin{bmatrix} [Z_1^{-1}]^\dagger & 0 \\ 0 & Z_2 \end{bmatrix} = \begin{pmatrix} [\chi_1^*]^\top [Z_1^{-1}]^\dagger & [\chi_1^*]^\top Z_2 \\ \chi_2^\top [Z_1^{-1}]^\dagger & \chi_2^\top Z_2 \end{pmatrix},
$$

and the corresponding matrix CMV second kind sequences are

$$
\begin{pmatrix} C_1^L \\ C_2^L \end{pmatrix} = \begin{pmatrix} C_{1,1}^L & C_{1,2}^L \\ C_{2,1}^L & C_{2,2}^L \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
$$

$$
\begin{pmatrix} C_1^R & C_2^R \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} C_{1,1}^R & C_{2,1}^R \\ C_{1,2}^R & C_{2,2}^R \end{pmatrix}.
$$

Complementary to the above definition

Definition 8. The associated CMV Fourier series are

$$
\begin{pmatrix} \Gamma_{2,1}^L & \Gamma_{2,2}^L \\ \Gamma_{1,1}^L & \Gamma_{1,2}^L \end{pmatrix} := \begin{bmatrix} [g^L]^\dagger & 0 \\ 0 & g^L \end{bmatrix} \begin{pmatrix} \chi_1^* & \chi_2 \\ \chi_1^* & \chi_2 \end{pmatrix} = \begin{pmatrix} [g^L]^\dagger \chi_1^* & [g^L]^\dagger \chi_2 \\ g^L \chi_1^* & g^L \chi_2 \end{pmatrix},
$$

$$
\begin{pmatrix} \Gamma_{2,1}^R & \Gamma_{1,1}^R \\ \Gamma_{2,2}^R & \Gamma_{1,2}^R \end{pmatrix} := \begin{pmatrix} [\chi_1^*]^\top & [\chi_1^*]^\top \\ \chi_2^\top & \chi_2^\top \end{pmatrix} \begin{bmatrix} [g^R]^\dagger & 0 \\ 0 & g^R \end{bmatrix} = \begin{pmatrix} [\chi_1^*]^\top [g^R]^\dagger & [\chi_1^*]^\top g^R \\ \chi_2^\top [g^R]^\dagger & \chi_2^\top g^R \end{pmatrix},
$$

for which we have:

Proposition 8.

(1) *The elements Γ^H and C^H, $H = L, R$, are related in the following way*

$$
\begin{pmatrix} \Gamma_2^L \\ \Gamma_1^L \end{pmatrix} = \begin{bmatrix} [S_2]^\dagger & 0 \\ 0 & S_1^{-1} \end{bmatrix} \begin{pmatrix} C_1^L \\ C_2^L \end{pmatrix}, \qquad \begin{pmatrix} \Gamma_2^R & \Gamma_1^R \end{pmatrix} = \begin{pmatrix} C_1^R & C_2^R \end{pmatrix} \begin{bmatrix} [Z_2]^\dagger & 0 \\ 0 & Z_1^{-1} \end{bmatrix}.
$$

(2) *The second kind functions can be expressed as Schur complements as follows*

$$
[(C_1^L)^{(l)}(z)]^\dagger = \mathrm{SC} \left(\begin{array}{c|c} (g^L)^{[l]} & \begin{array}{c} (g^L)_{0,l} \\ \vdots \\ (g^L)_{l-1,l} \end{array} \\ \hline (\Gamma_2^L(z)^\dagger)^{[l]} & (\Gamma_2^L(z)^\dagger)^{(l)} \end{array} \right) (D_L^{-1})_l,
$$

$$
(C_2^L)^{(l)}(z) = \mathrm{SC} \left(\begin{array}{ccc|c} & (g^L)^{[l]} & & (\Gamma_1(z))^{[l]} \\ \hline (g^L)_{l,0} & \cdots & (g^L)_{l,l-1} & (\Gamma_1(z))^{(l)} \end{array} \right),
$$

$$\left[\left(C_1^R\right)^{(l)}(z)\right]^\dagger = \text{SC}\left(\frac{(g^R)^{[l]}}{(g^R)_{l,0} \quad \cdots \quad (g^R)_{l,l-1}} \,\middle|\, \frac{(\Gamma_2^R(z)^\dagger)^{[l]}}{(\Gamma_2^R(z)^\dagger)^{(l)}}\right),$$

$$\left(C_2^R\right)^{(l)}(z) = \text{SC}\left(\begin{array}{c|c} (g^R)^{[l]} & \begin{array}{c} (g^R)_{0,l} \\ \vdots \\ (g^R)_{l-1,l} \end{array} \\ \hline (\Gamma_1^R(z))^{[l]} & (\Gamma_1^R(z))^{(l)} \end{array}\right) (D_R^{-1})_l.$$

(3) *In terms of the matrix Laurent orthogonal polynomials and the Fourier series of the matrix measure, we have*

$$\left(C_1^L\right)^{(l)}(z) = 2\pi z^{-1}\left(\varphi_2^L\right)^{(l)}\left(z^{-1}\right)F_\mu^\dagger(z),$$

$$\left(C_2^L\right)^{(l)}(z) = 2\pi z^{-1}\left(\varphi_1^L\right)^{(l)}\left(z^{-1}\right)F_\mu\left(z^{-1}\right),$$

$$\left(C_1^R\right)^{(l)}(z) = 2\pi z^{-1}F_\mu^\dagger(z)\left(\varphi_2^R\right)^{(l)}\left(z^{-1}\right),$$

$$\left(C_2^R\right)^{(l)}(z) = 2\pi z^{-1}F_\mu\left(z^{-1}\right)\left(\varphi_1^R\right)^{(l)}\left(z^{-1}\right). \tag{23}$$

Proof. The first part of the proposition follows directly from comparison of the structure of the relations from the previous lemma with the definitions of the CMV matrix polynomials. For example

$$\Gamma_1^L = S_1^{-1}C_2^L \;\Rightarrow\; C_2^L = S_1\Gamma_1^L \text{ same structure as } \phi_1^L = S_1\chi \text{ replacing } \Gamma_1^L \longleftrightarrow \chi.$$

For the second part of the proposition, we shall only prove one of the cases since the rest of them can be proven following the same procedure. First, from the definition of the second kind functions, we have

$$\begin{aligned} C_1^R(z) &= \left(\chi^*(z)\right)^\top \left[Z_1^{-1}\right]^\dagger \\ &= \left(\chi^*(z)\right)^\top \left[g^R\right]^\dagger \left(Z_2^{-1}\right)^\dagger \\ &= \left[\chi^*(z)\right]^\top \oint_{\mathbb{T}} \left[\chi^\top(u)\right]^\dagger \left[\frac{\mathrm{d}\mu(u)}{\mathrm{i}u}\right]^\dagger \left[\chi(u)\right]^\top Z_2^{-1} \\ &= \left[\chi^*(z)\right]^\top \oint_{\mathbb{T}} \left[\chi^\top(u)\right]^\dagger \left[\frac{\mathrm{d}\mu(u)}{\mathrm{i}u}\right]^\dagger \phi_2^R(u). \end{aligned}$$

Taking the l-th component of this vector of matrices, we get

$$\left(C_1^R\right)^{(l)}(z) = \int_0^{2\pi} \sum_{n=-\infty}^{\infty} z^{n-1}\mathrm{e}^{\mathrm{i}n\theta}\left[\mathrm{d}\mu(\theta)\right]^\dagger \left(\varphi_2^R\right)^{(l)}\left(\mathrm{e}^{\mathrm{i}\theta}\right)$$

$$= \sum_{k,n=-\infty} z^{n-1} \int_0^{2\pi} e^{i(n+k)\theta} \big[d\mu(\theta)\big]^\dagger \big(\varphi_{2,k}^R\big)^{(l)}$$

$$= 2\pi z^{-1} \bigg(\sum_{k=-\infty} \big(\varphi_{2,k}^R\big)^{(l)} z^{-k} \bigg) \bigg(\sum_{n=-\infty} c_{n+k}^\dagger z^{n+k} \bigg)$$

$$= 2\pi z^{-1} F_\mu^\dagger(z) \big(\varphi_2^R\big)^{(l)}\big(z^{-1}\big). \qquad \square$$

Recalling the previously stated relation between the Γ^H and the C^H, it follows from Proposition 8 that

Proposition 9. *The associated CMV Fourier series satisfy*

$$\Gamma_{1,j}^L = 2\pi z^{-1} F_\mu\big(z^{-1}\big)\chi^{(j)}\big(z^{-1}\big), \qquad \Gamma_{2,j}^L = 2\pi z^{-1} F_\mu^\dagger(z)\chi^{(j)}\big(z^{-1}\big),$$

$$\Gamma_{1,j}^R = 2\pi z^{-1} F_\mu\big(z^{-1}\big)\chi^{(j)}\big(z^{-1}\big), \qquad \Gamma_{2,j}^R = 2\pi z^{-1} F_\mu^\dagger(z)\chi^{(j)}\big(z^{-1}\big).$$

Another interesting representation of these functions is

Proposition 10. *The second kind functions have the following Cauchy integral type formulae*

$$\big[C_{1,1}^L(z)\big]^\dagger = \oint_{\mathbb{T}} \bigg[z^{-1}\frac{u}{u-z^{-1}}\bigg]^\dagger \frac{d\mu(u)}{iu}\big[\phi_2^L(u)\big]^\dagger,$$

$$C_{2,1}^L(z) = \oint_{\mathbb{T}} \phi_1^L(u)\frac{d\mu(u)}{iu}\bigg[z^{-1}\frac{u}{u-z^{-1}}\bigg], \quad |z| > 1,$$

$$\big[C_{1,2}^L(z)\big]^\dagger = \oint_{\mathbb{T}} \bigg[-z^{-1}\frac{u}{u-z^{-1}}\bigg]^\dagger \frac{d\mu(u)}{iu}\big[\phi_2^L(u)\big]^\dagger,$$

$$C_{2,2}^L(z) = \oint_{\mathbb{T}} \phi_1^L(u)\frac{d\mu(u)}{iu}\bigg[-z^{-1}\frac{u}{u-z^{-1}}\bigg], \quad |z| < 1,$$

$$\big[C_{1,1}^R(z)\big]^\dagger = \oint_{\mathbb{T}} \big[\phi_2^R(u)\big]^\dagger \frac{d\mu(u)}{iu}\bigg[z^{-1}\frac{u}{u-z^{-1}}\bigg]^\dagger,$$

$$\big[C_{2,1}^R(z)\big] = \oint_{\mathbb{T}} \bigg[z^{-1}\frac{u}{u-z^{-1}}\bigg]\frac{d\mu(u)}{iu}\phi_1^R(u), \quad |z| > 1,$$

$$\big[C_{1,2}^R(z)\big]^\dagger = \oint_{\mathbb{T}} \big[\phi_2^R(u)\big]^\dagger \frac{d\mu(u)}{iu}\bigg[-z^{-1}\frac{u}{u-z^{-1}}\bigg]^\dagger,$$

$$\big[C_{2,1}^R(z)\big] = \oint_{\mathbb{T}} \bigg[-z^{-1}\frac{u}{u-z^{-1}}\bigg]\frac{d\mu(u)}{iu}\phi_1^R(u), \quad |z| < 1.$$

Proof. Direct substitution leads to

$$\left[C_{1,1}^{L}\right](z)^{\dagger} = \oint_{\mathbb{T}} \left[\sum_{n=0} z^{-1}(uz)^{-n}\right]^{\dagger} \frac{\mathrm{d}\mu(u)}{iu} \left[\phi_2^L(u)\right]^{\dagger},$$

$$C_{2,1}^{L}(z) = \oint_{\mathbb{T}} \phi_1^L(u) \frac{\mathrm{d}\mu(u)}{iu} \left[\sum_{n=0} z^{-1}(uz)^{-n}\right],$$

$$\left[C_{1,2}^{L}(z)\right]^{\dagger} = \oint_{\mathbb{T}} \left[\sum_{n=0} u(uz)^{n}\right]^{\dagger} \frac{\mathrm{d}\mu(u)}{iu} \left[\phi_2^L(u)\right]^{\dagger},$$

$$C_{2,2}^{L}(z) = \oint_{\mathbb{T}} \phi_1^L(u) \frac{\mathrm{d}\mu(u)}{iu} \left[\sum_{n=0} u(uz)^{n}\right],$$

$$\left[C_{1,1}^{R}(z)\right]^{\dagger} = \oint_{\mathbb{T}} \left[\phi_2^R(u)\right]^{\dagger} \frac{\mathrm{d}\mu(u)}{iu} \left[\sum_{n=0} z^{-1}(uz)^{-n}\right]^{\dagger},$$

$$C_{2,1}^{R}(z) = \oint_{\mathbb{T}} \left[\sum_{n=0} z^{-1}(uz)^{-n}\right] \frac{\mathrm{d}\mu(u)}{iu} \phi_1^R(u),$$

$$\left[C_{1,2}^{R}(z)\right]^{\dagger} = \oint_{\mathbb{T}} \left[\phi_2^R(u)\right]^{\dagger} \frac{\mathrm{d}\mu(u)}{iu} \left[\sum_{n=0} u(uz)^{n}\right]^{\dagger},$$

$$C_{2,1}^{R}(z) = \oint_{\mathbb{T}} \left[\sum_{n=0} u(uz)^{-n}\right] \frac{\mathrm{d}\mu(u)}{iu} \phi_1^R(u).$$

But these are the series expansions of the functions of the proposition. We will not deal here with convergence problems since their discussion follows the ideas of [13]. □

2.4. Recursion relations

In order to get the recursion relations we introduce the following

Definition 9. For each pair $i, j \in \mathbb{Z}_+$, we consider the block semi-infinite matrix $E_{i,j}$ whose only non-zero $m \times m$ block is the (i, j)-th block where the identity of $\mathbb{M}_m$ appears. Then, we define the projectors

$$\Pi_1 := \sum_{j=0}^{\infty} E_{2j,2j}, \qquad \Pi_2 := \sum_{j=0}^{\infty} E_{2j+1,2j+1},$$

and the following matrices

$$\Lambda_1 := \sum_{j=0}^{\infty} E_{2j,2+2j}, \qquad \Lambda_2 := \sum_{j=0}^{\infty} E_{1+2j,3+2j}, \qquad \Lambda := \sum_{j=0}^{\infty} E_{j,j+1},$$

$$\Upsilon := \Lambda_1 + \Lambda_2^{\top} + E_{1,1}\Lambda^{\top}.$$

The matrix Υ, which can be written more explicitly as follows

$$\Upsilon = \begin{pmatrix} 0 & 0 & \mathbb{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \mathbb{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mathbb{I} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \mathbb{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \mathbb{I} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \mathbb{I} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

satisfies

$$\Upsilon^\dagger = \Upsilon^{-1} = \Upsilon^\top$$

and has the following properties

Proposition 11. *The next* eigen-value *type relations hold true*

$$\Upsilon\chi(z) = z\chi(z), \qquad\qquad \Upsilon^{-1}\chi(z) = z^{-1}\chi(z), \tag{24}$$

$$\chi(z)^\top\Upsilon^{-1} = z\chi(z)^\top, \qquad \chi(z)^\top\Upsilon = z^{-1}\chi(z)^\top. \tag{25}$$

Proof. It follows from the relations

$$\Lambda_1\chi(z) = z\Pi_1\chi(z), \qquad\qquad \Lambda_2\chi(z) = z^{-1}\Pi_2\chi(z),$$

$$\Lambda_1^\top\chi(z) = \left(z^{-1}\Pi_1 - E_{0,0}\Lambda\right)\chi(z), \qquad \Lambda_2^\top\chi(z) = \left(z\Pi_2 - E_{1,1}\Lambda^\top\right)\chi(z). \qquad \square$$

From these, the following *symmetry* relations are obtained

Proposition 12. *The moment matrices commute with* Υ*; i.e.,*

$$\Upsilon g^H = g^H\Upsilon, \quad H = L, R. \tag{26}$$

Proof. It is a consequence of

$$\Upsilon g^L = \oint_{\mathbb{T}} z\chi(z)\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}\chi(z)^\dagger = \oint_{\mathbb{T}} \chi(z)\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}\left(z^{-1}\chi(z)\right)^\dagger = g^L\Upsilon,$$

$$\Upsilon g^R = \oint_{\mathbb{T}} \overline{z\chi(z)}\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}\chi(z)^\top = \oint_{\mathbb{T}} \overline{\chi(z)}\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}z^{-1}\chi(z)^\top = g^R\Upsilon. \qquad \square$$

We now introduce another important matrix in the CMV theory

Definition 10. The intertwining matrix η is

$$
\eta := \begin{pmatrix}
\mathbb{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \mathbb{I} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \mathbb{I} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \mathbb{I} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \mathbb{I} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I} & \cdots \\
0 & 0 & 0 & 0 & 0 & \mathbb{I} & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
$$

which, as the reader can easily check, has the following properties

$$
\eta^{-1} = \eta, \qquad \eta\chi(z) = \chi\big(z^{-1}\big), \qquad \chi(z)^\top \eta = \chi\big(z^{-1}\big)^\top.
$$

When $z \in \mathbb{T}$, we have that $\eta\chi = \bar{\chi}$ and $\chi^\top \eta = \chi^\dagger$ which lead to the intertwining property

Proposition 13. *The left and right moment matrices satisfy the intertwining type property*

$$
\eta g^R = g^L \eta.
$$

Proof. It is straightforward to realize that

$$
\eta g^L \eta = \oint_{\mathbb{T}} \eta\chi(z) \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} \chi(z)^\dagger \eta = \oint_{\mathbb{T}} \chi(\bar{z}) \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} \chi\big(\bar{z}^{-1}\big)^\top
$$

$$
= \oint_{\mathbb{T}} \overline{\chi(z)} \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} \chi(z)^\top = g^R. \qquad \square
$$

Proposition 14. *The matrices Υ and η are related by*

$$
\eta\Upsilon = \Upsilon^{-1}\eta.
$$

Now we proceed to the *dressing* of Υ and η. We first notice that

Proposition 15. *The following equations hold*

$$
S_1 \Upsilon S_1^{-1} = S_2 \Upsilon S_2^{-1},
$$
$$
Z_1^{-1} \Upsilon Z_1 = Z_2^{-1} \Upsilon Z_2,
$$
$$
Z_2^{-1} \eta \Upsilon^p S_1^{-1} = Z_1^{-1} \eta \Upsilon^p S_2^{-1}, \quad p \in \mathbb{Z}.
$$

Those equations allow us to define

Definition 11. Let us define

$$J^L := S_1 \Upsilon S_1^{-1} = S_2 \Upsilon S_2^{-1}, \qquad J^R := Z_1^{-1} \Upsilon Z_1 = Z_2^{-1} \Upsilon Z_2, \qquad (27)$$

and for any $p \in \mathbb{Z}$, introduce

$$C_{[p]} = Z_2^{-1} \eta \Upsilon^p S_1^{-1} = Z_1^{-1} \eta \Upsilon^p S_2^{-1}. \qquad (28)$$

Observations.

(1)

$$C_{[-|p|]} = Z_2^{-1} \eta \Upsilon^{-|p|} S_1^{-1} = Z_2^{-1} \Upsilon^{|p|} \eta S_1^{-1}.$$

(2) In the Hermitian case,

$$C_{[0]} = Z_2^{-1} \eta S_1^{-1} = Z_1^{-1} \eta S_2^{-1} = D^R \widehat{Z_1}^{-1} \eta \widehat{S_2}^{-1} D^L = D^R Z_2^\dagger \eta S_1^\dagger D^L$$

$$\implies \quad C_{[0]}^\dagger = D^L C_{[0]}^{-1} D^R.$$

Proposition 16. *Powers of J_H can be expressed as follows*

$$\left(J^R \right)^{l-p} = C_{[p]} [C_{[l]}]^{-1}, \qquad \left(J^L \right)^{p-l} = [C_{[l]}]^{-1} C_{[p]}.$$

Now we give the schematic shape of some of these matrices

$$J^H = \begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & * & * & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & * & * & * & * & 0 & 0 & 0 & 0 & \cdots \\ 0 & * & * & * & * & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & * & * & * & * & 0 & 0 & \cdots \\ 0 & 0 & 0 & * & * & * & * & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \left(J^H \right)^{-1} = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & * & * & * & 0 & 0 & 0 & 0 & 0 & \cdots \\ & * & * & * & * & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & * & * & * & * & 0 & 0 & 0 & \cdots \\ 0 & 0 & * & * & * & * & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & * & * & * & * & 0 & \cdots \\ 0 & 0 & 0 & 0 & * & * & * & * & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$
C_{[0]} = \begin{pmatrix} *\,|\,* & 0\,|\,0 & 0\,|\,0 & 0 & \cdots \\ \hline *\,|\,* & *\,|\,0 & 0\,|\,0 & 0 & \cdots \\ 0\,|\,* & *\,|\,* & 0\,|\,0 & 0 & \cdots \\ \hline 0\,|\,0 & *\,|\,* & *\,|\,0 & 0 & \cdots \\ 0\,|\,0 & 0\,|\,* & *\,|\,* & 0 & \cdots \\ \hline 0\,|\,0 & 0\,|\,0 & *\,|\,* & * & \cdots \\ 0\,|\,0 & 0\,|\,0 & 0\,|\,* & * & \cdots \\ \vdots\,|\,\vdots & \vdots\,|\,\vdots & \vdots\,|\,\vdots & \vdots & \ddots \end{pmatrix}, \qquad
C_{[-1]} = \begin{pmatrix} *\,|\,* & 0\,|\,0 & 0\,|\,0 & 0 & \cdots \\ \hline *\,|\,* & *\,|\,0 & 0\,|\,0 & 0 & \cdots \\ 0\,|\,* & *\,|\,* & 0\,|\,0 & 0 & \cdots \\ \hline 0\,|\,0 & *\,|\,* & *\,|\,0 & 0 & \cdots \\ 0\,|\,0 & 0\,|\,* & *\,|\,* & 0 & \cdots \\ \hline 0\,|\,0 & 0\,|\,0 & *\,|\,* & * & \cdots \\ 0\,|\,0 & 0\,|\,0 & 0\,|\,* & * & \cdots \\ \vdots\,|\,\vdots & \vdots\,|\,\vdots & \vdots\,|\,\vdots & \vdots & \ddots \end{pmatrix}.
$$

Here the $*$ are non-zero $m \times m$ blocks that, thanks to the factorization problem, can be written in terms of the Verblunsky coefficients as we will see later. The shape of each matrix is a consequence of the two possible definitions (in terms of upper or lower block-triangular matrices). For the explicit form of these matrices, see Appendix B.

A first consequence is the following relations among Verblunsky coefficients and the *matrix quasi-norms* of the Szegő polynomials

Proposition 17. *The following relations are fulfilled*

$$
\begin{aligned}
h_m^R \big[\alpha_{2,m}^L\big]^\dagger &= \big[\alpha_{2,m}^R\big]^\dagger h_m^L, & h_m^L \alpha_{1,m+1}^R &= \alpha_{1,m+1}^L h_m^R, \\
\alpha_{1,m}^L h_m^R &= h_m^L \alpha_{1,m}^R, & \big[\alpha_{2,m+1}^R\big]^\dagger h_m^L &= h_m^R \big[\alpha_{2,m+1}^L\big]^\dagger, \\
h_k^L &= \big(\mathbb{I} - \alpha_{1,k}^L \big[\alpha_{2,k}^R\big]^\dagger\big) h_{k-1}^L, & h_k^R &= \big(\mathbb{I} - \big[\alpha_{2,k}^R\big]^\dagger \alpha_{1,k}^L\big) h_{k-1}^R, \\
h_k^L &= h_{k-1}^L \big(\mathbb{I} - \alpha_{1,k}^R \big[\alpha_{2,k}^L\big]^\dagger\big), & h_k^R &= h_{k-1}^R \big(\mathbb{I} - \big[\alpha_{2,k}^L\big]^\dagger \alpha_{1,k}^R\big),
\end{aligned}
$$

Proof. Just compare the two possible definitions of $C_{[0]}^{\pm 1}$ and $C_{[-1]}^{\pm 1}$. $\quad\square$

Notice that the two relations in each column coincide in the Hermitian case.

Proposition 18. *The next* eigen-value *properties hold*

$$
\begin{aligned}
J^L \Phi_1^L &= z \Phi_1^L, & \big(J^L\big)^{-1} \Phi_1^L &= z^{-1} \Phi_1^L, \\
\big[J^L\big]^\dagger \Phi_2^L &= z^{-1} \Phi_2^L, & \big(\big[J^L\big]^\dagger\big)^{-1} \Phi_2^L &= z \Phi_2^L, \\
\Phi_2^R \big[J^R\big]^\dagger &= z \Phi_2^R, & \Phi_2^R \big(\big[J^R\big]^\dagger\big)^{-1} &= z^{-1} \Phi_2^R, \\
\Phi_1^R J^R &= z^{-1} \Phi_1^R, & \Phi_1^R (J^R)^{-1} &= z \Phi_1^R,
\end{aligned}
$$

and the following properties are fulfilled

$$
C_{[p]} \Phi_1^L(z) = z^p \big(\Phi_2^R\big(\bar{z}^{-1}\big)\big)^\dagger, \qquad \Phi_1^R(z) C_{[p]} = z^p \big(\Phi_2^L\big(\bar{z}^{-1}\big)\big)^\dagger.
$$

Proof. The results follow directly from the action of $\Upsilon^{\pm 1}$ and η on χ and the definitions of $J^H, C_{[p]}$ and Φ_i^H. For example

$$J^L\Phi_1^L = S_1\Upsilon S_1^{-1}S_1\chi(z) = S_1\Upsilon\chi(z) = zS_1\chi(z) = z\Phi_1^L,$$

$$C_{[p]}\Phi_1^L = Z_2^{-1}\eta\Upsilon^p S_1^{-1}S_1\chi(z) = Z_2^{-1}\eta z^p\chi(z) = z^p Z_2^{-1}\chi(z^{-1}) = z^p\big(\Phi_2^R(\bar{z}^{-1})\big)^\dagger.$$

For the remaining relations, one proceeds in a similar way. $\square$

This last proposition implies

Proposition 19. *The following recursion relations for the left Laurent polynomials hold*

$$z\big(\varphi_1^L\big)^{(2k)} = -\alpha_{1,2k+1}^L\big(\mathbb{I} - \big[\alpha_{2,2k}^R\big]^\dagger\alpha_{1,2k}^L\big)\big(\varphi_1^L\big)^{(2k-1)} - \alpha_{1,2k+1}^L\big[\alpha_{2,2k}^R\big]^\dagger\big(\varphi_1^L\big)^{(2k)}$$
$$- \alpha_{1,2k+2}^L\big(\varphi_1^L\big)^{(2k+1)} + \big(\varphi_1^L\big)^{(2k+2)},$$

$$z\big(\varphi_1^L\big)^{(2k+1)} = \big(\mathbb{I} - \big[\alpha_{2,2k+1}^R\big]^\dagger\alpha_{1,2k+1}^L\big)\big(\mathbb{I} - \big[\alpha_{2,2k}^R\big]^\dagger\alpha_{1,2k}^L\big)\big(\varphi_1^L\big)^{(2k-1)}$$
$$+ \big(\mathbb{I} - \big[\alpha_{2,2k+1}^R\big]^\dagger\alpha_{1,2k+1}^L\big)\big[\alpha_{2,2k}^R\big]^\dagger\big(\varphi_1^L\big)^{(2k)}$$
$$- \big[\alpha_{2,2k+1}^R\big]^\dagger\alpha_{1,2k+2}^L\big(\varphi_1^L\big)^{(2k+1)} + \big[\alpha_{2,2k+1}^R\big]^\dagger\big(\varphi_1^L\big)^{(2k+2)},$$

$$z\big(\varphi_2^L\big)^{(2k)}(z) = -\alpha_{2,2k+1}^R\big(\varphi_2^L\big)^{(2k-1)}(z) - \alpha_{2,2k+1}^R\big[\alpha_{1,2k}^L\big]^\dagger\big(\varphi_2^L\big)^{(2k)}$$
$$- \big(\mathbb{I} - \alpha_{2,2k+1}^R\big[\alpha_{1,2k+1}^L\big]^\dagger\big)\big(\varphi_2^L\big)^{(2k+1)}(z)$$
$$+ \big(\mathbb{I} - \alpha_{2,2k+1}^R\big[\alpha_{1,2k+1}^L\big]^\dagger\big)\big(\mathbb{I} - \alpha_{2,2k+2}^R\big[\alpha_{1,2k+2}^L\big]^\dagger\big)\big(\varphi_2^L\big)^{(2k+2)}(z),$$

$$z\big(\varphi_2^L\big)^{(2k+1)}(z) = \big(\varphi_2^L\big)^{(2k-1)}(z) + \big[\alpha_{1,2k}^L\big]^\dagger\big(\varphi_2^L\big)^{(2k)}(z) - \big[\alpha_{1,2k+1}^L\big]^\dagger\alpha_{2,2k+2}^R\big(\varphi_2^L\big)^{(2k+1)}(z)$$
$$+ \big[\alpha_{1,2k+1}^L\big]^\dagger\big(\mathbb{I} - \alpha_{2,2k+2}^R\big[\alpha_{1,2k+2}^L\big]^\dagger\big)\big(\varphi_2^L\big)^{(2k+2)}(z),$$

while for the right polynomials, relations are

$$z\big(\varphi_1^R\big)^{(2k)} = -\big(\varphi_1^R\big)^{(2k-1)}\alpha_{1,2k+1}^L - \big(\varphi_1^R\big)^{(2k)}\big[\alpha_{2,2k}^R\big]^\dagger\alpha_{1,2k+1}^L$$
$$- \big(\varphi_1^R\big)^{(2k+1)}\alpha_{1,2k+2}^L\big(\mathbb{I} - \big[\alpha_{2,2k+1}^R\big]^\dagger\alpha_{1,2k+1}^L\big)$$
$$+ \big(\varphi_1^R\big)^{(2k+2)}\big(\mathbb{I} - \big[\alpha_{2,2k+2}^R\big]^\dagger\alpha_{1,2k+2}^L\big)\big(\mathbb{I} - \big[\alpha_{2,2k+1}^R\big]^\dagger\alpha_{1,2k+1}^L\big),$$

$$z\big(\varphi_1^R\big)^{(2k+1)} = \big(\varphi_1^R\big)^{(2k-1)} + \big(\varphi_1^R\big)^{(2k)}\big[\alpha_{2,2k}^R\big]^\dagger - \big(\varphi_1^R\big)^{(2k+1)}\alpha_{1,2k+2}^L\big[\alpha_{2,2k+1}^R\big]^\dagger$$
$$+ \big(\varphi_1^R\big)^{(2k+2)}\big(\mathbb{I} - \big[\alpha_{2,2k+2}^R\big]^\dagger\alpha_{1,2k+2}^L\big)\big[\alpha_{2,2k+1}^R\big]^\dagger,$$

$$z\big(\varphi_2^R\big)^{(2k)} = -\big(\varphi_2^R\big)^{(2k-1)}\big(\mathbb{I} - \alpha_{2,2k}^R\big[\alpha_{1,2k}^L\big]^\dagger\big)\alpha_{2,2k+1}^R - \big(\varphi_2^R\big)^{(2k)}\big[\alpha_{1,2k}^L\big]^\dagger\alpha_{2,2k+1}^R$$
$$- \big(\varphi_2^R\big)^{(2k+1)}\alpha_{2,2k+2}^R + \big(\varphi_2^R\big)^{(2k+2)},$$

$$z\big(\varphi_2^R\big)^{(2k+1)} = \big(\varphi_2^R\big)^{(2k-1)}\big(\mathbb{I} - \alpha_{2,2k}^R\big[\alpha_{1,2k}^L\big]^\dagger\big)\big(\mathbb{I} - \alpha_{2,2k+1}^R\big[\alpha_{1,2k+1}^L\big]^\dagger\big)$$

$$+ \big(\varphi_2^R\big)^{(2k)}\big[\alpha_{1,2k}^L\big]^\dagger\big(\mathbb{I} - \alpha_{2,2k+1}^R\big[\alpha_{1,2k+1}^L\big]^\dagger\big)$$

$$- \big(\varphi_2^R\big)^{(2k+1)}\alpha_{2,2k+1}^R\big[\alpha_{1,2k+1}^L\big]^\dagger + \big(\varphi_2^R\big)^{(2k+2)}\big[\alpha_{1,2k+1}^L\big]^\dagger.$$

We have written down just the recursion relations for z and not those for z^{-1}, which can be derived similarly to these ones. For the complete recursion expressions, see Appendix C.

Proposition 20. *The following relations hold true*

$$\big(\big(\varphi_2^R\big)^{(2k)}\big(\bar{z}^{-1}\big)\big)^\dagger = \big(\mathbb{I} - \big[\alpha_{2,2k}^R\big]^\dagger\alpha_{1,2k}^L\big)\big(\varphi_1^L\big)^{(2k-1)}(z) + \big[\alpha_{2,2k}^R\big]^\dagger\big(\varphi_1^L\big)^{(2k)}(z),$$

$$\big(\big(\varphi_2^R\big)^{(2k+1)}\big(\bar{z}^{-1}\big)\big)^\dagger = -\alpha_{1,2k+2}^L\big(\varphi_1^L\big)^{(2k+1)}(z) + \big(\varphi_1^L\big)^{(2k+2)}(z),$$

$$\big(\big(\varphi_2^L\big)^{(2k)}\big(\bar{z}^{-1}\big)\big)^\dagger = \big(\varphi_1^R\big)^{(2k-1)}(z) + \big(\varphi_1^R\big)^{(2k)}(z)\big[\alpha_{2,2k}^R\big]^\dagger,$$

$$\big(\big(\varphi_2^L\big)^{(2k+1)}\big(\bar{z}^{-1}\big)\big)^\dagger$$

$$= -\big(\varphi_1^R\big)^{(2k+1)}(z)\alpha_{1,2k+2}^L + \big(\varphi_1^R\big)^{(2k+2)}(z)\big(\mathbb{I} - \big[\alpha_{2,2k+2}^R\big]^\dagger\alpha_{1,2k+2}^L\big),$$

$$\frac{1}{z}\big(\big(\varphi_2^R\big)^{(2k)}\big(\bar{z}^{-1}\big)\big)^\dagger = -\big[\alpha_{2,2k+1}^R\big]^\dagger\big(\varphi_1^L\big)^{(2k)}(z) + \big(\varphi_1^L\big)^{(2k+1)}(z),$$

$$\frac{1}{z}\big(\big(\varphi_2^R\big)^{(2k+1)}\big(\bar{z}^{-1}\big)\big)^\dagger$$

$$= \big(\mathbb{I} - \alpha_{1,2k+1}^L\big[\alpha_{2,2k+1}^R\big]^\dagger\big)\big(\varphi_1^L\big)^{(2k)}(z) + \alpha_{1,2k+1}^L\big(\varphi_1^L\big)^{(2k+1)}(z),$$

$$\frac{1}{z}\big(\big(\varphi_2^L\big)^{(2k+1)}\big(\bar{z}^{-1}\big)\big)^\dagger = \big(\varphi_1^R\big)^{(2k)}(z) + \big(\varphi_1^R\big)^{(2k)}(z)\alpha_{1,2k+1}^L,$$

$$\frac{1}{z}\big(\big(\varphi_2^L\big)^{(2k)}\big(\bar{z}^{-1}\big)\big)^\dagger$$

$$= -\big(\varphi_1^R\big)^{(2k)}(z)\big[\alpha_{2,2k+1}^R\big]^\dagger + \big(\varphi_1^R\big)^{(2k+1)}(z)\big(\mathbb{I} - \alpha_{1,2k+1}^L\big[\alpha_{2,2k+1}^R\big]^\dagger\big).$$

Proof. These relations appear just by substituting into (18) the expressions of the blocks of $(J_H)^{\pm 1}$, $C_{[0]}$, $C_{[-1]}$. □

Using Proposition 19 and the matrix CMV recursion relations in Proposition 20, one derives the recursion relations for the matrix Szegő polynomials:

$$zP_{1,2l+1}^L(z) - P_{1,2l+2}^L(z) = -\alpha_{1,2l+2}^L\big(P_{2,2l+1}^R(z)\big)^*$$

$$\big(P_{2,2l}^R(z)\big)^* - \big(\mathbb{I} - \big(\alpha_{2,2l}^R\big)^\dagger\alpha_{1,2l}^L\big)\big(P_{2,2l-1}^R(z)\big)^* = \big(\alpha_{2,2l}^R\big)^\dagger P_{1,2l}^L(z)$$

$$\big(P_{2,2l}^L(z)\big)^* - \big(P_{2,2l-1}^L(z)\big)^*\big(\mathbb{I} - \alpha_{1,2l}^R\big(\alpha_{2,2l}^L\big)^\dagger\big) = P_{1,2l}^R(z)\big(\alpha_{2,2l}^L\big)^\dagger$$

$$zP_{1,2l+1}^R(z) - P_{1,2l+2}^R(z) = -\big(P_{2,2l+1}^L(z)\big)^*\alpha_{1,2l+2}^L$$

$$\big(P_{2,2l+1}^R(z)\big)^* - \big(P_{2,2l}^R(z)\big)^* = \big(\alpha_{2,2l+1}^R\big)^\dagger zP_{1,2l}^L(z)$$

$$P_{1,2l+1}^L(z) - \left(\mathbb{I} - \alpha_{1,2l+1}^L\left(\alpha_{2,2l+1}^R\right)^\dagger\right)zP_{1,2l}^L(z) = \alpha_{1,2l+1}^L\left(P_{2,2l+1}^R(z)\right)^*$$

$$P_{1,2l+1}^R(z) - zP_{1,2l}^R(z)\left(\mathbb{I} - \left(\alpha_{2,2l+1}^L\right)^\dagger\alpha_{1,2l+1}^R\right) = \left(P_{2,2l+1}^L(z)\right)^*\alpha_{1,2l+1}^R$$

$$\left(P_{2,2l}^L(z)\right)^* - \left(P_{2,2l+1}^L(z)\right)^* = -zP_{1,2l}^R(z)\left(\alpha_{2,2l+1}^L\right)^\dagger$$

which after the prescription

$$x_N^l := \alpha_{1,N}^L \qquad x_N^r := \alpha_{1,N}^R$$

$$y_N^l := \left(\alpha_{2,N}^L\right)^\dagger \qquad y_N^r := \left(\alpha_{2,N}^R\right)^\dagger$$

coincide with the formulae in [27].

2.5. Christoffel–Darboux theory

To conclude this section, we show how the Gaussian factorization leads to the Christoffel–Darboux theorem for the matrix Laurent polynomials on the unit circle context. In this particular situation we must consider two different cases. As we are working in a non-Abelian situation, we first have projections in the corresponding modules, "orthogonal" in the ring (our blocks) context. Secondly, when the matrix measures are Hermitian and positive definite, we will have a scalar product, and the projections to consider are orthogonal indeed.

2.5.1. Projections in modules

Given a right or left $\mathbb{M}_m$ module M, any idempotent endomorphism $\pi \in \text{End}_{\mathbb{M}_m}(M)$, $\pi^2 = \pi$, is called a projection. For any given projection π, we have $\text{Ker}\,\pi = \text{Im}(1-\pi)$, $\text{Ker}(1-\pi) = \text{Im}\,\pi$, and the following direct decomposition holds: $M = \text{Im}\,\pi \oplus \text{Im}(1-\pi)$. Two projections π and π' are said to be orthogonal if $\pi\pi' = 0$; observe that $(1-\pi)$ is idempotent and moreover orthogonal to π. Orthogonality is not related here to any inner product so far, it is just a construction in the module. In particular, in our discussion of matrix Laurent polynomials, we introduce the following free modules

$$\Lambda_{[l]} := \mathbb{M}_m\left\{\chi^{(j)}\right\}_{j=0}^l = \begin{cases} \Lambda_{m,[-k,k]}, & l = 2k, \\ \Lambda_{m,[-k-1,k]}, & l = 2k+1. \end{cases}$$

That we can consider as a left free module, when multiplied by the left, and denoted by $V_{[l+1]}$, or as a right free module (when multiplication by matrices is performed by the right) and denoted by $W_{[l+1]}$. We will denote by $V = \varinjlim V_{[l]}$ and $W = \varinjlim W_{[l]}$ the corresponding direct limits, the left and right modules of matrix Laurent polynomials. The bilinear form

$$G(f,g) = \left\langle\!\left\langle g^\dagger, f\right\rangle\!\right\rangle_L = \left\langle\!\left\langle f^\dagger, g\right\rangle\!\right\rangle_R = \oint_{\mathbb{T}} f(z)\frac{\mathrm{d}\mu(z)}{\mathrm{i}z}g(z),$$

$$G_{i,j} := \oint_{\mathbb{T}} \chi^{(i)}(z) \frac{\mathrm{d}\mu(z)}{\mathrm{i}z} \big(\chi^{(j)}(z)\big)^\top,$$

fulfills

$$G = \eta g^R = g^L \eta.$$

This can be understood as a change of basis in the left and right modules $W_{[l]}$ and $V_{[l]}$; the left moment matrix can be understood as the matrix of the bilinear form G when on the left module $W_{[l]}$ we apply the isomorphism or change of basis represented by the η matrix. Similarly, the right moment matrix can be understood as the matrix of the bilinear form G when on the right module $V_{[l]}$ we apply the isomorphism represented by the η matrix. Observe that the G dual vectors introduced in Appendix D are of the form

$$\big((\varphi_1^L)_j\big)^* = (\varphi_2^L)_j^\dagger, \qquad \big((\varphi_2^L)_j^\dagger\big)^* = (\varphi_1^L)_j,$$
$$\big((\varphi_1^R)_j\big)^* = (\varphi_2^R)_j^\dagger, \qquad \big((\varphi_2^R)_j^\dagger\big)^* = (\varphi_1^R)_j.$$

Thus, following Appendix D, we consider the ring of G projections in these left and right modules

Definition 12.

(1) The Christoffel–Darboux projectors

$$\pi_L^{[l]} : V \longrightarrow V_{[l]}, \qquad \pi_R^{[l]} : W \longrightarrow W_{[l]},$$

are the ring left and right projections associated to the bilinear form G.

(2) The matrix Christoffel–Darboux kernels are

$$K^{L,[l]}(z, z') := \sum_{k=0}^{l-1} \big[(\varphi_2^L)^{(k)}(z)\big]^\dagger (\varphi_1^L)^{(k)}(z'),$$

$$K^{R,[l]}(z, z') := \sum_{k=0}^{l-1} (\varphi_1^R)^{(k)}(z) \big[(\varphi_2^R)^{(k)}(z')\big]^\dagger. \tag{29}$$

Proposition 21. *For the projections and matrix Christoffel–Darboux kernels introduced in Definition 12, we have the following relations*

$$\big(\pi_L^{[l]} f\big)(z) = \int_{\mathbb{T}} f(z') \frac{\mathrm{d}\mu(z')}{\mathrm{i}z'} K^{L,[l]}(z', z) = \sum_{k=0}^{l-1} \big\langle\!\big\langle (\varphi_2^L)^{(k)}, f \big\rangle\!\big\rangle_L (\varphi_1^L)^{(k)}(z),$$

$$\forall f \in V,$$

$$\left[\left(\pi_L^{[l]}f\right)(z)\right]^{\dagger} = \oint_{\mathbb{T}} K^{L,[l]}(z,z')\frac{\mathrm{d}\mu(z')}{\mathrm{i}z'}\left[f(z')\right]^{\dagger} = \sum_{k=0}^{l-1}\left[\left(\varphi_2^L\right)^{(k)}(z)\right]^{\dagger}\left\langle\!\left\langle f,\left(\varphi_1^L\right)^{(k)}\right\rangle\!\right\rangle_L,$$

$$\forall f \in V,$$

$$\left(\pi_R^{[l]}f\right)(z) = \oint_{\mathbb{T}} K^{R,[l]}(z,z')\frac{\mathrm{d}\mu(z')}{\mathrm{i}z'}f(z') = \sum_{k=0}^{l-1}\left(\varphi_1^R\right)^{(k)}(z)\left\langle\!\left\langle\left(\varphi_2^R\right)^{(k)},f\right\rangle\!\right\rangle_R,$$

$$\forall f \in W,$$

$$\left[\left(\pi_R^{[l]}f\right)(z)\right]^{\dagger} = \oint_{\mathbb{T}}\left[f(z')\right]^{\dagger}\frac{\mathrm{d}\mu(z')}{\mathrm{i}z}K^{R,[l]}(z',z) = \sum_{k=0}^{l-1}\left\langle\!\left\langle f,\left(\varphi_1^R\right)^{(k)}\right\rangle\!\right\rangle_R\left[\left(\varphi_2^R\right)^{(k)}(z)\right]^{\dagger},$$

$$\forall f \in W.$$

Proposition 22. *The Christoffel–Darboux kernels have the reproducing property*

$$K^{H,[l]}(z,y) = \oint_{\mathbb{T}} K^{H,[l]}(z,z')\frac{\mathrm{d}\mu(z')}{\mathrm{i}z'}K^{H,[l]}(z',y), \quad H = L, R. \tag{30}$$

Proof. This follows from the idempotency property of the π's. $\quad\square$

Moreover,

Proposition 23. *If the matrix measure μ is Hermitian, then*

(1) *the followings expansions are satisfied*

$$\left(\pi_L^{[l]}f\right)(z) = \sum_{k=0}^{l-1}\left\langle\!\left\langle\left(\varphi_1^L\right)^{(k)},f\right\rangle\!\right\rangle_L\left(h_k^L\right)^{-1}\left(\varphi_1^L\right)^{(k)}(z), \quad \forall f \in V,$$

$$\left(\pi_R^{[l]}f\right)(z) = \sum_{k=0}^{l-1}\left(\varphi_1^R\right)^{(k)}(z)\left(h_k^R\right)^{-1}\left\langle\!\left\langle\left(\varphi_1^R\right)^{(k)},f\right\rangle\!\right\rangle_R, \quad \forall f \in W;$$

(2) *the following Hermitian type property holds for the projectors*

$$\left\langle\!\left\langle \pi_H^{[l]}f, g\right\rangle\!\right\rangle_H = \left\langle\!\left\langle f, \pi_H^{[l]}g\right\rangle\!\right\rangle_H, \quad H = R, L.$$

When the matrix measure is Hermitian and positive definite, we have a standard scalar product and a complex Hilbert space, and the projections $\pi_H^{[l]}$ are orthogonal projections — not only in the module but in the geometrical sense as well — to the subspaces of truncated matrix Laurent polynomials; notice that there are two different, however equivalent, scalar products and distances involved. In this situation, as is well known, these projections give the best approximation within the truncated Laurent polynomials and the corresponding left and right distances.

2.5.2. The Christoffel–Darboux type formulae

Theorem 2. *For $\bar{z}z' \neq 1$, the matrix Christoffel–Darboux kernels fulfill*

$$
K^{L,[2l]}(z,z')(1 - \bar{z}z')
$$
$$
= (\varphi_1^R)^{(2l)}(\bar{z}^{-1})h_{2l}^R(h_{2l-1}^R)^{-1}(\varphi_1^L)^{(2l-1)}(z') - (\varphi_1^R)^{(2l-1)}(\bar{z}^{-1})(\varphi_1^L)^{(2l)}(z')
$$
$$
= \big[[z(\varphi_2^L)^{(2l+1)}(z)]^\dagger h_{2l+1}^R - [z(\varphi_2^L)^{(2l)}(z)]^\dagger h_{2l}^L \alpha_{1,2l+1}^R\big](h_{2l-1}^R)^{-1}(\varphi_1^L)^{(2l-1)}(z')
$$
$$
\quad - \big[[z(\varphi_2^L)^{(2l-2)}(z)]^\dagger + [z(\varphi_2^L)^{(2l-1)}(z)]^\dagger[\alpha_{2,2l-1}^R]^\dagger\big](\varphi_1^L)^{(2l)}(z'),
$$
$$
K^{L,[2l+1]}(z,z')(1 - \bar{z}z')
$$
$$
= [z(\varphi_2^L)^{(2l+1)}(z)]^\dagger h_{2l+1}^R(h_{2l}^R)^{-1}[(\varphi_2^R)^{(2l)}]^\dagger(z'^{-1})
$$
$$
\quad - [z(\varphi_2^L)^{(2l)}(z)]^\dagger[(\varphi_2^R)^{(2l+1)}]^\dagger(z'^{-1})
$$
$$
= [z(\varphi_2^L)^{(2l+1)}(z)]^\dagger h_{2l+1}^R\big[(h_{2l-1}^R)^{-1}(\varphi_1^L)^{(2l-1)}(z')
$$
$$
\quad + [\alpha_{2,2l}^L]^\dagger(h_{2l}^L)^{-1}(\varphi_1^L)^{(2l)}(z')\big]
$$
$$
\quad - [z(\varphi_2^L)^{(2l)}(z)]^\dagger\big[(\varphi_1^L)^{(2l+2)}(z') - \alpha_{1,2l+2}^L(\varphi_1^L)^{(2l+1)}(z')\big], \tag{31}
$$
$$
K^{R,[2l]}(z,z')(1 - \bar{z}'z)
$$
$$
= [(\varphi_2^L)^{(2l)}]^\dagger(z^{-1})h_{2l}^L(h_{2l-1}^L)^{-1}[(\varphi_2^R)^{(2l-1)}(z')]^\dagger
$$
$$
\quad - [(\varphi_2^L)^{(2l-1)}]^\dagger(z^{-1})[(\varphi_2^R)^{(2l)}(z')]^\dagger
$$
$$
= z\big[(\varphi_1^R)^{(2l+1)}(z)h_{2l+1}^L - (\varphi_1^R)^{(2l)}(z)h_{2l}^R[\alpha_{2,2l+1}^L]^\dagger\big](h_{2l+1}^L)^{-1}[(\varphi_2^R)^{(2l-1)}(z')]^\dagger
$$
$$
\quad - z\big[(\varphi_1^R)^{(2l-2)}(z) - (\varphi_1^R)^{(2l-1)}(z)\alpha_{1,2l-1}^L\big][(\varphi_2^R)^{(2l)}(z')]^\dagger,
$$
$$
K^{R,[2l+1]}(z,z')(1 - \bar{z}'z)
$$
$$
= z(\varphi_1^R)^{(2l+1)}(z)h_{2l+1}^L(h_{2l}^L)^{-1}(\varphi_1^L)^{(2l-1)}(\bar{z}'^{-1}) - z(\varphi_1^R)^{(2l)}(z)(\varphi_1^L)^{(2l+1)}(\bar{z}'^{-1})
$$
$$
= z(\varphi_1^R)^{(2l+1)}(z)h_{2l+1}^L\big[(h_{2l-1}^L)^{-1}(\varphi_2^R)^{(2l-1)}(z')\big]^\dagger + \alpha_{1,2l}^R(h_{2l}^R)^{-1}[(\varphi_2^R)^{(2l)}(z')]^\dagger
$$
$$
\quad - z(\varphi_1^R)^{(2l)}(z)\big[(\varphi_2^R)^{(2l+2)}(z')\big]^\dagger - [\alpha_{2,2l+2}^R]^\dagger[(\varphi_2^R)^{(2l+1)}(z')]^\dagger. \tag{32}
$$

Proof. See Appendix A. □

In terms of the matrix Szegő polynomials, we have

Corollary 2. *The matrix Christoffel–Darboux kernels can be expressed in terms of the matrix Szegő polynomials as follows*

$$
K^{L,[2l]}(z,z') = \frac{(\bar{z}z'^{-1})^l}{1 - \bar{z}z'}\big[P_{1,2l}^R(\bar{z}^{-1})(h_{2l-1}^R)^{-1}(P_{2,2l-1}^R)^*(z')
$$
$$
\quad - (P_{2,2l-1}^L)^*(\bar{z}^{-1})(h_{2l-1}^L)^{-1}P_{1,2l}^L(z')\big],
$$

$$K^{L,[2l+1]}(z,z') = \frac{\bar{z}^{l+1}z'^{-l}}{1-\bar{z}z'}\big[P_{1,2l+1}^R(\bar{z}^{-1})(h_{2l}^R)^{-1}(P_{2,2l}^R)^*(z')$$
$$- (P_{2,2l}^L)^*(z^{-1})(h_{2l}^L)^{-1}P_{1,2l+1}^L(z')\big],$$

$$K^{R,[2l]}(z,z') = \frac{(z\bar{z}'^{-1})^l}{1-\bar{z}'z}\big[[P_{2,2l}^L(\bar{z}^{-1})]^\dagger(h_{2l-1}^L)^{-1}[(P_{1,2l-1}^L)^*(z')]^\dagger$$
$$- [(P_{1,2l-1}^R)^*(\bar{z}^{-1})]^\dagger(h_{2l-1}^R)^{-1}[P_{2,2l}^R(z')]^\dagger\big],$$

$$K^{R,[2l+1]}(z,z') = \frac{z^{l+1}\bar{z}'^{-l}}{1-\bar{z}'z}\big[[P_{2,2l+1}^L(\bar{z}^{-1})]^\dagger(h_{2l}^L)^{-1}[(P_{1,2l}^L)^*(z')]^\dagger$$
$$- [(P_{1,2l}^R)^*(\bar{z}^{-1})]^\dagger(h_{2l}^R)^{-1}[P_{1,2l+1}^R(z')]^\dagger\big],$$

where we assume that $\bar{z}z' \neq 1$.

As we have just seen, letting an operator act to the left or to the right and comparing the two results has been very successful with J_K. Actually we still have the operators C_0, C_{-1} to which we can also apply the same procedure to get some other interesting relations for the CD kernels.

Proposition 24. *The next relations between K^L and K^R hold*

$$K^{R,[2l+1]}\left(z,\frac{1}{\bar{z}'}\right) = K^{L,[2l+1]}\left(\frac{1}{\bar{z}},z'\right),$$

$$K^{R,[2l+2]}\left(z,\frac{1}{\bar{z}'}\right) - K^{L,[2l+2]}\left(\frac{1}{\bar{z}},z'\right)$$
$$= (\varphi_1^R)^{(2k+1)}(z)(\varphi_1^L)^{(2k+2)}(z')$$
$$- (\varphi_1^R)^{(2k+2)}(z)\big(\mathbb{I} - [\alpha_{2,2k+2}^R]^\dagger\alpha_{1,2k+2}^L\big)(\varphi_1^L)^{(2k+1)}(z'),$$

$$\frac{1}{z'}K^{R,[2l+1]}\left(z,\frac{1}{\bar{z}'}\right) - \frac{1}{z}K^{L,[2l+1]}\left(\frac{1}{\bar{z}},z'\right)$$
$$= (\varphi_1^R)^{(2k)}(z)(\varphi_1^L)^{(2k+1)}(z')$$
$$- (\varphi_1^R)^{(2k+1)}(z)\big(\mathbb{I} - \alpha_{1,2k+1}^L[\alpha_{2,2k+1}^R]^\dagger\big)(\varphi_1^L)^{(2k)}(z'),$$

$$\frac{1}{z'}K^{R,[2l+2]}\left(z,\frac{1}{\bar{z}'}\right) = \frac{1}{z}K^{L,[2l+2]}\left(\frac{1}{\bar{z}},z'\right).$$

Proof. The first two relations arise when comparing the action of C_0 to the left or to the right in

$$\phi_1^R(z)C_0\phi_1^L(z')$$

and truncating the expressions up to $[2k+1]$ (first relation) or $[2k+2]$ (second one). The other two relations are obtained proceeding in the same way but using C_{-1} instead. $\square$

3. MOLPUC and two dimensional Toda type hierarchies

Once we have explored how the Gauss–Borel factorization of block CMV moment matrices leads to the algebraic theory of MOLPUC, we are ready to show how this approach also connects these polynomials to integrable hierarchies of Toda type. We first introduce convenient deformations of the moment matrices, that as we will show correspond to deformations of the matrix measure. With these we will construct wave functions, Lax equations, Zakharov–Shabat equations, discrete flows and Darboux transformations and Miwa transformations. These last transformations will lead to interesting relations between the matrix Christoffel–Darboux kernels, Miwa shifted MOLPUC and their "norms". The integrable equations that we derive are a non-Abelian version of the Toeplitz lattice or non-Abelian ALL equations that extend, in the partial flows case, those of [27] — appearing these last ones in what we denominate total flows.

3.1. 2D Toda continuous flows

In order to construct deformation matrices which will act on the moment matrices (resulting in a deformation of the matrix measure) we first introduce some definitions.

Definition 13.

(1) Given the diagonal matrices $t_j^H = \mathrm{diag}(t_{j,1}^H, \ldots, t_{j,m}^H) \in \mathrm{diag}_m$, $j = 0, 1, 2, \ldots$, $H = L, R$ and $t_{j,a}^H \in \mathbb{C}$, we introduce

$$t^L := \big(t_0^L, t_1^L, t_2^L, \ldots \big), \qquad t^H := \big(t_0^R, t_1^R, t_2^R, \ldots \big)^\top,$$

we also impose $t_0^H = 0$.

(2) We also consider the CMV ordered Fourier monomial vector but evaluated in Υ

$$\big[\chi(\Upsilon) \big]^\top = \big(\mathbb{I}, \Upsilon, \Upsilon^{-1}, \Upsilon^2, \Upsilon^{-2}, \ldots \big).$$

(3) With this we introduce

$$t^L * \chi(\Upsilon) := \sum_{j=0}^\infty t_j^L \chi(\Upsilon)^{(j)}, \qquad \big[\chi(\Upsilon) \big]^\top * t^R := \sum_{j=0}^\infty \big[\chi(\Upsilon)^{(j)} \big]^\top t_j^R.$$

The products in the above expressions are by blocks; i.e., the factors in $\mathbb{M}_m$ multiply $\mathbb{M}_m$ block of the $\mathbb{M}_\infty$ block matrix.

(4) The deformation matrices are

$$W_0\big(t^L\big) := \exp\big(t^L * \chi(\Upsilon)\big), \qquad V_0\big(t^R\big) := \exp\big(\big[\chi(\Upsilon)\big]^\top * t^R\big).$$

(5) The t-dependent deformation of moment matrices, $g^H(t)$, $H = L, R$, and their Gauss–Borel factorization are considered

$$g^L(t) := W_0\big(t^L\big)g^L\big[V_0\big(-\eta t^R\big)\big]^{-1}, \qquad g^L(t) = \big(S_1(t)\big)^{-1}S_2(t), \qquad (33)$$

$$g^R(t) := \big[W_0\big(-t^L\eta\big)\big]^{-1}g^R V_0\big(t^R\big), \qquad g^R(t) = Z_2(t)\big(Z_1(t)\big)^{-1}. \qquad (34)$$

Proposition 25.

(1) *The deformed moment matrices can be understood as the moment matrices for a deformed (t-dependent) measure given by*

$$\mathrm{d}\mu(z,t) := \exp\big(t_L\chi(z)\big)\mathrm{d}\mu(z)\exp\big(\chi(z)^\top t_R\big),$$

with the deformed Fourier series of the evolved matrix measure given by

$$F_{\mu(t)}(z) := \exp\big(t_L\chi(z)\big)F(z)\exp\big(\chi(z)^\top t_R\big). \qquad (35)$$

(2) *The Hermitian character of the matrix measure is preserved by the deformation whenever $t^L = (t^R)^\dagger \eta$.*

Observe that in this paper we introduce a slightly different set of flows or deformations of the measure than those in the scalar case [13]. Despite that in that scalar situation both definitions give the very same flows that is not the case in this non-Abelian scenario, as in this case we have deformation matrices multiplying at the left and right of the initial matrix measure, and the order is relevant.

3.1.1. The Gauss–Borel approach to integrability
We consider the elements that enable us to construct the integrable hierarchy

Definition 14.

(1) Left and right wave matrices

$$W_1^L(t) := S_1(t)W_0\big(t^L\big), \qquad W_2^L(t) := S_2(t)V_0\big(-\eta t^R\big),$$
$$W_1^R(t) := V_0\big(t^R\big)Z_1(t), \qquad W_2^R(t) := W_0\big(-t^L\eta\big)Z_2(t). \qquad (36)$$

(2) Left and right wave and adjoint wave functions

$$\Psi_1^L(z,t) = W_1^L(t)\chi(z), \qquad \big(\Psi_1^L\big)^*(z,t) = \big[\big(W_1^L\big)^{-1}(t)\big]^\dagger \chi^*(z),$$
$$\Psi_2^L(z,t) = W_2^L(t)\chi^*(z), \qquad \big(\Psi_2^L\big)^*(z,t) = \big[\big(W_2^L\big)^{-1}(t)\big]^\dagger \chi(z),$$
$$\Psi_1^R(z,t) = \chi(z)^\top W_1^R(t), \qquad \big(\Psi_1^R\big)^*(z,t) = \chi^*(z)^\top \big[\big(W_1^R\big)^{-1}(t)\big]^\dagger,$$
$$\Psi_2^R(z,t) = \chi^*(z)^\top W_2^R(t), \qquad \big(\Psi_2^R\big)^*(z,t) = \chi(z)^\top \big[\big(W_2^R\big)^{-1}(t)\big]^\dagger.$$

(3) Left and right Jacobi vector of matrices (using our previous notation)

$$\chi\big(J_H(t)\big) := \begin{pmatrix} \mathbb{I} \\ (J^H(t))^{-1} \\ J^H(t) \\ (J^H(t))^{-2} \\ (J^H(t))^2 \\ \vdots \end{pmatrix}, \quad H = L, R.$$

(4) Projection operators, $a = 1, \dots, m$

$$P_a^{(H,H')} := \begin{cases} S_1 E_{aa} S_1^{-1}, & H = L, \ H' = L, \\ S_2 E_{aa} S_2^{-1}, & H = R, \ H' = L, \\ Z_2^{-1} E_{aa} Z_2, & H = L, \ H' = R, \\ Z_1^{-1} E_{aa} Z_1, & H = R, \ H' = R. \end{cases} \tag{37}$$

(5) Left and right Lax matrices

$$\begin{aligned}
L_1(t) &:= S_1(t)\Upsilon S_1(t)^{-1} = S_2(t)\Upsilon S_2(t)^{-1} = J^L(t), \\
R_1(t) &:= Z_1(t)^{-1}\Upsilon^{-1} Z_1(t) = Z_2(t)^{-1}\Upsilon^{-1} Z_2(t) = J^R(t), \tag{38} \\
L_2(t) &:= S_2(t)\Upsilon^{-1} S_2(t)^{-1} = S_1(t)\Upsilon^{-1} S_1(t)^{-1} = \big(J^L(t)\big)^{-1} \\
R_2(t) &:= Z_2(t)^{-1}\Upsilon Z_2(t) = Z_1(t)^{-1}\Upsilon Z_1(t) = \big(J^R(t)\big)^{-1}. \tag{39}
\end{aligned}$$

(6) Zakharov–Shabat matrices

$$B_{j,a}^{(H,H')} := \begin{cases} (S_1 E_{aa}(\chi(\Upsilon))^{(j)} S_1^{-1})_+, & H = L, \ H' = L, \\ -(S_2 E_{aa}(\chi(\Upsilon))^{(j)} S_2^{-1})_-, & H = R, \ H' = L, \\ -(Z_2^{-1} E_{aa}(\chi(\Upsilon^{-1}))^{(j)} Z_2)_+, & H = L, \ H' = R, \\ (Z_1^{-1} E_{aa}(\chi(\Upsilon^{-1}))^{(j)} Z_1)_-, & H = R, \ H' = R, \end{cases}$$

$$B_j^{(H,H')} := \begin{cases} ((\chi(J^L))^{(j)})_+, & H = L, \ H' = L, \\ -((\chi(J^L))^{(j)})_-, & H = R, \ H' = L, \\ -(\chi((J^R)^{-1})^{(j)})_+, & H = L, \ H' = R, \\ (\chi((J^R)^{-1})^{(j)})_-, & H = R, \ H' = R. \end{cases} \tag{40}$$

(7) A time dependent intertwining operator

$$C_{[p]}(t) = Z_2(t)^{-1}\eta\Upsilon^p S_1(t)^{-1} = Z_1(t)^{-1}\eta\Upsilon^p S_2(t)^{-1}. \tag{41}$$

Observe that

$$g^L = \big(W_1^L(t)\big)^{-1} W_2^L(t), \qquad g^R = W_2^R(t)\big(W_1^R\big)^{-1}(t). \tag{42}$$

Definition 15. For $H = R, L$ we introduce the total derivatives

$$\partial_{H,j} := \sum_{a=1}^{m} \frac{\partial}{\partial t_{j,a}^{H}}.$$

We now present the linear systems, Lax equations and Zakharov–Shabat equations that characterize integrability

Proposition 26. *The following equations hold:*

(1) *Linear systems for the wave matrices*

$$\frac{\partial W_i^L}{\partial t_{j,a}^H} = B_{j,a}^{H,L} W_i^L, \qquad \partial_{H,j} W_i^L = B_j^{H,L} W_i^L,$$

$$\frac{\partial W_i^R}{\partial t_{j,a}^H} = W_i^R B_{j,a}^{H,R}, \qquad \partial_{H,j} W_i^R = W_i^R B_j^{H,R},$$

for $i = 1, 2$, $H = L, R$, $a = 1, \ldots, m$, $j = 0, 1, \ldots$.

(2) *Lax equations*

$$\frac{\partial J^{H'}}{\partial t_{j,a}^H} = \left[B_{j,a}^{H,H'}, J^{H'} \right], \qquad \partial_{H,j} J^{H'} = \left[B_j^{H,H'}, J^{H'} \right],$$

$$\frac{\partial P_b^{H',H''}}{\partial t_{j,a}^H} = \left[B_{j,a}^{H,H''}, P_b^{H',H''} \right], \qquad \partial_{H,j} P_b^{H',H''} = \left[B_j^{H,H''}, P_b^{H',H''} \right]$$

with $H, H', H'' = L, R$, $a, b = 1, \ldots, m$ and $j = 0, 1, \ldots$.

(3) *Evolution of the dressed intertwining operator*

$$\frac{\partial C_{[p]}}{\partial t_{j,a}^H} = -B_{j,a}^{H,R} C_{[p]} - C_{[p]} B_{j,a}^{H,L}, \qquad \frac{\partial C_{[p]}}{\partial t_j^H} = -B_j^{H,R} C_{[p]} - C_{[p]} B_j^{H,L}.$$

(4) *Zakharov-Shabat equations*

$$\frac{\partial B_{j_1,b_1}^{H_1,H'}}{\partial t_{j_2,a_2}^{H_2}} - \frac{\partial B_{j_2,b_2}^{H_2,H'}}{\partial t_{j_1,a_1}^{H_1}} + \left[B_{j_1,b_1}^{H_1,H'}, B_{j_2,b_2}^{H_2,H'} \right] = 0.$$

Proof. See Appendix A □

From the definitions of the wave functions, the action of Υ on χ, the expression (35), and the relations (23), it follows that

Proposition 27. *The wave functions are linked to the CMV polynomials and the Fourier series of the measure as follows*

$$\Psi_1^L(z,t) = \phi_1^L(z,t)\exp\left(t^L\chi(z)\right),$$

$$\left(\Psi_1^L\right)^*(z,t) = 2\pi z^{-1}\phi_2^L\left(z^{-1},t\right)F_\mu^\dagger(z)\exp\left(-\overline{t^L}\chi(z)\right),$$

$$\Psi_2^L(z,t) = 2\pi z^{-1}\phi_1^L\left(z^{-1},t\right)F_\mu\left(z^{-1}\right)\exp\left(-\chi\left(z^{-1}\right)^\top t^R\right),$$

$$\left(\Psi_2^L\right)^*(z,t) = \phi_2^L(z,t)\exp\left(\chi\left(z^{-1}\right)^\top \overline{t^R}\right),$$

$$\Psi_1^R(z,t) = \exp\left(\chi(z)^\top t^R\right)\phi_1^R(z,t),$$

$$\left(\Psi_1^R\right)^*(z,t) = 2\pi z^{-1}\exp\left(-\chi(z)^\top \overline{t^R}\right)F_\mu^\dagger(z)\left(\phi_2^R\right)\left(z^{-1},t\right),$$

$$\Psi_2^R(z,t) = 2\pi z^{-1}\exp\left(-t^L\chi\left(z^{-1}\right)\right)F_\mu\left(z^{-1}\right)\phi_1^R\left(z^{-1},t\right),$$

$$\left(\Psi_2^R\right)^*(z,t) = \exp\left(\overline{t^L}\chi\left(z^{-1}\right)\right)\phi_2^R(z,t). \tag{43}$$

These wave functions are also eigen-functions of the Lax matrices (38) L_i, R_i, *for* $i = 1, 2,$

$$L_i\Psi_i^L = z\Psi_i^L, \qquad\qquad \Psi_i^R R_i = z\Psi_i^R,$$

$$L_i^\dagger\left(\Psi_i^L\right)^* = z\left(\Psi_i^L\right)^*, \qquad \left(\Psi_i^R\right)^* R_i^\dagger = z\left(\Psi_i^R\right)^*.$$

3.1.2. CMV matrices and matrix Toeplitz lattice

For the CMV ordering of the Laurent basis, the Lax equations acquire a dynamical non-linear system form that is the matrix version, in the CMV context, of the Toeplitz lattice developed in [3]. In [27] Mattia Cafasso presented a non-Abelian extension of the TL which corresponds to our total flows. The partial flows presented here are, to our knowledge, new in the literature.

Proposition 28. *The Lax equations result in the following non-linear dynamical system for the matrix Verblunsky coefficients* $H = L, R$:

- *Partial flows*

$$\frac{\partial}{\partial t_{1,a}^L}\alpha_{1,k}^R = -\left(h_{k-1}^L\right)^{-1}\alpha_{1,k-1}^L E_{a,a}h_k^R,$$

$$\frac{\partial}{\partial t_{1,a}^L}\left[\alpha_{2,k}^L\right]^\dagger = \left(h_{k-1}^R\right)^{-1}E_{a,a}\left[\alpha_{2,k+1}^R\right]^\dagger h_k^L,$$

$$\frac{\partial}{\partial t_{1,a}^R}\left[\alpha_{2,k}^R\right]^\dagger = h_k^R\left[\alpha_{2,k+1}^L\right]^\dagger E_{a,a}\left(h_{k-1}^L\right)^{-1},$$

$$\frac{\partial}{\partial t_{1,a}^R}\alpha_{1,k}^L = -\left(h_k^L\right)E_{a,a}\alpha_{1,k-1}^R\left(h_{k-1}^R\right)^{-1},$$

$$\frac{\partial}{\partial t_{2,a}^L}\alpha_{1,k}^R = \left(h_{k-1}^L\right)^{-1}E_{a,a}\alpha_{1,k+1}^L h_k^R,$$

$$\frac{\partial}{\partial t_{2,a}^L}\left[\alpha_{2,k}^L\right]^\dagger = -\left(h_{k-1}^R\right)^{-1}\left[\alpha_{2,k-1}^R\right]^\dagger E_{a,a}h_k^L,$$

$$\frac{\partial}{\partial t_{2,a}^R}\big[\alpha_{2,k}^R\big]^\dagger = -h_k^R E_{a,a}\big[\alpha_{2,k-1}^L\big]^\dagger\big(h_{k-1}^L\big)^{-1},$$

$$\frac{\partial}{\partial t_{2,a}^R}\alpha_{1,k}^L = h_k^L \alpha_{1,k+1}^R E_{a,a}\big(h_{k-1}^R\big)^{-1},$$

$$\frac{\partial}{\partial t_{1,a}^L}h_k^L = -\alpha_{1,k}^L E_{a,a}\big(\alpha_{2,k+1}^R\big)^\dagger h_k^L,$$

$$\frac{\partial}{\partial t_{1,a}^R}h_k^R = -h_k^R\big(\alpha_{2,k-1}^L\big)^\dagger E_{a,a}\alpha_{1,k}^R,$$

$$\frac{\partial}{\partial t_{2,a}^L}h_k^R = -\big(\alpha_{2,k}^R\big)^\dagger E_{a,a}\alpha_{1,k+1}^L h_k^R,$$

$$\frac{\partial}{\partial t_{2,a}^R}h_k^L = -h_k^L \alpha_{1,k+1}^R E_{a,a}\big(\alpha_{2,k}^L\big)^\dagger.$$

- *Total flows [27]*

$$\partial_{H,1}\big[\alpha_{2,k}^R\big]^\dagger = \big[\alpha_{2,k+1}^R\big]^\dagger\big(\mathbb{I} - \alpha_{1,k}\big[\alpha_{2,k}^R\big]^\dagger\big),$$

$$\partial_{H,1}\alpha_{1,k}^L = -\big(\mathbb{I} - \alpha_{1,k}^L\big[\alpha_{2,k}^R\big]^\dagger\big)\alpha_{1,k-1}^L,$$

$$\partial_{H,1}\big[\alpha_{2,k}^L\big]^\dagger = \big(\mathbb{I} - \big[\alpha_{2,k}^L\big]^\dagger\alpha_{1,k}^R\big)\big[\alpha_{2,k+1}^L\big]^\dagger,$$

$$\partial_{H,1}\alpha_{1,k}^R = -\alpha_{1,k-1}^R\big(\mathbb{I} - \big[\alpha_{2,k}^L\big]^\dagger\alpha_{1,k}^R\big),$$

$$\partial_{H,2}\big[\alpha_{2,k}^R\big]^\dagger = -\big(\mathbb{I} - \big[\alpha_{2,k}^R\big]^\dagger\alpha_{1,k}^L\big)\big[\alpha_{2,k-1}^R\big]^\dagger,$$

$$\partial_{H,2}\alpha_{1,k}^L = \alpha_{1,k+1}^L\big(\mathbb{I} - \big[\alpha_{2,k}^R\big]^\dagger\alpha_{1,k}^L\big),$$

$$\partial_{H,2}\big[\alpha_{2,k}^L\big]^\dagger = -\big[\alpha_{2,k-1}^L\big]^\dagger\big(\mathbb{I} - \alpha_{1,k}^R\big[\alpha_{2,k}^L\big]^\dagger\big),$$

$$\partial_{H,2}\alpha_{1,k}^R = \big(\mathbb{I} - \alpha_{1,k}^R\big[\alpha_{2,k}^L\big]^\dagger\big)\alpha_{1,k+1}^R,$$

$$\partial_{H,1}h_k^L = -\alpha_{1,k}^L\big(\alpha_{2,k+1}^R\big)^\dagger h_k^L,$$

$$\partial_{H,1}h_k^R = -h_k^R\big(\alpha_{2,k-1}^L\big)^\dagger\alpha_{1,k}^R,$$

$$\partial_{H,2}h_k^R = -\big(\alpha_{2,k}^R\big)^\dagger\alpha_{1,k+1}^L h_k^R,$$

$$\partial_{H,2}h_k^L = -h_k^L \alpha_{1,k+1}^R\big(\alpha_{2,k}^L\big)^\dagger.$$

Proof. To obtain the partial flows, it is enough to use the Lax equations for $j, p = 1, 2$ and operate. In order to obtain the total flows, we go back to the partial flows, and sum in a. From the Lax equations, we know that in this total case we no longer need to distinguish between R, L. This procedure leads to the result that is finally rewritten using the relations in Proposition 17. □

3.1.3. Bilinear equations

Bilinear equations are an alternative way of expressing an integrable hierarchy developed by the Japanese school, see [40–42]. We are going to show that these MOLPUC also fulfill a particular type of bilinear equations. These results are the matrix extensions of the scalar situation described in [13]. Let us start by considering the wave semi-infinite matrices $W_i^H(t)$ 36 associated to the moment matrix g^H, $H = L, R$. Since the last one is time independent, the reader can easily check that

Proposition 29.

(1) *The wave matrices associated to different times satisfy*

$$
\begin{aligned}
W_1^L(t)\big(W_1^L(t')\big)^{-1} &= W_2^L(t)\big(W_2^L(t')\big)^{-1}, \\
\big(W_1^R(t)\big)^{-1} W_1^R(t') &= \big(W_2^R(t)\big)^{-1} W_2^R(t').
\end{aligned}
\tag{44}
$$

(2) *The vectors χ, χ^* fulfill*

$$
\mathrm{Res}_{z=0}\big[\chi(z)\big(\chi^*(\bar z)\big)^\dagger\big] = \mathrm{Res}_{z=0}\big[\chi^*(z)\big(\chi(\bar z)\big)^\dagger\big] = \mathbb{I}.
$$

(3) *One has that the product of two matrices can be expressed as*

$$
\begin{aligned}
UV &= \mathrm{Res}_{z=0}\big[U\chi(z)\big(V^\dagger\chi^*(\bar z)\big)^\dagger\big] = \mathrm{Res}_{z=0}\big[U\chi^*(z)\big(V^\dagger\chi(\bar z)\big)^\dagger\big] \\
&= \mathrm{Res}_{z=0}\big[\big(\chi^*(\bar z)^\top U^\dagger\big)^\dagger\chi(z)^\top V\big] = \mathrm{Res}_{z=0}\big[\big(\chi(\bar z)^\top U^\dagger\big)^\dagger\chi^*(z)^\top V\big]
\end{aligned}
$$

From where we derive

Theorem 3. *For two different set of times $t, \tilde t$ the wave functions satisfy*

$$
\mathrm{Res}_{z=0}\big[\Psi_1^L(z,t)\big[(\Psi_1^L)^*(\bar z,\tilde t)\big]^\dagger\big] = \mathrm{Res}_{z=0}\big[\Psi_2^L(z,t)\big[(\Psi_2^L)^*(\bar z,\tilde t)\big]^\dagger\big],
\tag{45}
$$

$$
\mathrm{Res}_{z=0}\big[\big[(\Psi_1^R)^*(\bar z,t)\big]^\dagger\Psi_1^R(z,\tilde t)\big] = \mathrm{Res}_{z=0}\big[\big[(\Psi_2^R)^*(\bar z,t)\big]^\dagger\Psi_2^R(z,\tilde t)\big].
\tag{46}
$$

From the identities in (43) the previous theorem can be rewritten in terms of CMV polynomials as

$$
\begin{aligned}
&\mathrm{Res}_{z=0}\big[(\varphi_1^L)^{(l)}(z,t)\big(\exp\big((t^L - \tilde t^L)\chi(z)\big)z^{-1}F_\mu(z)\big)\big[(\varphi_2^L)^{(m)}(\bar z^{-1},\tilde t)\big]^\dagger\big] \\
&= -\mathrm{Res}_{z=\infty}\big[(\varphi_1^L)^{(l)}(z,t)\big(z^{-1}F_\mu(z)\exp\big(\chi(z)^\top(\tilde t^R - t^R)\big)\big)\big[(\varphi_2^L)^{(m)}(\bar z^{-1},\tilde t)\big]^\dagger\big], \\
&\mathrm{Res}_{z=0}\big[\big[(\varphi_2^R)^{(l)}(\bar z^{-1},t)\big]^\dagger\big(z^{-1}F_\mu(z)\exp\big(\chi(z)^\top(t'_R - t_R)\big)\big)(\varphi_1^R)^{(m)}(z,\tilde t)\big] \\
&= -\mathrm{Res}_{z=\infty}\big[\big[(\varphi_2^R)^{(l)}(\bar z^{-1},t)\big]^\dagger\big(\exp\big((t_L - t'_L)\chi(z)\big)z^{-1}F_\mu(z)\big)(\varphi_1^R)^{(m)}(z,\tilde t)\big].
\end{aligned}
$$

Here we have used that $\mathrm{Res}_{z=\infty} F(z) = -\mathrm{Res}_{z=0} z^{-2} F(z^{-1})$. Alternatively, we can write all the previous expressions using integrals instead of using residues. To do this, let us denote by γ_0 and γ_∞ two positively oriented circles around $z=0$ and $z=\infty$, respectively, included in the annulus of convergence of the Fourier series of the matrix measure, such that they do not include different simple poles that $z=0,\infty$, respectively. Then,

$$\oint_{\gamma_0} \Psi_1^L(z,t) \big[(\Psi_1^L)^*(\bar{z},\tilde{t})\big]^\dagger \mathrm{d}z = \oint_{\gamma_0} \Psi_2^L(z,t) \big[(\Psi_2^L)^*(\bar{z},\tilde{t})\big]^\dagger \mathrm{d}z, \tag{47}$$

$$\oint_{\gamma_0} \big[(\Psi_1^R)^*(\bar{z},t)\big]^\dagger \Psi_1^L(z,\tilde{t}) \mathrm{d}z = \oint_{\gamma_0} \big[(\Psi_2^R)^*(\bar{z},t)\big]^\dagger \Psi_2^R(z,\tilde{t}) \mathrm{d}z \tag{48}$$

or, in terms of matrix Laurent orthogonal polynomials and Fourier series of the matrix measure:

Proposition 30. *The evolved MOLPUC satisfy*

$$\oint_{\gamma_0} \big(\varphi_1^L\big)^{(l)}(z,t) \big(\exp\big((t^L - \tilde{t}^L)\chi(z)\big) z^{-1} F_\mu(z)\big) \big[(\varphi_2^L)^{(m)}\big(\bar{z}^{-1},\tilde{t}\big)\big]^\dagger \mathrm{d}z$$

$$= \oint_{\gamma_\infty} \big(\varphi_1^L\big)^{(l)}(z,t) \big(z^{-1} F_\mu(z) \exp\big(\chi(z)^\top (\tilde{t}^R - t^R)\big)\big) \big[(\varphi_2^L)^{(m)}\big(\bar{z}^{-1},\tilde{t}\big)\big]^\dagger \mathrm{d}z,$$

$$\oint_{\gamma_0} \big[(\varphi_2^R)^{(l)}\big(\bar{z}^{-1},t\big)\big]^\dagger \big(z^{-1} F_\mu(z) \exp\big(\chi(z)^\top (\tilde{t}^R - t^R)\big)\big) \big(\varphi_1^R\big)^{(m)}(z,\tilde{t}) \mathrm{d}z$$

$$= \oint_{\gamma_\infty} \big[(\varphi_2^R)^{(l)}\big(\bar{z}^{-1},t\big)\big]^\dagger \big(\exp\big((t^L - \tilde{t}^L)\chi(z)\big) z^{-1} F_\mu(z)\big) \big(\varphi_1^R\big)^{(m)}(z,\tilde{t}) \mathrm{d}z.$$

3.2. 2D Toda discrete flows

Given a couple of sequences of diagonal matrices

$$d = \{d_+, d_-\}, \quad d_\pm = \{d_{\pm,0} = 0, d_{\pm,1}, d_{\pm,2}, \ldots\}, \ d_{\pm,j} \in \mathrm{diag}_m,$$

and a pair of non-negative integers $n = \{n_+, n_-\} \in \mathbb{Z}_+^2$, we consider the next semi-infinite block matrices

$$\Delta_d^L(n) = \big(\mathbb{I} - d_{-,0}\Upsilon^{-1}\big) \cdots \big(\mathbb{I} - d_{-,n_-}\Upsilon^{-1}\big)\big(\mathbb{I} - d_{+,0}\Upsilon\big) \cdots \big(\mathbb{I} - d_{+,n_+}\Upsilon\big),$$

$$\Delta_d^R(n) = \big(\mathbb{I} - d_{-,0}\Upsilon\big) \cdots \big(\mathbb{I} - d_{-,n_-}\Upsilon\big)\big(\mathbb{I} - d_{+,0}\Upsilon^{-1}\big) \cdots \big(\mathbb{I} - d_{+,n_+}\Upsilon^{-1}\big).$$

Observe that the order of the factors does not alter the product as each of them commutes with the others.

Definition 16. Given two couples of sequences of diagonal matrices, say $d^H = \{d_+^H, d_-^H\}$, $H = L, R$, we introduce the discrete flows for the right and left moment matrices

$$g^H(n^L, n^R) = \Delta_{d^L}^H(n^L) g^H \Delta_{d^R}^H(n^R), \qquad n^H = \{n_+^H, n_-^H\} \in \mathbb{Z}_+^2,$$

$$g^H(0,0) = g^H, \quad H = L, R.$$

The property $\eta g^L(n^L, n^R) = g^R(n^L, n^R)\eta$ is easily checked and it follows that we have an associated measure of which these are the corresponding left and right moment matrices given by

$$d\mu(n^L, n^R) = \left[\prod_{i=0}^{n_-^L} (\mathbb{I} - d_{-,i}^L z^{-1}) \prod_{j=0}^{n_+^L} (\mathbb{I} - d_{+,j}^L z) \right]$$

$$\times d\mu \left[\prod_{i=0}^{n_-^R} (\mathbb{I} - d_{-,i}^R z^{-1}) \prod_{j=0}^{n_+^R} (\mathbb{I} - d_{+,j}^R z) \right].$$

The measure is Hermitian if the following conditions are fulfilled

$$\left[d_{\mp,j}^R \right]^\dagger = d_{\pm,j}^L = d_{\pm,j}, \qquad n_\pm^L = n_\mp^R = n_\pm,$$

being the evolved measure

$$d\mu(n_+, n_-) = \left[\prod_{i=0}^{n_-} (\mathbb{I} - d_{-,i} z^{-1}) \prod_{j=0}^{n_+} (\mathbb{I} - d_{+,j} z) \right]$$

$$\times d\mu \left[\prod_{i=0}^{n_-} (\mathbb{I} - d_{-,i} z^{-1}) \prod_{j=0}^{n_+} (\mathbb{I} - d_{+,j} z) \right]^\dagger.$$

Positive definiteness for the Hermitian situation can be ensured if we request $d_i := d_{+,i} = [d_{-,i}]^\dagger$ and $n := n_+ = n_-$ so that

$$d\mu(n) = \left[\sum_{a=1}^m \left(\prod_{j=0}^n |1 - d_{j,a} z|^2 \right) E_{a,a} \right] d\mu \left[\sum_{a=1}^m \left(\prod_{j=0}^n |1 - d_{j,a} z|^2 \right) E_{a,a} \right].$$

As in the continuous case, we introduce

Definition 17. The wave matrices, depending on discrete variables $n^L, n^R \in \mathbb{Z}_+^2$, are defined as

$$W_1^L(n^L, n^R) = S_1(n^L, n^R) \Delta_{d^L}^L(n^L), \qquad W_2^L(n^L, n^R) = S_2(n^L, n^R) \left(\Delta_{d^R}^L(n^R) \right)^{-1},$$

$$W_1^R(n^L, n^R) = \Delta_{d^R}^R(n^R) Z_1(n^L, n^R), \qquad W_2^R(n^L, n^R) = \left(\Delta_{d^L}^R(n^L) \right)^{-1} Z_2(n^L, n^R).$$

Hence

$$g^L = \left[W_1^L\big(n^L, n^R\big)\right]^{-1} W_2^L\big(n^L, n^R\big), \qquad g^R = W_2^R\big(n^L, n^R\big)\left[W_1^R\big(n^L, n^R\big)\right]^{-1}.$$

We also need to introduce the following objects

Definition 18.

(1) Given a diagonal matrix $d \in \mathrm{diag}_m$, we define the semi-infinite block matrices

$$\delta_\pm^{H,H'}(d) = \begin{cases} S_1(\mathbb{I} - d\Upsilon^{\pm 1})S_1^{-1}, & H = L, \ H' = L, \\ S_2(\mathbb{I} - d\Upsilon^{\pm 1})S_2^{-1}, & H = R, \ H' = L, \\ Z_2^{-1}(\mathbb{I} - d\Upsilon^{\mp 1})Z_2, & H = L, \ H' = R, \\ Z_1^{-1}(\mathbb{I} - d\Upsilon^{\mp 1})Z_1, & H = R, \ H' = R. \end{cases}$$

(2) The shifts are

$$T_+^L : \begin{pmatrix} (n_+^L, n_-^L) \longrightarrow (n_+^L + 1, n_-^R) \\ n^R \longrightarrow n^R \end{pmatrix}, \qquad T_-^L : \begin{pmatrix} (n_+^L, n_-^R) \longrightarrow (n_+^L, n_-^L + 1) \\ n^R \longrightarrow n^R \end{pmatrix},$$

$$T_+^R : \begin{pmatrix} n_L \longrightarrow n_L \\ (n_+^R, n_-^R) \longrightarrow (n_+^R + 1, n_-^R) \end{pmatrix}, \qquad T_-^R : \begin{pmatrix} n_L \longrightarrow n_L \\ (n_+^R, n_-^R) \longrightarrow (n_+^R, n_-^R + 1) \end{pmatrix}.$$

For any diagonal matrix $d = \sum_{a=1}^m d_a E_{a,a} \in \mathrm{diag}_m$, $d_a \in \mathbb{C}$, we introduce the semi-infinite matrices

$$d^{H,H'} = \sum_{a=1}^m d_a P_a^{H,H'},$$

where $P_a^{H,H'}$ was defined in (37); observe that when $d = c\mathbb{I}_m$, $c \in \mathbb{C}$, we have $d^{H,H'} = c\mathbb{I}$.

Notice that the $\delta_\pm^{H,H'}$ are just particular combinations of the block Jacobi matrices J^H

$$\delta_\pm^{H,H'}(d) = \mathbb{I} - d^{H,H'}\big(J^{H'}\big)^{\pm 1}.$$

Proposition 31. *If $g^H(n^L, n^R)$, $(T_\pm^H g^{H'})(n^L, n^R)$, $H, H' = L, R$, admit a block LU factorization, then the δ matrices introduced in Definition 18(1) also admit a block LU factorization.*

Proof. We have

$$T_\pm^L g^L = \big(\mathbb{I} - d_{\pm, n_\pm^L + 1}^L \Upsilon^{\pm 1}\big) g^L$$

$$\implies \quad S_1\big(T_\pm^L S_1\big)^{-1}\big(T_\pm^L S_2\big)S_2^{-1} = S_1\big(\mathbb{I} - d_{\pm, n_\pm^L + 1}^L \Upsilon^{\pm 1}\big)S_1^{-1} = \delta_\pm^{L,L}\big(d_{\pm, n_\pm^L + 1}^L\big),$$

$$T_\pm^L g^R = \big(\mathbb{I} - d_{\pm, n_\pm^L + 1}^L \Upsilon^{\mp 1}\big) g^R$$

$$\implies \quad Z_2^{-1}\bigl(T_\pm^L Z_2\bigr)\bigl(T_\pm^L Z_1\bigr)^{-1} Z_1 = Z_2^{-1}\bigl(\mathbb{I} - d_{\pm,n_\pm^L+1}^L \Upsilon^{\mp 1}\bigr) Z_2 = \delta_\pm^{L,R}\bigl(d_{\pm,n_\pm^L+1}^L\bigr),$$

$$T_\pm^R g^L = g^L\bigl(\mathbb{I} - d_{\pm,n_\pm^R+1}^R \Upsilon^{\pm 1}\bigr)$$

$$\implies \quad S_1\bigl(T_\pm^R S_1\bigr)^{-1}\bigl(T_\pm^R S_2\bigr) S_2^{-1} = S_2\bigl(\mathbb{I} - d_{\pm,n^{R,\pm}+1}^R \Upsilon^{\pm 1}\bigr) S_2^{-1} = \delta_\pm^{R,L}\bigl(d_{\pm,n_\pm^R+1}^R\bigr),$$

$$T_\pm^R g^R = g^R\bigl(\mathbb{I} - d_{\pm,n_\pm^R+1}^R \Upsilon^{\mp 1}\bigr)$$

$$\implies \quad Z_2^{-1}\bigl(T_\pm^R Z_2\bigr)\bigl(T_\pm^R Z_1\bigr)^{-1} Z_1 = Z_1^{-1}\bigl(\mathbb{I} - d_{\pm,n_\pm^R+1}^R \Upsilon^{\mp 1}\bigr) Z_1 = \delta_\pm^{R,R}\bigl(d_{\pm,n_\pm^R+1}^R\bigr).$$

Therefore, for $H = L, R$,

$$\delta_\pm^{H,L} = \bigl(\delta_\pm^{H,L}\bigr)_-^{-1} \cdot \bigl(\delta_\pm^{H,L}\bigr)_+, \qquad \bigl(\delta_\pm^{H,L}\bigr)_- = \bigl(T_\pm^H S_1\bigr) S_1^{-1} \in \mathscr{L},$$

$$\bigl(\delta_\pm^{H,L}\bigr)_+ = \bigl(T_\pm^H S_2\bigr) S_2^{-1} \in \mathscr{U}, \qquad \delta_\pm^{H,R} = \bigl(\delta_\pm^{H,R}\bigr)_- \cdot \bigl(\delta_\pm^{H,R}\bigr)_+^{-1},$$

$$\bigl(\delta_\pm^{H,R}\bigr)_- = Z_2^{-1}\bigl(T_\pm^H Z_2\bigr) \in \mathscr{L}, \qquad \bigl(\delta_\pm^{H,R}\bigr)_+ = Z_1^{-1}\bigl(T_\pm^H Z_1\bigr) \in \mathscr{U}. \qquad \square$$

Definition 19. We define

$$\omega_\pm^{H,H'} := \begin{cases} (\delta_\pm^{L,L})_+ = \bigl(T_\pm^L S_2\bigr) S_2^{-1}, & H = L, \ H' = L, \\ (\delta_\pm^{R,L})_- = \bigl(T_\pm^R S_1\bigr) S_1^{-1}, & H = R, \ H' = L, \\ (\delta_\pm^{L,R})_- = Z_1^{-1}\bigl(T_\pm^L Z_1\bigr), & H = L, \ H' = R, \\ (\delta_\pm^{R,R})_+ = Z_2^{-1}\bigl(T_\pm^R Z_2\bigr), & H = R, \ H' = R. \end{cases}$$

We are ready to derive discrete integrability.

Theorem 4.

- *The discrete linear systems*

$$T_\pm^H W_i^L = \omega_\pm^{H,L} W_i^L, \qquad T_\pm^H W_i^R = W_i^R \omega_\pm^{H,R}, \quad i = 1, 2, \ H = L, R.$$

- *Discrete Lax equations hold*

$$T_\pm^H J^L = \omega_\pm^{H,L} J^L \bigl(\omega_\pm^{H,L}\bigr)^{-1}, \qquad T_\pm^H J^R = \bigl(\omega_\pm^{H,R}\bigr)^{-1} J^R \omega_\pm^{H,R}, \quad H = L, R.$$

- *Intertwining matrix*

$$T_\pm^H C_{[p]} = \bigl(\omega_\pm^{H,R}\bigr)^{-1} C_{[p]} \bigl(\omega_\pm^{H,L}\bigr)^{-1}, \quad H = L, R.$$

- *Zakharov–Shabat equations*

$$\big(T_a^H \omega_b^{H',L}\big)\omega_a^{H,L} = \big(T_b^{H'} \omega_a^{H,L}\big)\omega_b^{H',L}, \qquad \omega_a^{H,R}\big(T_a^H \omega_b^{H',R}\big) = \omega_b^{H',R}\big(T_b^{H'} \omega_a^{H,R}\big),$$
$$a,b = \pm, \ H,H' = L,R.$$

- *Continuous-discrete equations*

$$\frac{\partial \omega_\pm^{H',L}}{\partial t_{j,a}^H} + \omega_\pm^{H',L} B_{j,a}^{H,L} = \big(T_\pm^{H'} B_{j,a}^{H,L}\big)\omega_\pm^{H',L},$$

$$\frac{\partial \omega_\pm^{H',R}}{\partial t_{j,a}^H} + B_{j,a}^{H,R} \omega^{H',R}\pm = \omega_\pm^{H',R}\big(T_\pm^{H'} B_{j,a}^{H,R}\big),$$

with $H,H' = L,R$, $a = 1,\dots,m$ and $j = 0,1,\dots$.

From these results, one may derive discrete matrix equations for the Verblunsky coefficients.

It also follows that these flows are extensions of Darboux transformations, see [13] for the scalar case. Each of these discrete shifts is generalization of the typical Darboux transformation corresponding to the flip of the upper and lower triangular factors of the operators $\delta_\pm^{H,H'}$. These flips occur in some specific cases as follows. Let us assume that the diagonal matrices $d_{\pm,j}^H$ do not depend on j; then,

$$\delta_\pm^{H,H'} = \begin{cases} (\delta^{H,L})_-^{-1}(\delta_\pm^{H,L})_+, & H' = L, \\ (\delta^{H,R})_-(\delta_\pm^{H,R})_+^{-1}, & H' = L, \end{cases}$$

$$\xrightarrow{T_\pm^H} \quad T_\pm^H \delta_\pm^{H,H'} = \begin{cases} (\delta_\pm^{H,L})_+(\delta^{H,L})_-^{-1}, & H' = L, \\ (\delta_\pm^{H,R})_+^{-1}(\delta^{H,R})_-, & H' = L. \end{cases}$$

It is clear that the shift corresponds to the flip of the factors in the Gaussian factorization of the $\delta_\pm^{H,H'}$ matrices, just as in the Darboux transformations. When the constant sequences $d_{\pm,j}^H = c_\pm^H \mathbb{I}_m$, with $c_\pm^H \in \mathbb{C}$ scalars, we have that $\delta_\pm^{H,H'}$ are pentadiagonal block matrices (main diagonal and the two next diagonals above and below it), and therefore the Gauss–Borel factorizations give upper or lower block tridiagonal matrices, $(\delta_\pm^{H,H'})_+$ and $(\delta_\pm^{H,H'})_-$, respectively. This is quite close to some results in the talk [28].

3.3. Miwa shifts

In our unsuccessful search for a neat τ-function theory in this matrix scenario, we have studied the action of Miwa shifts. Despite we did not find appropriate τ-functions, we found interesting relations among Christoffel–Darboux kernels and the Miwa transformations of the MOLPUC. These relations do in fact lead in the scalar case to the τ-function representation of MOLPUC. Unfortunately, apparently that is not the case in the matrix scenario.

Miwa shifts are coherent time translations that lead to discrete type flows. Given a diagonal matrix $w = \mathrm{diag}(w_1, \ldots, w_m) \in \mathbb{C}^{m \times m}$, we introduce four different $\mathfrak{M}_w^{H,\pm}$, $H = L, R$, coherent shifts

$$\mathfrak{M}_w^{L,+} : t_{2k}^L \mapsto t_{2k}^L - \frac{w^k}{k}, \qquad \mathfrak{M}_w^{L,-} : t_{2k-1}^L \mapsto t_{2k-1}^L - \frac{w^k}{k},$$

$$\mathfrak{M}_w^{R,+} : t_{2k}^R \mapsto t_{2k}^R - \frac{w^k}{k}, \qquad \mathfrak{M}_w^{R,-} : t_{2k-1}^R \mapsto t_{2k-1}^R - \frac{w^k}{k}.$$

For each Miwa shift, we only write down those times with a non-trivial transformation. When these shifts act on the deformed matrix measure, we get new matrix measures

$$\mathrm{d}\mathfrak{M}_w^{L,\pm}[\mu] = \left(1 - wz^{\pm 1}\right)\mathrm{d}\mu, \qquad \mathrm{d}\mathfrak{M}_w^{R,\pm}[\mu] = \mathrm{d}\mu\left(1 - wz^{\pm 1}\right), \tag{49}$$

with corresponding left and right moment matrices given by

$$\mathfrak{M}_w^{L,\pm}\big[g^L\big] = \left(\mathbb{I} - w\Upsilon^{\pm 1}\right)g^L, \qquad \mathfrak{M}_w^{L,\pm}\big[g^R\big] = \left(\mathbb{I} - w\Upsilon^{\mp 1}\right)g^R,$$

$$\mathfrak{M}_w^{R,\pm}\big[g^L\big] = g^L\left(\mathbb{I} - w\Upsilon^{\pm 1}\right), \qquad \mathfrak{M}_w^{R,\pm}\big[g^R\big] = g^R\left(\mathbb{I} - w\Upsilon^{\mp 1}\right). \tag{50}$$

From these we can deduce the next

Theorem 5. *For every diagonal matrix $w \in \mathrm{diag}_m$, the following relations between Miwa shifted and non-shifted Christoffel–Darboux kernels and MOLPUC hold*

$$K^{L,[2l+1]}(z,u) = \mathfrak{M}_w^{L,+}\big[K^{L,[2l]}\big](z,u)(\mathbb{I} - wu)$$
$$+ \left(\mathfrak{M}_w^{L,+}\big[\big(\varphi_2^L\big)^{(2l)}\big](z)\right)^\dagger \mathfrak{M}_w^{L,+}\big[h_{2l}^L\big]\big(h_{2l}^L\big)^{-1}\big(\varphi_1^L\big)^{(2l)}(u),$$

$$K^{R,[2l]}(z,u) = \mathfrak{M}_w^{L,+}\big[K^{R,[2l-1]}\big](z,u)\big(\mathbb{I} - w\bar{u}^{-1}\big)$$
$$+ \mathfrak{M}_w^{L,+}\big[\big(\varphi_1^R\big)^{(2l-1)}\big](z)\mathfrak{M}_w^{L,+}\big[h_{2l-1}^L\big]\big(h_{2l-1}^L\big)^{-1}\big(\big(\varphi_2^R\big)^{(2l-1)}(u)\big)^\dagger,$$

$$K^{L,[2l]}(z,u) = \mathfrak{M}_w^{L,-}\big[K^{L,[2l-1]}\big](z,u)\big(\mathbb{I} - wu^{-1}\big)$$
$$+ \left(\mathfrak{M}_w^{L,-}\big[\big(\varphi_2^L\big)^{(2l-1)}\big](z)\right)^\dagger \mathfrak{M}_w^{L,-}\big[h_{2l-1}^R\big]\big(h_{2l-1}^R\big)^{-1}\big(\varphi_1^L\big)^{(2l-1)}(u),$$

$$K^{R,[2l+1]}(z,u) = \mathfrak{M}_w^{L,-}\big[K^{R,[2l]}\big](z,u)(\mathbb{I} - w\bar{u})$$
$$+ \mathfrak{M}_w^{L,-}\big[\big(\varphi_1^R\big)^{(2l)}\big](z)\mathfrak{M}_w^{L,-}\big[h_{2l}^L\big]\big(h_{2l}^L\big)^{-1}\big(\big(\varphi_2^R\big)^{(2l)}(u)\big)^\dagger,$$

$$K^{L,[2l]}(z,u) = \big(\mathbb{I} - w\bar{z}^{-1}\big)\mathfrak{M}_w^{R,+}\big[K^{L,[2l-1]}\big](z,u)$$
$$+ \big(\big(\varphi_2^L\big)^{(2l-1)}(z)\big)^\dagger \mathfrak{M}_w^{R,+}\big[\big(\varphi_1^L\big)^{(2l-1)}\big](u),$$

$$K^{R,[2l+1]}(z,u) = (\mathbb{I} - wz)\mathfrak{M}_w^{R,+}\big[K^{R,[2l]}\big](z,u)$$
$$+ \big(\varphi_1^R\big)^{(2l)}(z)\big(\mathfrak{M}_w^{R,+}\big[\big(\varphi_2^R\big)^{(2l)}\big](u)\big)^\dagger,$$

$$K^{L,[2l+1]}(z,u) = (\mathbb{I} - w\bar{z})\mathfrak{M}_w^{R,-}\big[K^{L,[2l]}\big](z,u)$$
$$+ \big((\varphi_2^L)^{(2l)}(z)\big)^\dagger \mathfrak{M}_w^{R,-}\big[(\varphi_1^L)^{(2l)}\big](u),$$
$$K^{R,[2l]}(z,u) = \big(\mathbb{I} - wz^{-1}\big)\mathfrak{M}_w^{R,-}\big[K^{R,[2l-1]}\big](z,u)$$
$$+ \big(\varphi_1^R\big)^{(2l-1)}(z)\big(\mathfrak{M}_w^{R,-}\big[(\varphi_2^R)^{(2l-1)}\big](u)\big)^\dagger.$$

Proof. We just give the main ideas of the proof not dealing with details. Let us consider (50) at the light of the Gauss–Borel factorizations (7) and (8)

$$\mathfrak{M}_w^{L,\pm}[S_2]S_2^{-1} = \mathfrak{M}_w^{L,\pm}[S_1]\big[\mathbb{I} - w\Upsilon^{\pm 1}\big]S_1^{-1},$$
$$\big(\mathfrak{M}_w^{L,\pm}[Z_1]\big)^{-1}Z_1 = \big(\mathfrak{M}_w^{L,\pm}[Z_2]\big)^{-1}\big[\mathbb{I} - w\Upsilon^{\mp 1}\big]Z_2,$$
$$S_1\big(\mathfrak{M}_w^{R,\pm}[S_1]\big)^{-1} = S_2\big[\mathbb{I} - w\Upsilon^{\pm 1}\big]\big(\mathfrak{M}_w^{R,\pm}[S_2]\big)^{-1},$$
$$Z_2^{-1}\mathfrak{M}_w^{R,\pm}[Z_2] = Z_1^{-1}\big[\mathbb{I} - w\Upsilon^{\mp 1}\big]\mathfrak{M}_w^{R,\pm}[Z_1].$$

Each of these equalities defines a semi-infinite matrix relating shifted and non-shifted polynomials. At this point it is important to stress that the LHS in the two first equations are upper triangular semi-infinite matrices, while the two last equations have in the RHS upper triangular semi-infinite matrices. Observe also that in the two first equations, because of the RHS only the main, the first and the second block diagonals over the first have non-zero blocks while in the LHS of the two last equations only the main diagonal and the two immediate diagonals below it have non-zero blocks. Then we proceed as in the proof of the Christoffel–Darboux formula in Theorem 2. To get a glance of the technique, let us illustrate it for the first equation. On the one hand, we have for the $2l$-th and $(2l+1)$-th block rows

$$\mathfrak{M}_w^{L,+}[S_2]S_2^{-1}$$
$$= \mathfrak{M}_w^{L,+}[S_1][\mathbb{I} - w\Upsilon]S_1^{-1}$$
$$= \begin{pmatrix} \ddots & \ddots & & & & \\ \cdots & 0 & \mathfrak{M}_w^{L,+}[h_{2l}^L](h_{2l}^L)^{-1} & w\alpha_{1,2l+2}^L & -w & 0 & \cdots \\ \cdots & 0 & 0 & \mathfrak{M}_w^{L,+}[h_{2l+1}^R](h_{2l+1}^R)^{-1} & -(\mathfrak{M}_w^{L,+}[\alpha_{2,2l+1}^R])^\dagger w & * & 0 \\ & & & & & \ddots \end{pmatrix}.$$

On the other hand,

$$\big(\mathfrak{M}_w^{L,+}\big[\phi_2^L\big](z)\big)^\dagger \mathfrak{M}_w^{L,+}[S_2]S_2^{-1} = \big(\phi_2^L(z)\big)^\dagger,$$
$$\mathfrak{M}_w^{L,+}[S_1][\mathbb{I} - w\Upsilon]S_1^{-1}\phi_1^L(u) = \mathfrak{M}_w^{L,+}\big[\phi_1^L\big](u)(\mathbb{I} - wu).$$

Then, by appropriate scalar product pairings, we get the result. $\square$

An appropriated choice of the variables allows us to express the rows or columns of the kernel in terms of the rows or columns of a product of a shifted and a non-shifted polynomial

Corollary 3. *If $w = \mathrm{diag}(w_1, \ldots, w_m)$, $w_k \in \mathbb{C}$, we have*

$$K^{L,[2l+1]}(z, w_k^{-1}) E_{k,k}$$
$$= \left(\mathfrak{M}_w^{L,+}\left[(\varphi_2^L)^{(2l)}\right](z)\right)^{\dagger} \mathfrak{M}_w^{L,+}\left[h_{2l}^L\right](h_{2l}^L)^{-1}(\varphi_1^L)^{(2l)}(w_k^{-1}) E_{k,k},$$
$$K^{R,[2l]}(z, \bar{w}_k) E_{k,k}$$
$$= \mathfrak{M}_w^{L,+}\left[(\varphi_1^R)^{(2l-1)}\right](z) \mathfrak{M}_w^{L,+}\left[h_{2l-1}^L\right](h_{2l-1}^L)^{-1}\left((\varphi_2^R)^{(2l-1)}(\bar{w}_k)\right)^{\dagger} E_{k,k},$$
$$K^{L,[2l]}(z, w_k) E_{k,k}$$
$$= \left(\mathfrak{M}_w^{L,-}\left[(\varphi_2^L)^{(2l-1)}\right](z)\right)^{\dagger} \mathfrak{M}_w^{L,-}\left[h_{2l-1}^R\right](h_{2l-1}^R)^{-1}(\varphi_1^L)^{(2l-1)}(w_k) E_{k,k},$$
$$K^{R,[2l+1]}(z, \bar{w}_k^{-1}) E_{k,k}$$
$$= \mathfrak{M}_w^{L,-}\left[(\varphi_1^R)^{(2l)}\right](z) \mathfrak{M}_w^{L,-}\left[h_{2l}^L\right](h_{2l}^L)^{-1}\left((\varphi_2^R)^{(2l)}(\bar{w}_k^{-1})\right)^{\dagger} E_{k,k},$$
$$E_{k,k} K^{L;[2l]}(\bar{w}_k, u) = E_{k,k}\left((\varphi_2^L)^{(2l-1)}(\bar{w}_k)\right)^{\dagger} \mathfrak{M}_w^{R,+}\left[(\varphi_1^L)^{(2l-1)}\right](u),$$
$$E_{k,k} K^{R,[2l+1]}(w_k^{-1}, u) = E_{k,k}(\varphi_1^R)^{(2l)}(w_k^{-1})\left(\mathfrak{M}_w^{R,+}\left[(\varphi_2^R)^{(2l)}\right](u)\right)^{\dagger},$$
$$E_{k,k} K^{L,[2l+1]}(\bar{w}_k^{-1}, u) = E_{k,k}\left((\varphi_2^L)^{(2l)}(\bar{w}_k^{-1})\right)^{\dagger} \mathfrak{M}_w^{R,-}\left[(\varphi_1^L)^{(2l)}\right](u),$$
$$E_{k,k} K^{R,[2l]}(w_k, u) = E_{k,k}(\varphi_1^R)^{(2l-1)}(w_k)\left(\mathfrak{M}_w^{R,-}\left[(\varphi_2^R)^{(2l-1)}\right](u)\right)^{\dagger}.$$

Let us consider what happens when instead of a diagonal matrix w is proportional to the identity matrix. In this case, (49) informs us that left and right handed Miwa shifts coincide. We only have two Miwa shifts $\mathfrak{M}_w^{\pm}$ where now $w \in \mathbb{C}$

$$\mathrm{d}\mathfrak{M}_w^{\pm}[\mu] = (1 - wz^{\pm 1})\mathrm{d}\mu. \tag{51}$$

In this case, Corollary 3 would be written in much a simpler way (closer to the scalar case):

Proposition 32. *The following relations hold*

$$\left[(\varphi_2^L)^{(2l-1)}(\bar{w})\right]^{\dagger} \mathfrak{M}_w^+(h_{2l-1}^R) = (\varphi_1^R)^{(2l)}(w^{-1}) h_{2l}^R,$$
$$\mathfrak{M}_w^-(h_{2l-1}^R)(h_{2l-1}^R)^{-1}(\varphi_1^L)^{(2l-1)}(w) = \left[(\varphi_2^R)^{(2l)}(\bar{w}^{-1})\right]^{\dagger},$$
$$(\varphi_1^R)^{(2l-1)}(w) \mathfrak{M}_w^-(h_{2l-1}^L) = \left[(\varphi_2^L)^{(2l)}(\bar{w}^{-1})\right]^{\dagger} h_{2l}^L,$$
$$\mathfrak{M}_w^+(h_{2l-1}^L)(h_{2l-1}^L)^{-1}\left[(\varphi_2^R)^{(2l-1)}(\bar{w})\right]^{\dagger} = (\varphi_1^L)^{(2l)}(w^{-1}),$$
$$(\varphi_1^R)^{(2l)}(w^{-1}) \mathfrak{M}_w^+(h_{2l}^R) = \left[\bar{w}(\varphi_2^L)^{(2l+1)}(\bar{w})\right]^{\dagger} h_{2l+1}^R,$$

$$\mathfrak{M}_w^-\big(h_{2l}^R\big)\big(h_{2l}^R\big)^{-1}\big[\big(\varphi_2^R\big)^{(2l)}\big(\bar{w}^{-1}\big)\big]^\dagger = z\big(\varphi_1^L\big)^{(2l+1)}(w),$$

$$\big[\big(\varphi_2^L\big)^{(2l)}\big(\bar{w}^{-1}\big)\big]^\dagger \mathfrak{M}_w^-\big(h_{2l}^L\big) = w\big(\varphi_1^R\big)^{(2l+1)}(z)h_{2l+1}^L,$$

$$\mathfrak{M}_w^+\big(h_{2l}^L\big)\big(h_{2l}^L\big)^{-1}\big(\varphi_1^L\big)^{(2l)}\big(w^{-1}\big) = \big[\bar{w}\big(\varphi_2^R\big)^{(2l+1)}(\bar{w})\big]^\dagger.$$

Proof. See Appendix A □

Now, we can state

Theorem 6. *The CMV matrix Laurent orthogonal polynomials can be expressed as follows*

$$\big(\varphi_1^L\big)^{(2l)}(z) = z^l\big[\mathfrak{M}_{z^{-1}}^+\big(h_{2l-1}^L\big)\big(h_{2l-1}^L\big)^{-1}\big]\cdots\big[\mathfrak{M}_{z^{-1}}^+\big(h_0^L\big)\big(h_0^L\big)^{-1}\big], \tag{52}$$

$$\big(\varphi_1^L\big)^{(2l+1)}(z) = z^{-(l+1)}\big[\mathfrak{M}_z^-\big(h_{2l}^R\big)\big(h_{2l}^R\big)^{-1}\big]\cdots\big[\mathfrak{M}_z^-\big(h_0^R\big)\big(h_0^R\big)^{-1}\big], \tag{53}$$

$$\big[\big(\varphi_2^L\big)^{(2l)}\big(\bar{z}^{-1}\big)\big]^\dagger = z^{-l}\big[\big(h_0^L\big)^{-1}\mathfrak{M}_z^-\big(h_0^L\big)\big]\cdots\big[\big(h_{2l-1}^L\big)\mathfrak{M}_z^-\big(h_{2l-1}^L\big)\big]\big(h_{2l}^L\big)^{-1}, \tag{54}$$

$$\big[\big(\varphi_2^L\big)^{(2l+1)}\big(\bar{z}^{-1}\big)\big]^\dagger = z^{l+1}\big[\big(h_0^R\big)^{-1}\mathfrak{M}_{z^{-1}}^+\big(h_0^R\big)\big]\cdots\big[\big(h_{2l}^R\big)^{-1}\mathfrak{M}_{z^{-1}}^+\big(h_{2l}^R\big)\big]\big(h_{2l+1}^R\big)^{-1}, \tag{55}$$

$$\big(\varphi_1^R\big)^{(2l)}(z) = z^l\big[\big(h_0^R\big)^{-1}\mathfrak{M}_{z^{-1}}^+\big(h_0^{\ R}\big)\big]\cdots\big[\big(h_{2l-1}^R\big)^{-1}\mathfrak{M}_{z^{-1}}^+\big(h_{2l-1}^{\ R}\big)\big]\big(h_{2l}^R\big)^{-1}, \tag{56}$$

$$\big(\varphi_1^R\big)^{(2l+1)}(z) = z^{-(l+1)}\big[\big(h_0^L\big)^{-1}\mathfrak{M}_z^-\big(h_0^L\big)\big]\cdots\big[\big(h_{2l}^L\big)^{-1}\big]\mathfrak{M}_z^-\big(h_{2l}^L\big)\big(h_{2l+1}^L\big)^{-1}, \tag{57}$$

$$\big[\big(\varphi_2^R\big)^{(2l)}\big(\bar{z}^{-1}\big)\big]^\dagger = z^{-l}\big[\mathfrak{M}_z^-\big(h_{2l-1}^R\big)\big(h_{2l-1}^R\big)^{-1}\big]\cdots\big[\mathfrak{M}_z^-\big(h_0^R\big)\big(h_0^R\big)^{-1}\big], \tag{58}$$

$$\big[\big(\varphi_2^R\big)^{(2l+1)}\big(\bar{z}^{-1}\big)\big]^\dagger = z^{l+1}\big[\mathfrak{M}_{z^{-1}}^+\big(h_{2l}^L\big)\big(h_{2l}^L\big)^{-1}\big]\cdots\big[\mathfrak{M}_{z^{-1}}^+\big(h_0^L\big)\big(h_0^L\big)^{-1}\big]. \tag{59}$$

Proof. See Appendix A □

This is the furthest we have managed to take our τ description of the MOLPUC search. The reader may have noticed that forgetting about the R and L labels and the noncommutativity of the matrix norms we would be left with a quotient of Miwa shifted and non-shifted norms which in the scalar case coincides with the quotient of the determinants of the truncated Miwa shifted and non-shifted moment matrices.

Appendix A. Proofs

Proof of Proposition 1. Assuming $\det A \neq 0$ for any block matrix $M = \big(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\big)$, we can write in terms of Schur complements

$$M = \begin{pmatrix} \mathbb{I} & 0 \\ CA^{-1} & \mathbb{I} \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & M/A \end{pmatrix}\begin{pmatrix} \mathbb{I} & A^{-1}B \\ 0 & \mathbb{I} \end{pmatrix}.$$

Thus, as g^H is given for a matrix quasi-definite measure

$$\left(g^H\right)^{[l+1]} = \left(\begin{array}{c|c} \mathbb{I}_{l\times l} & 0 \\ \hline v_{[l]} & \mathbb{I} \end{array}\right) \left(\begin{array}{c|c} (g^H)^{[l]} & 0 \\ \hline 0 & (g^H)^{[l+1]}\big/(g^H)^{[l]} \end{array}\right) \left(\begin{array}{c|c} \mathbb{I}_{l\times l} & w^{[l]} \\ \hline 0 & \mathbb{I} \end{array}\right),$$

where $v_{[l]} = (v_0,\dots,v_{l-1})$ and $w^{[l]} = \begin{pmatrix} w^0 \\ w^1 \\ \vdots \\ w^{l-1} \end{pmatrix}$ are two matrix vectors. Applying the same factorization to $(g^H)^{[l]}$, we get

$$\left(g^H\right)^{[l+1]} = \left(\begin{array}{c|cc} \mathbb{I}_{(l-1)\times(l-1)} & 0 & 0 \\ \hline r_{[l-1]} & \mathbb{I} & 0 \\ v'_{[l-1]} & * & \mathbb{I} \end{array}\right) \left(\begin{array}{c|cc} (g^H)^{[l-1]} & 0 & 0 \\ \hline 0 & (g^H)^{[l]}\big/(g^H)^{[l-1]} & 0 \\ 0 & 0 & (g^H)^{[l+1]}\big/(g^H)^{[l]} \end{array}\right)$$
$$\times \left(\begin{array}{c|cc} \mathbb{I}_{(l-1)\times(l-1)} & s^{[l-1]} & w'^{[l-1]} \\ \hline 0 & \mathbb{I} & * \\ 0 & 0 & \mathbb{I} \end{array}\right).$$

Finally, the iteration of these factorizations leads to

$$\left(g^H\right)^{[l+1]} = \begin{pmatrix} \mathbb{I} & 0 & \dots & 0 \\ * & \mathbb{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \dots & * & \mathbb{I} \end{pmatrix}$$
$$\times \begin{pmatrix} (g^H)^{[1]}\big/(g^H)^{[0]} & 0 & \dots & 0 \\ 0 & (g^H)^{[2]}\big/(g^H)^{[1]} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & (g^H)^{[l+1]}\big/(g^H)^{[l]} \end{pmatrix}$$
$$\times \begin{pmatrix} \mathbb{I} & * & \dots & * \\ 0 & \mathbb{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & \mathbb{I} \end{pmatrix}, \tag{A.1}$$

for $H = L, R$. Since this would have been valid for any l, it would also hold for the direct limit $\varinjlim(g^H)^{[l]}$; i.e., for g^H with $H = L, R$. $\square$

Proof of Lemma 1. Notice is that the third equality of each expression is the second one written in terms of Schur complements. Therefore, just the first and second equalities of each expression need to be proven. The LU factorization leads to

$$S_1^{[\geq l,l]}\left(g^L\right)^{[l]} = -S_1^{[\geq l]}\left(g^L\right)^{[\geq l,l]},$$

$$S_2^{[l]}\big((g^L)^{[l]}\big)^{-1} = S_1^{[l]}$$

from where the result follows immediately. As an illustration, let us derive the first expression; on the one hand,

$$\big(\varphi^L\big)_1^{(l)}(z) = \sum_{j=0}^{l} (S_1)_{l,j} \chi^{(j)}$$

$$= \chi^{(l)} + \big(S_1^{[\geq l,l]} \chi^{[l]}\big)^{(0)}$$

$$= \chi^{(l)} - \sum_{m=0}^{l-1} \sum_{j=0}^{l} \big((g^L)^{[\geq l,l]}\big)_{0,m} \big(((g^L)^{[l]})^{-1}\big)_{m,j} \chi^{(j)}$$

$$= \chi^{(l)} - \big((g^L)_{l,0} \quad (g^L)_{l,1} \quad \cdots \quad (g^L)_{l,l-1}\big)\big((g^L)^{[l]}\big)^{-1} \chi^{[l]},$$

and on the other,

$$\big(\varphi^L\big)_1^{(l)}(z) = \sum_{m=0}^{l} \sum_{j=0}^{l} \big(S_2^{[l+1]}\big)_{l,m} \big(((g^L)^{[l+1]})^{-1}\big)_{m,j} \chi^{(j)}$$

$$= \big(S_2^{[l+1]}\big)_{l,l} \big([(g^L)^{[l+1]}]^{-1}\big)_{l,j} \chi^{(j)}$$

$$= (S_2)_{ll} \big(0 \quad 0 \quad \ldots \quad 0 \quad \mathbb{I}\big)\big((g^L)^{[l+1]}\big)^{-1} \chi^{[l+1]}.$$

Proceeding in a similar manner, one gets all the other identities.[3] $\quad\square$

Proof of Theorem 2. We will only prove the first equation as the other three are proven in a similar way. In particular, we first prove the second equality of the first equation. We are interested in evaluating the expression

$$\big[\big(\varphi_2^L\big)^{[2k]}(z)\big]^{\dagger}\big[\big(J^L\big)^{[2k]}\big]\big(\varphi_1^L\big)^{[2k]}(z')$$

in two different ways. On the one hand, we could first let J act to the right. Truncating the expression

$$J^L \varphi_1^L(z) = z \varphi_1^L(z),$$

we have

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[3] It is interesting to notice that in order to prove the right case expressions, once we have worked out the left ones, there is no need to go over the same calculations again. It is enough to realize that

$$\varphi_1^R(\bar{z}) = \chi^{\dagger}(z) Z_1 \text{ same structure as } [\varphi_2^L(z)]^{\dagger} = \chi^{\dagger}(z) S_2^{-1},$$

$$\varphi_1^L(z) = S_1 \chi(z) \text{ same structure as } [\varphi_2^R(\bar{z})]^{\dagger} = Z_2^{-1} \chi(z).$$

$$\left[\left(J^L\right)^{[2k]}\right]\left(\varphi_1^L\right)^{[2k]}\left(z'\right) =$$

$$\begin{pmatrix} z'(\varphi_1^L)^{(0)}(z') \\ z'(\varphi_1^L)^{(1)}(z') \\ \vdots \\ -\alpha_{1,2k-1}^L h_{2k-2}^R [h_{2k-3}^R]^{-1}(\varphi_1^L)^{(2k-3)}(z') - \alpha_{1,2k-1}^L [\alpha_{2,2k-2}^R]^\dagger(\varphi_1^L)^{(2k-2)}(z') - \alpha_{1,2k}^L(\varphi_1^L)^{(2k-1)}(z') \\ h_{2k-1}^R [h_{2k-3}^R]^{-1}(\varphi_1^L)^{(2k-3)}(z') + h_{2k-1}^R [h_{2k-2}^R]^{-1}[\alpha_{2,2k-2}^R]^\dagger(\varphi_1^L)^{(2k-2)}(z') - [\alpha_{2,2k-1}^R]^\dagger\alpha_{1,2k}^L(\varphi_1^L)^{(2k-1)}(z') \end{pmatrix}.$$

But

$$z'\left(\varphi_1^L\right)^{(2k-2)}\left(z'\right) = -\alpha_{1,2k-1}^L h_{2k-2}^R \left[h_{2k-3}^R\right]^{-1}\left(\varphi_1^L\right)^{(2k-3)}\left(z'\right)$$
$$- \alpha_{1,2k-1}^L \left[\alpha_{2,2k-2}^R\right]^\dagger\left(\varphi_1^L\right)^{(2k-2)}\left(z'\right)$$
$$- \alpha_{1,2k}^L \left(\varphi_1^L\right)^{(2k-1)}\left(z'\right) + \left(\varphi_1^L\right)^{(2k)}\left(z'\right)$$
$$z'\left(\varphi_1^L\right)^{(2k-1)}\left(z'\right) = h_{2k-1}^R \left[h_{2k-3}^R\right]^{-1}\left(\varphi_1^L\right)^{(2k-3)}\left(z'\right)$$
$$+ h_{2k-1}^R \left[h_{2k-2}^R\right]^{-1}\left[\alpha_{2,2k-2}^R\right]^\dagger\left(\varphi_1^L\right)^{(2k-2)}\left(z'\right)$$
$$- \left[\alpha_{2,2k-1}^R\right]^\dagger\alpha_{1,2k}^L\left(\varphi_1^L\right)^{(2k-1)}\left(z'\right) + \left[\alpha_{2,2k-1}^R\right]^\dagger\left(\varphi_1^L\right)^{(2k)}\left(z'\right),$$

so that we obtain

$$\left[\left(\varphi_2^L\right)^{[2k]}(z)\right]^\dagger\left[J_L^{[2k]}\right]\left(\varphi_1^L\right)^{[2k]}\left(z'\right)$$
$$= \left(\left[\left(\varphi_2^L\right)^{(0)}(z)\right]^\dagger, \left[\left(\varphi_2^L\right)^{(1)}(z)\right]^\dagger, \ldots, \left[\left(\varphi_2^L\right)^{(2k-1)}(z)\right]^\dagger\right)$$
$$\cdot \begin{pmatrix} z'(\varphi_1^L)^{(0)}(z') \\ z'(\varphi_1^L)^{(1)}(z') \\ \vdots \\ z'(\varphi_1^L)^{(2k-2)}(z') - (\varphi_1^L)^{(2k)}(z') \\ z'(\varphi_1^L)^{(2k-1)}(z') - [\alpha_{2,2k-1}^R]^\dagger(\varphi_1^L)^{(2k)}(z') \end{pmatrix}.$$

On the other hand, we could let J^L act to the left and remember that

$$\left[\left(\varphi_2^L\right)(z)\right]^\dagger J^L = \bar{z}^{-1}\left[\left(\varphi_2^L\right)(z)\right]^\dagger.$$

So, truncating the expression as we did when J^L acted to the right, we are left with

$$\left[\left(\varphi_2^L\right)^{[2k]}(z)\right]^\dagger\left(J^L\right)^{[2k]}$$
$$= \left(\bar{z}^{-1}\left[\left(\varphi_2^L\right)^{(0)}(z)\right]^\dagger, \ldots, \bar{z}^{-1}\left[\left(\varphi_2^L\right)^{(2k-2)}(z)\right]^\dagger,\right.$$
$$\left.\left[\left(\varphi_2^L\right)^{(2k-2)}(z)\right]^\dagger\left(-\alpha_{1,2k}^L\right) + \left[\left(\varphi_2^L\right)^{(2k-1)}(z)\right]^\dagger\left(-\left[\alpha_{2,2k-1}^R\right]^\dagger\alpha_{1,2k}^L\right)\right).$$

But we also have

$$\bar{z}^{-1}\big[\big(\varphi_2^L\big)^{(2k-1)}(z)\big]^\dagger = \big[\big(\varphi_2^L\big)^{(2k-2)}(z)\big]^\dagger\big(-\alpha_{1,2k}^L\big) + \big[\big(\varphi_2^L\big)^{(2k-1)}(z)\big]^\dagger\big(-\big[\alpha_{2,2k-1}^R\big]^\dagger\alpha_{1,2k}^L\big)$$
$$+ \big[\big(\varphi_2^L\big)^{(2k)}(z)\big]^\dagger\big[-\alpha_{1,2k+1}^L h_{2k}^R\big(h_{2k-1}^R\big)^{-1}\big]$$
$$+ \big[\big(\varphi_2^L\big)^{(2k+1)}(z)\big]^\dagger\big[h_{2k+1}^R\big(h_{2k-1}^R\big)^{-1}\big].$$

So, inserting it into the equation we are interested in, we have

$$\big[\big(\varphi_2^L\big)^{[2k]}(z)\big]^\dagger\big[\big(J^L\big)^{[2k]}\big]\big(\varphi_1^L\big)^{[2k]}(z')$$
$$= \big(\bar{z}^{-1}\big[\big(\varphi_2^L\big)^{(0)}(z)\big]^\dagger, \ldots, \bar{z}^{-1}\big[\big(\varphi_2^L\big)^{(2k-1)}(z)\big]^\dagger\big[\big(\varphi_2^L\big)^{(2k)}(z)\big]^\dagger\alpha_{1,2k+1}^L h_{2k}^R\big(h_{2k-1}^R\big)^{-1}$$
$$- \big[\big(\varphi_2^L\big)^{(2k+1)}(z)\big]^\dagger h_{2k+1}^R\big(h_{2k-1}^R\big)^{-1}\big)$$
$$\times \begin{pmatrix} \big(\varphi_1^L\big)^{(0)}(z') \\ \big(\varphi_1^L\big)^{(1)}(z') \\ \vdots \\ \big(\varphi_1^L\big)^{(2k-1)}(z') \end{pmatrix}.$$

Hence, we are left with the result we wanted to prove

$$\bar{z}^{-1}\big[\big(\varphi_2^L\big)^{[2k]}(z)\big]^\dagger \cdot \big(\varphi_1^L\big)^{[2k]}(z') + \big[\big[\big(\varphi_2^L\big)^{(2k)}(z)\big]^\dagger\alpha_{1,2k+1}^L h_{2k}^R\big(h_{2k-1}^R\big)^{-1}$$
$$- \big[\big(\varphi_2^L\big)^{(2k+1)}(z)\big]^\dagger h_{2k+1}^R\big(h_{2k-1}^R\big)^{-1}\big]\big(\varphi_1^L\big)^{(2k-1)}(z')$$
$$= \big[\big(\varphi_2^L\big)^{[2k]}(z)\big]^\dagger \cdot z'\big(\varphi_1^L\big)^{[2k]}(z') - \big[\big[\big(\varphi_2^L\big)^{(2k-2)}(z)\big]^\dagger$$
$$+ \big[\big(\varphi_2^L\big)^{(2k-1)}(z)\big]^\dagger\big[\alpha_{2,2k-1}^R\big]^\dagger\big]\big(\varphi_1^L\big)^{(2k)}(z').$$

Finally, the first equality in the first equation follows from the just proven result and Proposition 20. As was said at the beginning of this proof, the rest of the relations are proven in the exact same way. $\square$

Proof of Proposition 26. First of all, we have

$$\frac{\partial}{\partial t_{j,a}^L}W_0\big(t^L\big) = \big[E_{a,a}\chi(\Upsilon)^{(j)}\big]W_0\big(t^L\big) \quad\Longrightarrow\quad \partial_{L,j}W_0\big(t^L\big) = \chi(\Upsilon)^{(j)}W_0\big(t^L\big),$$

$$\frac{\partial}{\partial t_{j,a}^R}V_0\big(t^R\big) = V_0\big(t^R\big)\big[\chi\big(\Upsilon^{-1}\big)^{(j)}E_{a,a}\big] \quad\Longrightarrow\quad \partial_{R,j}V_0\big(t^R\big) = V_0\big(t^R\big)\chi\big(\Upsilon^{-1}\big)^{(j)}.$$

The previous derivatives make sense and are well defined since the two factors in the results commute. Hence, we have

$$\frac{\partial}{\partial t_{j,a}^L}W_1^L(t) = \Big[\Big(\frac{\partial}{\partial t_{j,a}^L}S_1(t)\Big)S_1(t)^{-1} + S_1(t)\big[E_{a,a}\chi(\Upsilon)^{(j)}\big]S_1(t)^{-1}\Big]W_1^L(t),$$

$$\frac{\partial}{\partial t_{j,a}^R}W_1^L(t) = \Big[\Big(\frac{\partial}{\partial t_{j,a}^R}S_1(t)\Big)S_1(t)^{-1}\Big]W_1^L(t),$$

$$\frac{\partial}{\partial t_{j,a}^R} W_2^L(t) = \left[\left(\frac{\partial}{\partial t_{j,a}^R} S_2(t) \right) S_2(t)^{-1} - S_2(t) \left[E_{a,a} \chi(\Upsilon)^{(j)} \right] S_2(t)^{-1} \right] W_2^L(t),$$

$$\frac{\partial}{\partial t_{j,a}^L} W_2^L(t) = \left[\left(\frac{\partial}{\partial t_{j,a}^L} S_2(t) \right) S_2(t)^{-1} \right] W_2^L(t),$$

$$\frac{\partial}{\partial t_{j,a}^L} W_2^R(t) = W_2^R(t) \left[Z_2(t)^{-1} \left(\frac{\partial}{\partial t_{j,a}^L} Z_2(t) \right) - Z_2(t)^{-1} \left[E_{a,a} \chi(\Upsilon^{-1})^{(j)} \right] Z_2(t) \right],$$

$$\frac{\partial}{\partial t_{j,a}^R} W_2^R(t) = W_2^R(t) \left[Z_2(t)^{-1} \left(\frac{\partial}{\partial t_{j,a}^R} Z_2(t) \right) \right],$$

$$\frac{\partial}{\partial t_{j,a}^R} W_1^R(t) = W_1^R(t) \left[Z_1(t)^{-1} \left(\frac{\partial}{\partial t_{j,a}^R} Z_1(t) \right) + Z_1(t)^{-1} \left[\chi(\Upsilon^{-1})^{(j)} E_{a,a} \right] Z_1(t) \right],$$

$$\frac{\partial}{\partial t_{j,a}^L} W_1^R(t) = W_1^R(t) \left[Z_1(t)^{-1} \left(\frac{\partial}{\partial t_{j,a}^L} Z_1(t) \right) \right].$$

Now, if we let $\frac{\partial}{\partial t_{j,a}^H}$ act on both sides of the first expression in (42), we obtain

$$\left(\frac{\partial}{\partial t_{j,a}^L} S_1(t) \right) S_1(t)^{-1} + S_1(t) \left[E_{a,a} \chi(\Upsilon)^{(j)} \right] S_1(t)^{-1} = \left(\frac{\partial}{\partial t_{j,a}^L} S_2(t) \right) S_2(t)^{-1},$$

$$\left(\frac{\partial}{\partial t_{j,a}^R} S_2(t) \right) S_2(t)^{-1} - S_2(t) \left[\chi(\Upsilon)^{(j)} E_{a,a} \right] S_2(t)^{-1} = \left(\frac{\partial}{\partial t_{j,a}^R} S_1(t) \right) S_1(t)^{-1},$$

which implies

$$\left(\frac{\partial}{\partial t_{j,a}^L} S_2(t) \right) S_2(t)^{-1} = \left(S_1(t) \left[E_{a,a} \chi(\Upsilon)^{(j)} \right] S_1(t)^{-1} \right)_+,$$

$$\left(\frac{\partial}{\partial t_{j,a}^L} S_1(t) \right) S_1(t)^{-1} = -\left(S_1(t) \left[E_{a,a} \chi(\Upsilon)^{(j)} \right] S_1(t)^{-1} \right)_-,$$

$$\left(\frac{\partial}{\partial t_{j,a}^R} S_2(t) \right) S_2(t)^{-1} = \left(S_2(t) \left[E_{a,a} \chi(\Upsilon)^{(j)} \right] S_2(t)^{-1} \right)_+,$$

$$\left(\frac{\partial}{\partial t_{j,a}^R} S_1(t) \right) S_1(t)^{-1} = -\left(S_2(t) \left[E_{a,a} \chi(\Upsilon)^{(j)} \right] S_2(t)^{-1} \right)_-.$$

Similarly, let $\frac{\partial}{\partial t_{j,a}^H}$ act on both sides of the second expression in (42)

$$Z_2(t)^{-1} \left(\frac{\partial}{\partial t_{j,a}^L} Z_2(t) \right) - Z_2(t)^{-1} \left[E_{a,a} \chi(\Upsilon^{-1})^{(j)} \right] Z_2(t) = Z_1^{-1} \left(\frac{\partial}{\partial t_{j,a}^L} Z_1(t) \right),$$

$$Z_1(t)^{-1} \left(\frac{\partial}{\partial t_{j,a}^R} Z_1(t) \right) + Z_1(t)^{-1} \left[\chi(\Upsilon^{-1})^{(j)} E_{a,a} \right] Z_1(t) = Z_2^{-1} \left(\frac{\partial}{\partial t_{j,a}^R} Z_2(t) \right).$$

This means

$$Z_2(t)^{-1}\left(\frac{\partial}{\partial t_{j,a}^L}Z_2(t)\right) = \left(Z_2(t)^{-1}\left[E_{a,a}\chi(\Upsilon^{-1})^{(j)}\right]Z_2(t)\right)_-,$$

$$Z_1(t)^{-1}\left(\frac{\partial}{\partial t_{j,a}^L}Z_1(t)\right) = -\left(Z_2(t)^{-1}\left[E_{a,a}\chi(\Upsilon^{-1})^{(j)}\right]Z_2(t)\right)_+,$$

$$Z_2(t)^{-1}\left(\frac{\partial}{\partial t_{j,a}^R}Z_2(t)\right) = \left(Z_1(t)^{-1}\left[E_{a,a}\chi(\Upsilon^{-1})^{(j)}\right]Z_1(t)\right)_-,$$

$$Z_1(t)^{-1}\left(\frac{\partial}{\partial t_{j,a}^R}Z_1(t)\right) = -\left(Z_1(t)^{-1}\left[E_{a,a}\chi(\Upsilon^{-1})^{(j)}\right]Z_1(t)\right)_+.$$

With all these results, it is easy to prove both the linear systems for the wave functions and the Lax equations. For the flows of the intertwining operators, we use these relations together with (41), the first expression for the right times and the second one for the left times; then just recall (40). Finally, the Zakharov–Shabat equations are just the compatibility conditions of the Lax equations. $\square$

Proof of Proposition 32. First we use (19) and apply a Miwa shift

$$\mathfrak{M}_w^+(h_{2l-1}^R) = \oint_{\mathbb{T}} \mathfrak{M}_w^+\big((\varphi_1^L)^{(2l-1)}\big)(u)\frac{\mathrm{d}\mathfrak{M}_w^+(\mu)(u)}{\mathrm{i}u}u^l.$$

Second, from Theorem 5 ($w_k = w$ for $k = 1,\ldots,m$), we get

$$\big(\big[(\varphi_2^L)^{(2l-1)}(\bar{w})\big]^\dagger\big)^{-1}K^{L,[2l]}(\bar{w},u) = \mathfrak{M}_w^+\big[(\varphi_1^L)^{(2l-1)}\big](u),$$

so that

$$\big[(\varphi_2^L)^{(2l-1)}(\bar{w})\big]^\dagger\mathfrak{M}_w^+(h_{2l-1}^R)$$

$$= \oint_{\mathbb{T}} K^{L,[2l]}(\bar{w},u)\frac{\mathrm{d}\mathfrak{M}_w^+(\mu)(u)}{\mathrm{i}u}u^l$$

$$= \oint_{\mathbb{T}} K^{L,[2l]}(\bar{w},u)(1-wu)\frac{\mathrm{d}\mu(u)}{\mathrm{i}u}u^l \quad \big(\text{by }(51)\big)$$

$$= \oint_{\mathbb{T}} \big((\varphi_1^R)^{(2l)}(w^{-1})h_{2l}^R(h_{2l-1}^R)^{-1}(\varphi_1^L)^{(2l-1)}(u)$$

$$\quad - (\varphi_1^R)^{(2l-1)}(w^{-1})(\varphi_1^L)^{(2l)}(u)\big)\frac{\mathrm{d}\mu(u)}{\mathrm{i}u}u^l \quad \big(\text{by }(31)\big)$$

$$= \oint_{\mathbb{T}} (\varphi_1^R)^{(2l)}(w^{-1})h_{2l}^Rh_{2l-1}^R(\varphi_1^L)^{(2l-1)}(u)\frac{\mathrm{d}\mu(u)}{\mathrm{i}u}u^l \quad \big(\text{by }(15)\big)$$

$$= (\varphi_1^R)^{(2l)}(w^{-1})h_{2l}^R \quad \big(\text{by }(19)\big).$$

This same procedure applies for the proof of the remaining formulae. $\square$

Proof of Theorem 6. Since we can take any value for w_1, w_2, let us consider them our variables and name them $w_1 = w_2 = z$. Now by iteration of the formulae in Proposition 32, we get

$$\left[\left(\varphi_2^R\right)^{(2l+1)}\left(\bar{z}^{-1}\right)\right]^\dagger = \mathfrak{M}_{z^{-1}}^+\left(h_{2l}^L\right)\left(h_{2l}^L\right)^{-1}\cdots\mathfrak{M}_{z^{-1}}^+\left(h_1^L\right)\left(h_1^L\right)^{-1}\left[\left(\varphi_2^R\right)^{(1)}\left(\bar{z}^{-1}\right)\right]^\dagger z^l,$$

$$\left(\varphi_1^L\right)^{(2l)}(z) = \mathfrak{M}_{z^{-1}}^+\left(h_{2l-1}^L\right)\left(h_{2l-1}^L\right)^{-1}\cdots\mathfrak{M}_{z^{-1}}^+\left(h_1^L\right)\left(h_1^L\right)^{-1}\mathfrak{M}_{z^{-1}}^+\left(h_0^L\right)\left(h_0^L\right)^{-1}z^l,$$

$$\left[\left(\varphi_2^L\right)^{(2l+1)}\left(\bar{z}^{-1}\right)\right]^\dagger = z^l\left[\left(\varphi_2^L\right)^{(1)}\left(\bar{z}^{-1}\right)\right]^\dagger\mathfrak{M}_{z^{-1}}^+\left(h_1^R\right)\cdots\left(h_{2l}^R\right)^{-1}\mathfrak{M}_{z^{-1}}^+\left(h_{2l}^R\right)\left(h_{2l+1}^R\right)^{-1},$$

$$\left(\varphi_1^R\right)^{(2l)}(z) = z^l\left[h_0^R\right]^{-1}\mathfrak{M}_2^z\left(h_0^R\right)\left(h_1^R\right)^{-1}\mathfrak{M}_{z^{-1}}^+\left(h_1^R\right)\cdots\left(h_{2l-2}^R\right)^{-1}\mathfrak{M}_2^z\left(h_{2l-2}^R\right)$$
$$\times\left(h_{2l-1}^R\right)^{-1}\mathfrak{M}_2^z\left(h_{2l-1}^R\right)\left(h_{2l}^R\right)^{-1},$$

$$\left(\varphi_1^L\right)^{(2l+1)}(z) = \mathfrak{M}_z^-\left(h_{2l}^R\right)\left(h_{2l}^R\right)^{-1}\mathfrak{M}_z^-\left(h_{2l-1}^R\right)\left(h_{2l-1}^R\right)^{-1}\cdots\mathfrak{M}_z^-\left(h_1^R\right)$$
$$\times\left(h_1^R\right)^{-1}\left(\varphi_1^L\right)^{(1)}(z)z^{-l},$$

$$\left[\left(\varphi_2^R\right)^{(2l)}\left(\bar{z}^{-1}\right)\right]^\dagger = \mathfrak{M}_z^-\left(h_{2l-1}^R\right)\left(h_{2l-1}^R\right)^{-1}\cdots\mathfrak{M}_z^-\left(h_0^R\right)\left(h_0^R\right)^{-1}z^{-l},$$

$$\left(\varphi_1^R\right)^{(2l+1)}(z) = z^{-l}\left(\varphi_1^R\right)^{(1)}(z)\mathfrak{M}_z^-\left(h_1^L\right)\cdots\left(h_{2l-1}^L\right)^{-1}\mathfrak{M}_z^-\left(h_{2l-1}^L\right)$$
$$\times\left(h_{2l}^L\right)^{-1}\mathfrak{M}_z^-\left(h_{2l}^L\right)\left(h_{2l+1}^L\right)^{-1},$$

$$\left[\left(\varphi_2^L\right)^{(2l)}\left(\bar{z}^{-1}\right)\right]^\dagger = z^{-l}\left(h_0^L\right)^{-1}\mathfrak{M}_z^-\left(h_0^L\right)\cdots\left(h_{2l-1}^L\right)^{-1}\mathfrak{M}_z^-\left(h_{2l-1}^L\right)\left(h_{2l}^L\right)^{-1}.$$

Finally, noticing that

$$\mathfrak{M}_{z^{-1}}^+\left(h_0^L\right) = \mathfrak{M}_{z^{-1}}^+\left(h_0^R\right) = \mathbb{I} - \frac{1}{z}\left(g^L\right)_{01} = \mathbb{I} - \frac{1}{z}\left(g^R\right)_{10},$$

$$\mathfrak{M}_z^-\left(h_0^L\right) = \mathfrak{M}_z^-\left(h_0^R\right) = \mathbb{I} - z\left(g^L\right)_{10} = \mathbb{I} - z\left(g^R\right)_{01},$$

we get

$$\left[\left(\varphi_2^R\right)^{(1)}\left(\bar{z}^{-1}\right)\right]^\dagger = z\left(\mathbb{I} - \frac{1}{z}\left(g^R\right)_{10}\right) = z\mathfrak{M}_{z^{-1}}^+\left(h_0^L\right),$$

$$\left[\left(\varphi_2^L\right)^{(1)}\left(\bar{z}^{-1}\right)\right]^\dagger = z\left(\mathbb{I} - \frac{1}{z}\left(g^L\right)_{01}\right)\left(h_1^R\right)^{-1} = z\mathfrak{M}_{z^{-1}}^+\left(h_0^R\right)\left(h_1^R\right)^{-1},$$

$$\left(\varphi_1^L\right)^{(1)}(z) = \frac{1}{z}\left(\mathbb{I} - z\left(g^L\right)_{10}\right) = \frac{1}{z}\mathfrak{M}_z^-\left(h_0^R\right),$$

$$\left(\varphi_1^R\right)^{(1)}(z) = \frac{1}{z}\left(\mathbb{I} - z\left(g^R\right)_{01}\right)\left(h_1^L\right)^{-1} = \frac{1}{z}\mathfrak{M}_z^-\left(h_0^L\right)\left(h_1^L\right)^{-1},$$

and the result is proven. $\square$

Appendix B. Explicit coefficients of J and C

Proposition 33. *The following expressions correspond to the block non-zero elements of $(J^H)^{\pm 1}$:*

$$\left(J^L\right)_{2k,2k-1} = -h^L_{2k}\alpha^R_{1,2k+1}\left(h^R_{2k-1}\right)^{-1}, \qquad \left(J^R\right)_{2k,2k-1} = h^R_{2k}\left[\alpha^L_{2,2k+1}\right]^\dagger\left(h^L_{2k-1}\right)^{-1},$$

$$\left(J^L\right)_{2k,2k} = -h^L_{2k}\alpha^R_{1,2k+1}\left[\alpha^L_{2,2k}\right]^\dagger\left(h^L_{2k}\right)^{-1}, \qquad \left(J^R\right)_{2k,2k} = -h^R_{2k}\left[\alpha^L_{2,2k+1}\right]^\dagger\alpha^R_{1,2k}\left(h^R_{2k}\right)^{-1},$$

$$\left(J^L\right)_{2k,2k+1} = -\alpha^L_{1,2k+2}, \qquad \left(J^R\right)_{2k,2k+1} = -\left[\alpha^R_{2,2k+2}\right]^\dagger,$$

$$\left(J^L\right)_{2k,2k+2} = \mathbb{I}, \qquad \left(J^R\right)_{2k,2k+2} = \mathbb{I},$$

$$\left(J^L\right)_{2k+1,2k-1} = h^R_{2k+1}\left(h^R_{2k-1}\right)^{-1}, \qquad \left(J^R\right)_{2k+1,2k-1} = h^L_{2k+1}\left(h^L_{2k-1}\right)^{-1},$$

$$\left(J^L\right)_{2k+1,2k} = h^R_{2k+1}\left[\alpha^L_{2,2k}\right]^\dagger\left(h^L_{2k}\right)^{-1}, \qquad \left(J^R\right)_{2k+1,2k} = h^L_{2k+1}\alpha^R_{1,2k}\left(h^R_{2k}\right)^{-1},$$

$$\left(J^L\right)_{2k+1,2k+1} = -\left[\alpha^R_{2,2k+1}\right]^\dagger\alpha^L_{1,2k+2}, \qquad \left(J^R\right)_{2k+1,2k+1} = -\alpha^L_{1,2k+1}\left[\alpha^R_{2,2k+2}\right]^\dagger,$$

$$\left(J^L\right)_{2k+1,2k+2} = \left[\alpha^R_{2,2k+1}\right]^\dagger, \qquad \left(J^R\right)_{2k+1,2k+2} = \alpha^L_{1,2k+1},$$

$$\left(J^L\right)_{0,0} = -h^L_0\alpha^R_{1,1}\left(h^L_0\right)^{-1}, \qquad \left(J^R\right)_{0,0} = -h^R_0\left[\alpha^L_{2,1}\right]\dagger\left(h^R_0\right)^{-1},$$

$$\left(J^L\right)_{0,1} = -\alpha^L_{1,2}, \qquad \left(J^R\right)_{0,1} = -\left[\alpha^R_{2,2}\right]^\dagger,$$

$$\left(J^L\right)_{0,2} = \mathbb{I}, \qquad \left(J^R\right)_{0,2} = \mathbb{I},$$

$$\left(J^L\right)_{1,0} = h^R_1\left(h^L_0\right)^{-1}, \qquad \left(J^R\right)_{1,0} = h^L_1\left(h^R_0\right)^{-1},$$

$$\left(J^L\right)_{1,1} = -\left[\alpha^R_{1,2}\right]^\dagger\alpha^L_{1,2}, \qquad \left(J^R\right)_{1,1} = -\alpha^L_{1,1}\left[\alpha^R_{2,2}\right]^\dagger,$$

$$\left(J^L\right)_{1,2} = \left[\alpha^R_{1,2}\right]^\dagger, \qquad \left(J^R\right)_{1,2} = \alpha^L_{1,2},$$

$$\left(\left(J^L\right)^{-1}\right)_{2k-1,2k} = -\left[\alpha^R_{2,2k+1}\right]^\dagger, \qquad \left(\left(J^R\right)^{-1}\right)_{2k-1,2k} = -\alpha^R_{1,2k+1},$$

$$\left(\left(J^L\right)^{-1}\right)_{2k,2k} = -\alpha^L_{1,2k}\left[\alpha^R_{2,2k+1}\right]^\dagger, \qquad \left(\left(J^R\right)^{-1}\right)_{2k,2k} = -\left[\alpha^R_{2,2k}\right]^\dagger\alpha^L_{1,2k+1},$$

$$\left(\left(J^L\right)^{-1}\right)_{2k+1,2k} = -h^R_{2k+1}\left[\alpha^L_{2,2k+2}\right]^\dagger\left(h^L_{2k}\right)^{-1},$$

$$\left(\left(J^R\right)^{-1}\right)_{2k+1,2k} = -h^L_{2k+1}\alpha^R_{1,2k+2}\left(h^R_{2k}\right)^{-1},$$

$$\left(\left(J^L\right)^{-1}\right)_{2k+2,2k} = \left(h^L\right)_{2k+2}\left(h^L_{2k}\right)^{-1}, \qquad \left(\left(J^R\right)^{-1}\right)_{2k+2,2k} = \left(h^R\right)_{2k+2}\left(h^R_{2k}\right)^{-1},$$

$$\left(\left(J^L\right)^{-1}\right)_{2k-1,2k+1} = \mathbb{I}, \qquad \left(\left(J^R\right)^{-1}\right)_{2k-1,2k+1} = \mathbb{I},$$

$$\left(\left(J^L\right)^{-1}\right)_{2k,2k+1} = \alpha^L_{1,2k}, \qquad \left(\left(J^R\right)^{-1}\right)_{2k,2k+1} = \left[\alpha^R_{2,2k}\right]^\dagger,$$

$$\left((J^L)^{-1}\right)_{2k+1,2k+1} = -h^R_{2k+1}\left[\alpha^L_{2,2k+2}\right]^\dagger \alpha^R_{1,2k+1}\left(h^R_{2k+1}\right)^{-1},$$

$$\left((J^R)^{-1}\right)_{2k+1,2k+1} = -h^L_{2k+1}\alpha^R_{1,2k+2}\left[\alpha^L_{2,2k+1}\right]^\dagger\left(h^L_{2k+1}\right)^{-1},$$

$$\left((J^L)^{-1}\right)_{2k+2,2k+1} = h^L_{2k+2}\alpha^R_{1,2k+1}\left(h^R_{2k+1}\right)^{-1},$$

$$\left((J^R)^{-1}\right)_{2k+2,2k+1} = \left(h^R\right)_{2k+2}\left[\alpha^L_{2,2k+1}\right]^\dagger\left(h^L_{2k+1}\right)^{-1},$$

$$\left((J^L)^{-1}\right)_{0,0} = -\left[\alpha^R_{2,1}\right]^\dagger, \qquad\qquad \left((J^R)^{-1}\right)_{0,0} = -\alpha^L_{1,1},$$

$$\left((J^L)^{-1}\right)_{1,0} = -h^R_1\left[\alpha^L_{2,2}\right]^\dagger\left(h^L_0\right)^{-1}, \qquad \left((J^R)^{-1}\right)_{1,0} = -h^L_1\alpha^R_{1,2}\left(h^R_0\right)^{-1},$$

$$\left((J^L)^{-1}\right)_{2,0} = h^L_2\left(h^L_0\right)^{-1}, \qquad\qquad \left((J^R)^{-1}\right)_{2,0} = h^R_2\left(h^R_0\right)^{-1},$$

$$\left((J^L)^{-1}\right)_{0,1} = \mathbb{I}, \qquad\qquad \left((J^R)^{-1}\right)_{0,1} = \mathbb{I},$$

$$\left((J^L)^{-1}\right)_{1,1} = -h^R_1\left[\alpha^L_{2,2}\right]^\dagger\alpha^R_{1,1}\left(h^R_1\right)^{-1}, \quad \left((J^R)^{-1}\right)_{1,1} = -h^L_1\alpha^R_{1,2}\left[\alpha^L_{2,1}\right]^\dagger\left(h^L_1\right)^{-1},$$

$$\left((J^L)^{-1}\right)_{2,1} = h^L_2\alpha^R_{1,1}\left(h^R_1\right)^{-1}, \qquad \left((J^R)^{-1}\right)_{2,1} = h^R_2\left[\alpha^L_{2,1}\right]^\dagger\left(h^L_1\right)^{-1}.$$

Proposition 34. *The following expressions correspond to the block non-zero elements of $C^{[\pm 1]}_{[0]}$:*

$$(C_{[0]})_{2k,2k-1} = h^R_{2k}\left[\left(h^R\right)_{2k-1}\right]^{-1} = \mathbb{I} - \left[\alpha^R_{2,2k}\right]^\dagger\alpha^L_{1,2k},$$

$$(C_{[0]})_{2k,2k} = \left[\alpha^R_{2,2k}\right]^\dagger = h^R_{2k}\left[\alpha^L_{2,2k}\right]^\dagger\left[h^L_{2k}\right]^{-1},$$

$$(C_{[0]})_{2k+1,2k+1} = -\alpha^L_{1,2k+2} = -h^L_{2k+1}\alpha^R_{1,2k+2}\left[h^R_{2k+1}\right]^{-1},$$

$$(C_{[0]})_{2k+1,2k+2} = \mathbb{I} = h^L_{2k+1}\left[\mathbb{I} - \alpha^R_{1,2k+2}\left[\alpha^L_{2,2k+2}\right]^\dagger\right]\left[\left(h^L\right)_{2k+2}\right]^{-1},$$

$$\left(C^{-1}_{[0]}\right)_{2k,2k-1} = h^L_{2k}\left[\left(h^L\right)_{2k-11}\right]^{-1} = \mathbb{I} - \alpha^L_{1,2k}\left[\alpha^R_{2,2k}\right]^\dagger,$$

$$\left(C^{-1}_{[0]}\right)_{2k,2k} = \alpha^L_{1,2k} = h^L_{2k}\alpha^R_{1,2k}\left[h^R_{2k}\right]^{-1},$$

$$\left(C^{-1}_{[0]}\right)_{2k+1,2k+1} = -\left[\alpha^R_{2,2k+2}\right]^\dagger = -h^R_{2k+1}\left[\alpha^L_{2,2k+2}\right]^\dagger\left[h^L_{2k+1}\right]^{-1},$$

$$\left(C^{-1}_{[0]}\right)_{2k+1,2k+2} = \mathbb{I} = h^R_{2k+1}\left[\mathbb{I} - \left[\alpha^L_{2,2k+2}\right]^\dagger\alpha^R_{1,2k+2}\right]\left[h^R_{2k+2}\right]^{-1}.$$

Proposition 35. *The following expressions correspond to the block non-zero elements of $C^{[\pm 1]}_{[-1]}$:*

$$(C_{[-1]})_{2k,2k} = -\left[\alpha^R_{2,2k+1}\right]^\dagger = -h^R_{2k}\left[\alpha^L_{2,2k+1}\right]^\dagger\left[h^L_{2k}\right]^{-1},$$

$$(C_{[-1]^{-1}})_{2k,2k} = -\alpha^L_{1,2k+1} = -h^L_{2k}\alpha^L_{2,2k+1}\left[h^R_{2k}\right]^{-1},$$

$$(C_{[-1]})_{2k,2k+1} = \mathbb{I} \quad \left(C^{-1}_{[-1]}\right)_{2k,2k+1} = \mathbb{I},$$

$$(C_{[-1]})_{2k+1,2k} = \mathbb{I} - \alpha_{1,2k+1}^L \big[\alpha_{2,2k+1}^R\big]^\dagger = h_{2k+1}^L \big[h_{2k}^L\big]^{-1},$$

$$\big(C_{[-1]}^{-1}\big)_{2k+1,2k} = \mathbb{I} - \big[\alpha_{2,2k+1}^R\big]^\dagger \alpha_{1,2k+1}^L = h_{2k+1}^R \big[h_{2k}^R\big]^{-1},$$

$$(C_{[-1]})_{2k+1,2k+1} = \alpha_{1,2k+1}^L = h_{2k+1}^L \alpha_{1,2k+1}^R \big[h_{2k+1}^R\big]^{-1},$$

$$\big(C_{[-1]}^{-1}\big)_{2k+1,2k+1} = \big[\alpha_{2,2k+1}^R\big]^\dagger = h_{2k+1}^R \big[\alpha_{2,2k+1}^L\big]^\dagger \big[h_{2k+1}^L\big]^{-1}.$$

Appendix C. Complete recursion relations

Here we give a more complete set of recursion relations for the MOLPUC.

Proposition 36. *The five term CMV recursion relations are*

$$z\big(\varphi_1^L\big)^{(2k)}(z) = -\alpha_{1,2k+1}^L\big(\mathbb{I} - \big[\alpha_{2,2k}^R\big]^\dagger \alpha_{1,2k}^L\big)\big(\varphi_1^L\big)^{(2k-1)} - \alpha_{1,2k+1}^L\big[\alpha_{2,2k}^R\big]^\dagger\big(\varphi_1^L\big)^{(2k)}$$
$$- \alpha_{1,2k+2}^L\big(\varphi_1^L\big)^{(2k+1)} + \big(\varphi_1^L\big)^{(2k+2)},$$

$$z\big(\varphi_1^L\big)^{(2k+1)}(z) = \big(\mathbb{I} - \big[\alpha_{2,2k+1}^R\big]^\dagger \alpha_{1,2k+1}^L\big)\big(\mathbb{I} - \big[\alpha_{2,2k}^R\big]^\dagger \alpha_{1,2k}^L\big)\big(\varphi_1^L\big)^{(2k-1)}$$
$$+ \big(\mathbb{I} - \big[\alpha_{2,2k+1}^R\big]^\dagger \alpha_{1,2k+1}^L\big)\big[\alpha_{2,2k}^R\big]^\dagger\big(\varphi_1^L\big)^{(2k)}$$
$$- \big[\alpha_{2,2k+1}^R\big]^\dagger \alpha_{1,2k+2}^L\big(\varphi_1^L\big)^{(2k+1)} + \big[\alpha_{2,2k+1}^R\big]^\dagger\big(\varphi_1^L\big)^{(2k+2)},$$

$$z\big(\varphi_1^L\big)^{(0)}(z) = -\alpha_{1,1}^R\big(\varphi_1^L\big)^{(0)} - \alpha_{1,2}^L\big(\varphi_1^L\big)^{(1)} + \big(\varphi_1^L\big)^{(2)},$$

$$z\big(\varphi_1^L\big)^{(1)}(z) = \big(\mathbb{I} - \big[\alpha_{2,1}^R\big]^\dagger \alpha_{1,1}^L\big)\big(\varphi_1^L\big)^{(0)} - \big[\alpha_{2,1}^R\big]^\dagger \alpha_{1,2}^L\big(\varphi_1^L\big)^{(1)} + \big[\alpha_{2,1}^R\big]^\dagger\big(\varphi_1^L\big)^{(2)},$$

$$z^{-1}\big(\varphi_1^L\big)^{(2k)}(z) = \big(\mathbb{I} - \alpha_{1,2k}^L\big[\alpha_{2,2k}^R\big]^\dagger\big)\big(\mathbb{I} - \alpha_{1,2k-1}^L\big[\alpha_{2,2k-1}^R\big]^\dagger\big)\big(\varphi_1^L\big)^{(2k-2)}$$
$$+ \big(\mathbb{I} - \alpha_{1,2k}^L\big[\alpha_{2,2k}^R\big]^\dagger\big)\alpha_{1,2k-1}^L\big(\varphi_1^L\big)^{(2k-1)} - \alpha_{1,2k}^L\big[\alpha_{2,2k+1}^R\big]^\dagger\big(\varphi_1^L\big)^{(2k)}$$
$$+ \big[\alpha_{1,2k}^L\big]^\dagger\big(\varphi_1^L\big)^{(2k+1)},$$

$$z^{-1}\big(\varphi_1^L\big)^{(2k+1)}(z) = -\big[\alpha_{2,2k+2}^R\big]^\dagger\big(\mathbb{I} - \alpha_{1,2k+1}^L\big[\alpha_{2,2k+1}^R\big]^\dagger\big)\big(\varphi_1^L\big)^{(2k)}$$
$$- \big[\alpha_{2,2k+2}^R\big]^\dagger \alpha_{1,2k+1}^L\big(\varphi_1^L\big)^{(2k+1)} - \big[\alpha_{2,2k+3}^R\big]^\dagger\big(\varphi_1^L\big)^{(2k+2)}$$
$$+ \big(\varphi_1^L\big)^{(2k+3)},$$

$$z^{-1}\big(\varphi_1^L\big)^{(0)}(z) = -\big[\alpha_{2,1}^R\big]^\dagger\big(\varphi_1^L\big)^{(0)} + \big(\varphi_1^L\big)^{(1)},$$

$$\big[z\big(\varphi_2^L\big)^{(2k)}(z)\big]^\dagger = -\big[\big(\varphi_2^L\big)^{(2k-1)}(z)\big]^\dagger\big[\alpha_{2,2k+1}^R\big]^\dagger - \big[\big(\varphi_2^L\big)^{(2k)}\big]^\dagger \alpha_{1,2k}^L\big[\alpha_{2,2k+1}^R\big]^\dagger$$
$$- \big[\big(\varphi_2^L\big)^{(2k+1)}(z)\big]^\dagger\big[\alpha_{2,2k+2}^R\big]^\dagger\big(\mathbb{I} - \alpha_{1,2k+1}^L\big[\alpha_{2,2k+1}^R\big]^\dagger\big)$$
$$+ \big[\big(\varphi_2^L\big)^{(2k+2)}(z)\big]^\dagger\big(\mathbb{I} - \alpha_{1,2k+2}^L\big[\alpha_{2,2k+2}^R\big]^\dagger\big)\big(\mathbb{I} - \alpha_{1,2k+1}^L\big[\alpha_{2,2k+1}^R\big]^\dagger\big),$$

$$\left[z\big(\varphi_2^L\big)^{(2k+1)}(z)\right]^\dagger = \left[\big(\varphi_2^L\big)^{(2k-1)}(z)\right]^\dagger + \left[\big(\varphi_2^L\big)^{(2k)}(z)\right]^\dagger \alpha_{1,2k}^L$$
$$- \left[\big(\varphi_2^L\big)^{(2k+1)}(z)\right]^\dagger \left[\alpha_{2,2k+2}^R\right]^\dagger \alpha_{1,2k+1}^L$$
$$+ \left[\big(\varphi_2^L\big)^{(2k+2)}(z)\right]^\dagger \big(\mathbb{I} - \alpha_{1,2k+2}^L \left[\alpha_{2,2k+2}^R\right]^\dagger\big) \alpha_{1,2k+1}^L,$$

$$\left[z^{-1}\big(\varphi_2^L\big)^{(2k)}(z)\right]^\dagger = \left[\big(\varphi_2^L\big)^{(2k-2)}(z)\right]^\dagger - \left[\big(\varphi_2^L\big)^{(2k-1)}(z)\right]^\dagger \left[\alpha_{2,2k-1}^R\right]^\dagger$$
$$- \left[\big(\varphi_2^L\big)^{(2k)}(z)\right]^\dagger \alpha_{1,2k+1}^L \left[\alpha_{2,2k}^R\right]^\dagger$$
$$+ \left[\big(\varphi_2^L\big)^{(2k+1)}(z)\right]^\dagger \big(\mathbb{I} - \left[\alpha_{2,2k+1}^R\right]^\dagger \alpha_{1,2k+1}^L\big) \left[\alpha_{2,2k}^R\right]^\dagger,$$
$$\left[z^{-1}\big(\varphi_2^L\big)^{(2k+1)}(z)\right]^\dagger = -\left[\big(\varphi_2^L\big)^{(2k)}(z)\right]^\dagger \alpha_{1,2k+2}^L - \left[\big(\varphi_2^L\big)^{(2k+1)}\right]^\dagger \left[\alpha_{2,2k+1}^R\right]^\dagger \alpha_{1,2k+2}^L$$
$$- \left[\big(\varphi_2^L\big)^{(2k+2)}(z)\right]^\dagger \alpha_{1,2k+3}^L \big(\mathbb{I} - \left[\alpha_{2,2k+2}^R\right]^\dagger \alpha_{1,2k+2}^L\big)$$
$$+ \left[\big(\varphi_2^L\big)^{(2k+3)}(z)\right]^\dagger \big(\mathbb{I} - \left[\alpha_{2,2k+3}^R\right]^\dagger \alpha_{1,2k+3}^L\big)$$
$$\times \big(\mathbb{I} - \left[\alpha_{2,2k+2}^R\right]^\dagger \alpha_{1,2k+2}^L\big),$$

$$z\big(\varphi_1^R\big)^{(2k)} = -\big(\varphi_1^R\big)^{(2k-1)} \alpha_{1,2k+1}^L - \big(\varphi_1^R\big)^{(2k)} \left[\alpha_{2,2k}^R\right]^\dagger \alpha_{1,2k+1}^L$$
$$- \big(\varphi_1^R\big)^{(2k+1)} \alpha_{1,2k+2}^L \big(\mathbb{I} - \left[\alpha_{2,2k+1}^R\right]^\dagger \alpha_{1,2k+1}^L\big)$$
$$+ \big(\varphi_1^R\big)^{(2k+2)} \big(\mathbb{I} - \left[\alpha_{2,2k+2}^R\right]^\dagger \alpha_{1,2k+2}^L\big) \big(\mathbb{I} - \left[\alpha_{2,2k+1}^R\right]^\dagger \alpha_{1,2k+1}^L\big),$$
$$z\big(\varphi_1^R\big)^{(2k+1)} = \big(\varphi_1^R\big)^{(2k-1)} + \big(\varphi_1^R\big)^{(2k)} \left[\alpha_{2,2k}^R\right]^\dagger - \big(\varphi_1^R\big)^{(2k+1)} \alpha_{1,2k+2}^L \left[\alpha_{2,2k+1}^R\right]^\dagger$$
$$+ \big(\varphi_1^R\big)^{(2k+2)} \big(\mathbb{I} - \left[\alpha_{2,2k+2}^R\right]^\dagger \alpha_{1,2k+2}^L\big) \left[\alpha_{2,2k+1}^R\right]^\dagger,$$

$$z^{-1}\big(\varphi_1^R\big)^{(2k)} = \big(\varphi_1^R\big)^{(2k-2)} + \big(\varphi_1^R\big)^{(2k-1)} \alpha_{1,2k-1}^L$$
$$- \big(\varphi_1^R\big)^{(2k)} \alpha_{1,2k}^L \left[\alpha_{2,2k+1}^R\right]^\dagger + \big(\varphi_1^R\big)^{(2k+1)} \big(\mathbb{I} - \alpha_{1,2k+1}^L \left[\alpha_{2,2k+1}^R\right]^\dagger\big) \alpha_{1,2k}^L,$$
$$z^{-1}\big(\varphi_1^R\big)^{(2k-1)} = -\big(\varphi_1^R\big)^{(2k-2)} \left[\alpha_{2,2k}^R\right]^\dagger - \big(\varphi_1^R\big)^{(2k-1)} \alpha_{1,2k-1}^L \left[\alpha_{2,2k}^R\right]^\dagger$$
$$- \big(\varphi_1^R\big)^{(2k)} \left[\alpha_{1,2k+1}^L\right]^\dagger \big(\mathbb{I} - \alpha_{1,2k}^L \left[\alpha_{2,2k}^R\right]^\dagger\big)$$
$$+ \big(\varphi_1^R\big)^{(2k+1)} \big(\mathbb{I} - \alpha_{1,2k+1}^L \left[\alpha_{2,2k+1}^R\right]^\dagger\big) \big(\mathbb{I} - \alpha_{1,2k}^L \left[\alpha_{2,2k}^R\right]^\dagger\big),$$

$$\left[z\big(\varphi_2^R\big)^{(2k)}(z)\right]^\dagger = -\left[\alpha_{2,2k+1}^R\right]^\dagger \big(\mathbb{I} - \alpha_{1,2k}^L \left[\alpha_{2,2k}^R\right]^\dagger\big) \left[\big(\varphi_2^R\big)^{(2k+1)}(z)\right]^\dagger$$
$$- \left[\alpha_{2,2k+1}^R\right]^\dagger \alpha_{1,2k}^L \left[\big(\varphi_2^R\big)^{(2k)}(z)\right]^\dagger - \left[\alpha_{2,2k+2}^R\right]^\dagger \left[\big(\varphi_2^R\big)^{(2k+1)}(z)\right]^\dagger$$
$$+ \left[\big(\varphi_2^R\big)^{(2k+2)}(z)\right]^\dagger,$$

$$\left[z\left(\varphi_2^R\right)^{(2k+1)}(z)\right]^\dagger = \left(\mathbb{I} - \alpha_{1,2k+1}^L\left[\alpha_{2,2k+1}^R\right]^\dagger\right)\left(\mathbb{I} - \alpha_{1,2k}^L\left[\alpha_{2,2k}^R\right]^\dagger\right)\left[\left(\varphi_2^R\right)^{(2k-1)}(z)\right]^\dagger$$
$$+ \left(\mathbb{I} - \alpha_{1,2k+1}^L\left[\alpha_{2,2k+1}^R\right]^\dagger\right)\alpha_{1,2k}^L\left[\left(\varphi_2^R\right)^{(2k)}(z)\right]^\dagger$$
$$- \alpha_{1,2k+1}^L\left[\alpha_{2,2k+1}^R\right]^\dagger\left[\left(\varphi_2^R\right)^{(2k+1)}(z)\right]^\dagger + \alpha_{1,2k+1}^L\left[\left(\varphi_2^R\right)^{(2k+2)}(z)\right]^\dagger,$$

$$\left[z\left(\varphi_2^R\right)^{(0)}(z)\right]^\dagger = -\left[\alpha_{2,1}^R\right]^\dagger\left[\left(\varphi_2^R\right)^{(0)}(z)\right]^\dagger - \left[\alpha_{2,2}^R\right]^\dagger\left[\left(\varphi_2^R\right)^{(1)}(z)\right]^\dagger + \left[\left(\varphi_2^R\right)^{(2)}(z)\right]^\dagger,$$

$$\left[z\left(\varphi_2^R\right)^{(1)}(z)\right]^\dagger = \left(\mathbb{I} - \alpha_{1,1}^L\left[\alpha_{2,1}^R\right]^\dagger\right)\left[\left(\varphi_2^R\right)^{(0)}(z)\right]^\dagger$$
$$- \alpha_{1,1}^L\left[\alpha_{2,2}^R\right]^\dagger\left[\left(\varphi_2^R\right)^{(1)}(z)\right]^\dagger \alpha_{1,1}^L\left[\left(\varphi_2^R\right)^{(2)}(z)\right]^\dagger,$$

$$\left[z^{-1}\left(\varphi_2^R\right)^{(2k)}(z)\right]^\dagger = \left(\mathbb{I} - \left[\alpha_{2,2k}^R\right]^\dagger\alpha_{1,2k}^L\right)\left(\mathbb{I} - \left[\alpha_{2,2k-1}^R\right]^\dagger\alpha_{1,2k-1}^L\right)\left[\left(\varphi_2^R\right)^{(2k-2)}(z)\right]^\dagger$$
$$+ \left(\mathbb{I} - \left[\alpha_{2,2k}^R\right]^\dagger\alpha_{1,2k}^L\right)\left[\alpha_{2,2k-1}^R\right]^\dagger\left[\left(\varphi_2^R\right)^{(2k-1)}(z)\right]^\dagger$$
$$- \left[\alpha_{2,2k}^R\right]^\dagger\alpha_{1,2k+1}^L\left[\left(\varphi_2^R\right)^{(2k)}(z)\right]^\dagger + \left[\alpha_{2,2k}^R\right]^\dagger\left[\left(\varphi_2^R\right)^{(2k+1)}(z)\right]^\dagger,$$

$$\left[z^{-1}\left(\varphi_2^R\right)^{(2k-1)}(z)\right]^\dagger = -\alpha_{1,2k}^L\left(\mathbb{I} - \left[\alpha_{2,2k-1}^R\right]^\dagger\alpha_{1,2k-1}^L\right)\left[\left(\varphi_2^R\right)^{(2k-2)}(z)\right]^\dagger$$
$$- \alpha_{1,2k}^L\left[\alpha_{2,2k-1}^R\right]^\dagger\left[\left(\varphi_2^R\right)^{(2k-1)}(z)\right]^\dagger - \alpha_{1,2k+1}^L\left[\left(\varphi_2^R\right)^{(2k)}(z)\right]^\dagger$$
$$+ \left[\left(\varphi_2^R\right)^{(2k+1)}(z)\right]^\dagger,$$

$$\left[z^{-1}\left(\varphi_2^R\right)^{(0)}(z)\right]^\dagger = -\alpha_{1,1}\left[\left(\varphi_2^R\right)^{(0)}(z)\right]^\dagger + \left[\left(\varphi_2^R\right)^{(1)}(z)\right]^\dagger.$$

Appendix D. Projections in modules

For a ring $\mathbb{M}$ and left and right modules V and W over $\mathbb{M}$, respectively, bilinear forms are applications

$$G : V \times W \longrightarrow \mathbb{M}$$

such that

$$G(m_1 v_1 + m_2 v_2, w) = m_1 G(v_1, w) + m_2 G(v_2, w), \quad \forall m_1, m_2 \in \mathbb{M}, \; v, v_1, v_2 \in V,$$
$$G(v, w_1 m_1 + w_2 m_2) = G(v, w_1)m_1 + G(v, w_2)m_2, \quad \forall m_1, m_2 \in \mathbb{M}, \; w, w_1, w_2 \in W.$$

In free modules, any such bilinear form can be represented by a unique $l \times r$ matrix denoted also by G, with coefficients in the ring $\mathbb{M}$, as follows

$$G : V \times W \longrightarrow \mathbb{M},$$

$$G(v, w) := \begin{pmatrix} v_0 & \ldots & v_{l-1} \end{pmatrix} G \begin{pmatrix} w_0 \\ \vdots \\ w_{l-1} \end{pmatrix}.$$

Given free submodules $\tilde{V} \subset V$ and $\tilde{W} \subset W$ of the modules (not necessarily free) V, W and two bases $\{e_0, \ldots, e_{\tilde{l}-1}\} \subset \tilde{V}$ and $\{f_0, \ldots, f_{\tilde{r}-1}\} \subset \tilde{W}$ of $\tilde{V}$ and $\tilde{W}$, respectively, we denote $G_{i,j} = G(e_i, f_j)$. For the same rank, $\tilde{l} = \tilde{r}$, the matrix $\tilde{G} = (G_{i,j})$ can be assumed to be invertible, $\tilde{G} \in \mathrm{GL}(\tilde{l}, \mathbb{M}) \cong \mathrm{GL}(\tilde{l}m, \mathbb{C})$. In such case, we introduce the G-dual vectors to e_i, f_j defined as

$$e_i^* = \sum_{j=0}^{\tilde{l}-1} f_j \big(\tilde{G}^{-1}\big)_{j,i}, \qquad f_j^* = \sum_{i=0}^{\tilde{r}-1} \big(\tilde{G}^{-1}\big)_{j,i} e_i.$$

These vectors have some interesting properties:

(1) If we change basis $\hat{e}_j = \sum_{i=0}^{\tilde{l}-1} a_{j,i} e_i$ and $\hat{f}_j = \sum_{i=0}^{\tilde{l}-1} f_i b_{i,j}$, then

$$\hat{e}_j^* = \sum_{i=0}^{\tilde{l}-1} e_i^* \big(a^{-1}\big)_{i,j}, \qquad \hat{f}_i^* = \sum_{i=0}^{\tilde{l}-1} \big(b^{-1}\big)_{i,j} f_j^*,$$

where we have used the matrices $a = (a_{i,j})$ and $b = (b_{i,j})$, $a, b \in \mathrm{GL}(\tilde{l}, \mathbb{M})$.

(2) The sets of dual vectors $\{e_i^*\}_{i=0}^{\tilde{l}-1}$ and $\{f_i^*\}_{i=0}^{\tilde{l}-1}$ are bases with duals given by

$$\big(e_i^*\big)^* = e_i, \qquad \big(f_j^*\big)^* = f_j.$$

(3) It is easy to see that they satisfy the biorthogonal type identity

$$G\big(e_i, e_j^*\big) = G\big(f_i^*, f_j\big) = \delta_{i,j}, \quad \forall i, j = 0, \ldots, \tilde{l} - 1.$$

Given the bilinear form G, we can construct the associated projections on these

$$p : V \to \tilde{V}, \qquad p(v) := \sum_{i=0}^{\tilde{l}-1} G\big(v, e_i^*\big) e_i,$$

$$q : W \to \tilde{W}, \qquad q(w) := \sum_{j=0}^{\tilde{l}-1} f_j G\big(f_j^*, w\big).$$

These constructions are relevant when considering the Christoffel–Darboux operators and formulae in the matrix context.

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