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Competition among auctioneers in large markets#

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Abstract

We analyse a multistage game of competition among auctioneers. First, the auctioneers commit to some reserve prices; second, the bidders enter one auction, if any; and finally, the auctions take place. We show that for any finite set of feasible reserve prices, each auctioneer announces a reserve price equal to his production cost if the numbers of auctioneers and bidders are sufficiently large, though finite. Our result supports the idea that optimal auctions may be quite simple. Our model also confirms previous results for some "limit" versions of the model by McAfee (Econometrica 61 (1993) 1281–1312), Peters (Rev. Econ. Stud. 64 (1997) 97–123), and Peters and Severinov (J. Econ. Theory 75 (1997) 141–179).

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1. Introduction

In this paper, we study a multistage game of competition among auctioneers. In the first stage auctioneers compete for a common pool of bidders by means of credible announcements of the minimum price accepted in a second price auction. In the second stage each bidder chooses an auction, if any, in which to participate. Finally, in the last stage each of the announced auctions takes place. This time structure was originally suggested by McAfee [3] in his pioneering work on competition among auctioneers.

We show that for any finite auctioneers' strategy space, if the number of auctioneers[1] is sufficiently large, and the set of production costs lies in the set of reserve prices, auctioneers announce a reserve price equal to their respective production costs in the unique symmetric equilibrium. Thus, our result supports the idea that optimal auctions may be easy to implement even when the seller has uncertainty about the market. In this case, we show that the optimal reserve price in a second price auction does not need to be fine-tuned to the auctioneer's beliefs about the market. Indeed, the optimal reserve price is quite simple, it equals the production cost.

Related results have been proven in previous papers under the assumption that the number of agents is infinite and with a continuous strategy space, for example by McAfee[2] [3], Peters [4], and Peters and Severinov [8]. The intuition that underlines these papers is that bidders' expected utility is always determined by the market, and hence, invariant to changes in one single auction if there is an infinite number of agents. Given that the bidders' expected utility is taken as given by the auctioneer, his best strategy is to announce a mechanism that creates the maximum surplus. For instance, if the auctioneer only chooses the reserve price of a second price auction, this means to fix the reserve price equal to the production cost.

However, the simplicity of this optimal rule seems to hinge on the assumption that the number of auctioneers is infinite. Even if the number of agents is large, changes in one single auction have an effect, albeit small, on the bidders' expected utility. Thus, auctioneers could profit by reducing the bidders' expected utility with a deviation from the mechanism that maximises the surplus. In fact, for the case in which all the auctioneers have the same production cost, we could show that the strategy *reserve price equal to production cost* is not an equilibrium when the number of auctioneers is finite. Actually, it is not even clear that the equilibrium of the finite game is close in any sense to the equilibrium of the limit game.

In general, we would expect that the auctioneers' equilibrium behaviour is more complex. For instance, each auctioneer in equilibrium may need to adjust his reserve

[1] We keep the proportion of bidders to auctioneers fix, hence the number of bidders grows at the same rate as the number of auctioneers.

[2] McAfee [3] does not exactly assume that the numbers of auctioneers and bidders are infinite. Instead, McAfee assumes that an auctioneer does not take into account that when he changes his mechanism, the expected utility that bidders can get in other auction mechanisms changes. McAfee justifies this assumption conjecturing that it should be true in the limit when the number of auctioneers and bidders tends to infinity.

price to his beliefs about the demand and supply primitives and possibly to his beliefs about the behaviour of the other auctioneers.

Our work shows that the intuition suggested in the last two paragraphs is somewhat misleading. The reason for the break down of the result when the number of auctioneers is finite is that by assuming that the strategy space is continuous we allow arbitrary small deviations. If we assume more realistically that the number of strategies is finite, then deviations from the surplus maximising strategy have losses bounded away from zero. Moreover, gains due to decreases in bidders' expected utility tend to zero as the number of auctioneers tends to infinity. Hence, the limit result price equal to production cost will also be an equilibrium for a large enough number of auctioneers.

Our work fills the gap in the literature that exists between the limit models with an infinite number of auctioneers and the models with only two auctioneers. The papers we mention above, [3,4,8], belong to the former group and an example of the latter is the paper by Burguet and Sákovics [2]. We also prove that there is a unique symmetric equilibrium, whereas the above papers only provide existence results. One exception is the paper by Peters and Severinov [8]. They prove uniqueness of equilibria for the case in which all the auctioneers have the same production cost. Our uniqueness result differs in that we look at the case in which auctioneers may have different production costs.

The problem of competition among auctioneers under limit assumptions about the number of auctioneers has also been studied by Peters [5,7] in other frameworks. For instance, the first paper deals with the private value model under the assumption of correlated types, and the second with the common value model. We restrict to the private value model with independent types.

Our model also relates to the model of price competition by Peters [6]. He studies the convergence of the exact equilibrium with a finite number of sellers to the equilibrium defined under different infinite number of sellers' assumptions. Our model differs in two aspects. First, we study a model of competition in auctions, not in prices. Second, we do not look to equilibria of limit games or approximate equilibria, but rather consider exact equilibria of large finite games.

The structure of the paper is as follows. We start with a description of the model in Section 2. Section 3 includes the main result of the paper. Section 4 concludes. We also include an Appendix with most of the proofs.

2. The model

We assume that there are $J \in \mathbb{N}$ auctioneers and $kJ \in \mathbb{N}$ bidders. We shall later consider the limit $J \to \infty$. When doing this, we shall keep the ratio $k > 0$ of bidders to auctioneers fixed.

Each auctioneer has the ability to produce a single indivisible unit of output at a private cost w. We allow w to differ across the auctioneers, although we assume that each auctioneer's private cost is common knowledge. This last assumption is done to keep coherency with the literature (e.g. [6]). The model could be extended with similar results to the case in which the production cost is private information.

We also take a given distribution H with support Π_W and assume that for each J the number of auctioneers with production cost less or equal than w equals the maximum integer j for which $j/J \leqslant H(w)$. Note that as J tends to infinity, the distribution of production costs converges point-wise, and thus weakly, to H.

Each bidder wishes to purchase exactly one unit of the commodity. Its value for each bidder, x, is privately observed before the beginning of the game. All other players only know that these values are independently drawn from the set $[0, 1]$ according to the same distribution function F with a density f and support[3] $[0, 1]$.

If an auctioneer with production cost w trades with a bidder with type x at a price p, they are assumed to obtain a von Neumann Morgenstern utility of $p - w$ and of $x - p$, respectively. In the case that there is no trade, both the auctioneer and the bidder get a von Neumann Morgenstern utility of 0. Notice that this assumption implies that the production occurs, and production costs are incurred, only once a trade has been agreed. The production cost could also be seen as an opportunity cost.

We consider a three stage game. In the first stage auctioneers simultaneously announce their reserve prices. In the second stage, the entry game, each bidder upon observing the auctioneers' announcements can either pick one and only one auction[4] in which she wants to participate, or she can choose to participate in no auction. In the final stage those bidders who have chosen to participate in some particular auction make their bids in their corresponding auctions. For the sake of simplicity, we shall only consider second price auctions with no entry fee.[5]

We shall assume that the auctioneers choose their reserve prices from a finite[6] set $\Pi \equiv \{r_1, r_2, \ldots, r_R\} \subset [0, 1)$, where $r_j > r_l$ if $j > l$. To allow for the possibility of the strategy reserve price equal to production cost we shall assume that the support of the production costs also lie in Π, i.e. $\Pi_W \subset \Pi$. We shall call a pure strategy for the auctioneers a map from Π_W to Π that gives the auctioneer's reserve price as a function of his production cost.

We also add two assumptions more. The first one is that the minimum reserve price equals the minimum production cost, i.e. $\min \Pi = \min \Pi_W$. The second one is that auctioneers' production costs are not too high in some sense. More precisely, that

[3] The assumption that the support of F equals $[0, 1]$ implies that we do not consider situations in which the production cost of an auctioneer is below the minimum valuation of the bidders. The same arguments provided by Peters [4] also imply here that this assumption is crucial for our results.

[4] We believe that our results could be easily extended to the case in which bidders can participate in more than one auction under the following additional assumptions. Each bidder has a constant marginal utility for a finite number of units and zero for additional units. The number of units from which the bidder obtains strictly positive utility is a finite number greater than the maximum number of auctions that the bidder can enter. Under these assumptions it is still true that it is weakly dominant for the bidder to bid her true value of the good. If these assumptions are not met then there is no straightforward solution for the bidding game, and hence, we cannot extend easily our analysis.

[5] We show in the working paper version that the optimality of the policy reserve price equal to production cost also holds when first price auctions are allowed. The reason is that it can be shown that first price and second price auctions with the same reserve price are revenue equivalent.

[6] See the discussion in the Conclusions about the role of the discretisation.

the unique solution of the system of equations of Lemma 3 for $G=H$ is such that each reserve price in the support of H has an associated solution (we shall call it limit cut-off) less than one. We explain the role of these two assumptions after Lemma 5.

3. The main result

We start with the main result of the paper and prove it in the rest of the section.

Proposition 1. *There exists a threshold $\bar{J}$ such that if $J \geqslant \bar{J}$, there is a unique symmetric Nash equilibrium in which auctioneers use pure strategies. In this equilibrium, each auctioneer announces a reserve price equal to his production cost.*

We prove this result using backward induction. We start noting that standard arguments show that the last stage, the bidding game, has a unique symmetric equilibrium strategy, to bid the true value. We continue with the study of the second stage, the entry game, assuming that bidders will bid in the third stage according to the former strategy.

We can describe an entry game with an increasing function $G_J : \mathbb{R} \to \{i \in \mathbb{N} : i \leqslant J\}$, with jump points in Π, and where $G_J(x)$ specifies the number of auctioneers that have announced a reserve price less or equal than x. We shall refer to the family of such functions as $\mathscr{G}_J$. One interesting feature that shall be used in our analysis is that G_J/J is a probability distribution function with support in Π.

To describe the equilibrium of the entry game we introduce what we call *cut-off strategies*. These are mixed strategies that can be characterised by a vector of cut-offs $\vec{y}$ and that have the following features: (i) each reserve price r_j in the support of G_J/J has an associated cut-off $y_j \in [0,1]$; (ii) if $r_j \geqslant r_l$, then $y_j \geqslant y_l$; (iii) the bidder enters an auction with reserve price r_j with positive probability if and only if her type is weakly higher than y_j; and (iv), the bidder randomises uniformly among all the auctions which she enters with positive probability.

For notational convenience we assume in the statement and proof of the following lemma that G_J/J has support Π.

Lemma 1. *There exists a unique symmetric equilibrium of the entry game. In this equilibrium bidders use a cut-off strategy characterised by the unique vector of cut-offs that solves the following conditions*:

(i) $y_1 = r_1$,
(ii) *for* $j \neq 1$, $\Psi(y_j; \vec{y}, G_J) = r_j$, *if* $y_j < 1$,
(iii) *for* $j \neq 1$, $\Psi(y_j; \vec{y}, G_J) \leqslant r_j$, *if* $y_j = 1$,

where for any $x \in (y_{j-1}, y_j]$,

$$\Psi(x; \vec{y}, G_J) \equiv \int_{y_{j-1}}^{x} \tilde{x} \frac{dz(\tilde{x}; \vec{y}, G_J)^{kJ-1}}{z(x; \vec{y}, G_J)^{kJ-1}} + r_{j-1} \left[\frac{z(y_{j-1}; \vec{y}, G_J)}{z(x; \vec{y}, G_J)} \right]^{kJ-1},$$

and,

$$z(x;\vec{y},G_J) \equiv 1 - \frac{F(y_j)-F(x)}{G_J(r_{j-1})} - \sum_{q=j}^{R} \frac{F(y_{q+1})-F(y_q)}{G_J(r_q)},$$

with $y_{R+1} \equiv 1$.

Proof in the appendix.

Condition (i) has an obvious interpretation. To understand the other conditions of the lemma, note first that the function $z(x;\vec{y},G_J)$ is the probability[7] that a bidder i that follows a cut-off strategy $\vec{y}$ either has a type below x or she does not bid in a given auction with a reserve price r_{j-1}.

Then, the function $\Psi(x;\vec{y},G_J)$ equals the expected price that a bidder i with type $x \in (y_{j-1}, y_j]$ pays conditional on winning in an auction with reserve price r_{j-1} when all the other bidders follow the same cut-off strategy defined by $\vec{y}$. To see why note the following. The probability that bidder i wins equals $z(x;\vec{y},G_J)^{kJ-1}$. This implies that for $\tilde{x} \in (y_{j-1}, x]$ the probability that the price is below $\tilde{x}$ given that bidder i wins equals $z(\tilde{x};\vec{y},G_J)^{kJ-1}/z(x;\vec{y},G_J)^{kJ-1}$. It also implies that the probability that no other bidder enters this auction conditional on bidder i winning equals $z(y_{j-1};\vec{y},G_J)^{kJ-1}/z(x;\vec{y},G_J)^{kJ-1}$. In this last case bidder i pays the reserve price r_{j-1}.

Note also that a bidder i with type y_j pays the reserve price r_j if she wins when all the other bidders follow the same cut-off strategy defined by $\vec{y}$. This is because bidder i only wins when no other bidder enters the same auction.

Consequently, conditions (ii) and (iii) compare the expected price paid by a bidder with a type equal to an arbitrary cut-off y_j conditional on winning an auction with reserve price r_j with the same conditional expected price in an auction with the reserve price immediately lower, this is r_{j-1}. When bidders use cut-off strategies, the probability of winning is the same in both auctions. Consequently, our conditions compare the expected utility of entering both auctions for bidders with cut-off values. These conditions are similar to the equilibrium conditions proposed by Peters and Severinov [8, Theorem 6, p. 173]. The contribution of Lemma 1 is to show that these conditions have a unique solution.

Our uniqueness proof uses the fact that conditions (i)–(ii) (condition (iii) is a boundary condition) define a map from equilibrium cut-offs to reserve prices. Then, our proof is based on the fact that conditions (i)–(iii) satisfies some continuity and monotonic properties that assure that this map is globally invertible. This also explains our convergence results, see Lemmas 3 and 4, and the comments in between.

[7] Note that the formula that we give has on the right-hand side one minus the probability of the complementary event to the one described in the text.

Lemma 2. *The expected payoffs of an auctioneer with production cost $w \in \Pi_W$ that sets reserve price $r_j \in \Pi$ equal*:

$$\Phi(r_j, w, \tilde{G}_{J-1}) \equiv \int_{y_j^*(G_J)}^{1} (\Psi(x; \vec{y}^*(G_J), G_J) - w)\, dz(x; \vec{y}^*(G_J), G_J)^{kJ}, \tag{1}$$

where $\tilde{G}_{J-1} \in \mathcal{G}_{J-1}$ describes the other auctioneers' reserve prices, G_J the distribution of reserve prices induced by $\tilde{G}_{J-1}$ and r_j, and $\vec{y}^*(G_J) = (y_1^*(G_J), y_2^*(G_J), \ldots, y_R^*(G_J))$ denotes the unique symmetric equilibrium of the entry game induced by G_J (see Lemma 1).

Proof in the appendix.

We shall concentrate on the analysis of the game that the auctioneers play when the number of auctioneers is large but finite. To do so we shall follow an indirect approach. We shall approximate the payoffs in the finite game with the limit payoffs when the number of auctioneers tends to infinity. The advantage of this approach is that the limit payoffs are more tractable than the finite version.

The first step to compute the limit payoffs is to compute the limit of the equilibrium cut-offs. We start providing some conditions that we shall show are the limit of conditions (i)–(iii). To introduce these conditions we also use some functions that are limit versions of the functions z and Ψ.

We shall denote by $\mathcal{G}$ the family of probability distribution functions that has support in Π. Then:

Lemma 3. *For any given distribution function $G \in \mathcal{G}$, there exists a unique vector of cut-offs $\vec{y} = (y_1, y_2, \ldots, y_R)$ that satisfies the following conditions*:

(i′) *for* $j \leqslant \underline{j}(G),\ y_j = r_j$.
(ii′) *for* $j > \underline{j}(G),\ \bar{\Psi}(y_j; \vec{y}, G) = r_j$, *if* $y_j < 1$,
(iii′) *for* $j > \underline{j}(G),\ \bar{\Psi}(y_j; \vec{y}, G) \leqslant r_j$, *if* $y_j = 1$,

where $\underline{j}(G)$ is such that $r_{\underline{j}(G)}$ is the minimum reserve price in the support of G, and where for any $x > y_{\underline{j}(G)}$ and $x \in (y_{j-1}, y_j]$,

$$\bar{\Psi}(x; \vec{y}, G) \equiv \int_{y_{j-1}}^{x} \tilde{x} \frac{d\bar{z}(\tilde{x}; \vec{y}, G)}{\bar{z}(x; \vec{y}, G)} + r_{j-1} \frac{\bar{z}(y_{j-1}; \vec{y}, G)}{\bar{z}(x; \vec{y}, G)}$$

and

$$\bar{z}(x; \vec{y}, G) \equiv e^{-k\left[\frac{F(y_j)-F(x)}{G(r_{j-1})} + \sum_{l=j}^{R} \frac{F(y_{l+1})-F(y_l)}{G(r_l)}\right]},$$

with $y_{R+1} \equiv 1$.

Proof in the appendix.

Once again, the uniqueness proof is implicitly based on the fact that conditions (i′)–(ii′) (condition (iii′) is a boundary condition) have some continuity and monotonicity properties that assure that these conditions define a globally invertible map from equilibrium cut-offs to reserve prices. Global invertibility under the same properties of continuity and monotonicity also explains that the equilibrium cut-offs converge to the limit cut-offs when the functions in conditions (i)–(iii) converge to the functions in conditions (i′)–(iii′). This is the core of the proof of the following lemma that states the limit of the auctioneer's payoffs:

Lemma 4. *Consider an infinite sequence* $\{\tilde{G}_{J-1}\}_J$ $(\tilde{G}_{J-1} \in \mathcal{G}_{J-1})$ *such that* $\tilde{G}_{J-1}/(J-1)$ *converges weakly*[8] *to* $G \in \mathcal{G}$ *when* J *tends to infinity. Then, for any* $w \in \Pi_W$ *and for any* $r_j \in \Pi$,

$$\Phi(r_j, w, \tilde{G}_{J-1}) \xrightarrow{J \to \infty} \bar{\Phi}(r_j, w, G),$$

where, $\bar{\Phi}(r_j, w, G)$ *is defined for* $r_j < r_{\underline{j}(G)}$,

$$\int_{r_{\underline{j}(G)}}^{1} (\bar{\Psi}(x; \vec{y}^*(G), G) - w)\, d\bar{z}(x; \vec{y}^*(G), G) + (r_{\underline{j}(G)} - w)\bar{z}(r_{\underline{j}(G)}; \vec{y}^*(G), G)$$

and for $r_j \geqslant r_{\underline{j}(G)}$,

$$\int_{y_j^*(G)}^{1} (\bar{\Psi}(x; \vec{y}^*(G), G) - w)\, d\bar{z}(x; \vec{y}^*(G), G),$$

where $\vec{y}^*(G)$ *denotes the unique vector of cut-offs that solves conditions* (i′)–(iii′), *see Lemma* 3.

Proof in the appendix.

We next show that for the limit payoffs it is weakly dominant to set the reserve price equal to the production cost. To state this result we denote by $r_{\bar{j}(G)}$ the minimum reserve price with an associated cut-off equal to one. Note that auctioneers with a reserve price $r_j \geqslant r_{\bar{j}(G)}$ obtain limit payoffs equal to zero.

Lemma 5. *For any* $G \in \mathcal{G}$, $w \in \Pi_W$, *and* $r_j \in \Pi \backslash w$:

(A) *If* $r_{\underline{j}(G)} < w < r_{\bar{j}(G)}$, *then* $\bar{\Phi}(w, w, G) > \bar{\Phi}(r_j, w, G)$.

(B) *If* $w \leqslant r_{\underline{j}(G)}$, *then* $\begin{cases} \bar{\Phi}(w, w, G) > \bar{\Phi}(r_j, w, G) & \text{for } r_j \geqslant r_{\underline{j}(G)}, \\ \bar{\Phi}(w, w, G) = \bar{\Phi}(r_j, w, G) & \text{otherwise}. \end{cases}$

(C) *If* $w \geqslant r_{\bar{j}(G)}$, *then* $\begin{cases} \bar{\Phi}(w, w, G) > \bar{\Phi}(r_j, w, G) & \text{for } r_j < r_{\bar{j}(G)}, \\ \bar{\Phi}(w, w, G) = \bar{\Phi}(r_j, w, G) & \text{otherwise}. \end{cases}$

Proof. Condition (ii) and the fact that $\bar{\Psi}(x; \vec{y}, G)$ strictly increases in x imply that $r_j = w$ weakly maximises the auctioneer's expected profits. It is a bit tedious, but mechanical, to check when the maximum is strict. □

[8] Since the elements of the sequence $\{G_J/J\}_J$ has support included in the finite set Π weak convergence, point-wise convergence and Euclidean convergence coincide.

We finish the proof of Proposition 1 in the appendix. Basically, we use the fact that the strict payoff comparisons in Lemma 5 should also hold for J sufficiently large. These conditions are sufficient to prove that reserve price equal to production cost is the unique equilibrium if J is large enough. However, it is important to note how this method depends on two assumptions that we have made and that get around two additional difficulties.

First, the limit payoff function becomes flat for reserve prices below $r_{\underline{j}(G)}$, and thus the limit payoffs do not provide strict comparisons for deviations below $r_{\underline{j}(G)}$, see Lemma 5(B). Our assumption that the auctioneers cannot announce reserve prices below the minimum production cost avoids this problem. Note that lower reserve prices are difficult to believe as they mean losses ex post for any auctioneer. This assumption was also implicity done by Peters and Severinov [8].

Second, if the auctioneers have production costs too high, it can be the case that the policy reserve price equal to production cost has an associated limit cut-off equal to one, i.e. there exists $w \in \Pi_W$ such that $w \geqslant r_{\bar{j}(G)}$. Intuitively, this means that fixing a reserve price equal to production cost attracts no bidder almost surely in the limit.[9] Thus, the auctioneer gets zero limit payoffs. Exactly the same as with any other reserve price above $r_{\bar{j}(G)}$, see Lemma 5(C). As a consequence, auctioneers may be indifferent between fixing a reserve price equal to production cost and another reserve price above $r_{\bar{j}(G)}$. This can cause multiplicity of equilibria. To avoid it, we have assumed that any reserve price in the support of H has an associated limit equilibrium cut-off less than one, when the limit distribution of reserve prices equals the distribution of production costs, i.e. $w < r_{\bar{j}(H)}$ for any $w \in \Pi_W$.

4. Conclusions

As we have argued in the Introduction, the analysis of the game with a continuous strategy space suggests that the auctioneer's optimal reserve price may depend on the auctioneer's beliefs about the market in a complex way. Our equilibrium analysis shows that this is not the case when the strategy space is finite. The optimal reserve price is quite simple and independent of the market characteristics whenever the number of auctioneers is sufficiently large.

This difference seems to suggest that the assumption that the strategy space is continuous may be somewhat misleading. We could reconcile the results of both models if we showed that the equilibrium strategy in the continuous strategy space model converge in some sense to reserve price equal to reserve price when the number of auctioneers tends to infinity. This seems a very difficult task. The payoff functions are so complex that a direct analysis of the finite game seems unfeasible.

The way we got around these difficulties in our model was by approximating the auctioneer's payoff function by its more tractable point-wise limit when the number

[9] The probability that no bidder enters an auction with reserve price r_j in equilibrium equals $z(y_j^*(G_J); \vec{y}^*(G_J), G_J)^{kJ}$. Lemma A.7(b) implies that if $y_j^*(G_J)$ tends to one, this probability tends to one.

of auctioneers tends to infinity. This approach works in our model because finiteness assures that for a sufficiently large number of auctioneers the exact payoff function can be arbitrary close to its point-wise limit not only for a proposed strategy but also for any possible deviation.[10] This is not the case when the strategy space is continuous. Then, the above result requires some kind of uniform convergence[11] of payoffs and this may be difficult to prove if not impossible. The reason is that uniform convergence of payoffs preserves continuity, see [10, Theorem 7.12, p. 150], and this is a contradiction with the fact that the auctioneers' payoff functions are continuous in the finite game whereas the limit payoffs are discontinuous.[12]

Moreover, uniform convergence with respect to the auctioneer's type and strategy may not be sufficient. It guarantees that in large markets the auctioneer's best response is close to the strategy reserve price equal to production cost when the other auctioneers use this strategy. This implies convergence in terms of ε-equilibria but not in terms of exact equilibria. To show the latter, we need to show that the auctioneer's best response is close to reserve price equal to production cost when the other auctioneers' strategies are close to reserve price equal to production cost. In our model, the assumption that the strategy space is discrete avoids this problem because the only reserve price sufficiently close to the production cost is the production cost itself.

Appendix

A.1. Proof of Lemma 1

We proceed in two steps.

Step 1: *Proof that conditions* (i)–(iii) *are necessary and sufficient for an equilibrium.* That the symmetric equilibria of the entry game are in cut-off strategies has been proved by Peters and Severinov [8, Theorem 5, p. 172]. They [8, Theorem 6, p. 173] also provide a system of equations on the vector of cut-offs whose solutions characterise the set of equilibria in cut-off strategies. These equations are basically indifference conditions. Each equation says that when all the bidders follow a given cut-off strategy, a bidder with value y_j must be indifferent between entering an auction with reserve price r_j and entering an auction with reserve price r_1.

As we have argued, our conditions state something similar: when all the bidders follow a given cut-off strategy, a bidder with value y_j must be indifferent between entering an auction with reserve price r_j and an auction with reserve price r_{j-1}. After some algebra, basically substituting recursively in our conditions (ii), we can easily

[10] I thank an anonymous referee for suggesting this point and most of the comments that follow.

[11] Note that when we allow for general heterogeneity among auctioneers we would require uniform convergence in two dimensions: strategies and types, i.e. in reserve prices and production costs.

[12] The continuity of the auctioneer's payoff function is proved in the working paper version of this paper, whereas the discontinuity has already been shown by Peters [4] and Peters and Severinov [8].

show that our conditions generate the same system of equations as the conditions by Peters and Severinov.

We also include a condition (iii) for the case in which some cut-offs are one, this is that some auctions do not attract bidders. This case was not considered by Peters and Severinov [8]. However, it is easy to show that condition (iii) must be included when we look for necessary and sufficient conditions for an asymmetric equilibrium in which some cut-offs equal one.

Step 2: *Proof that conditions* (i)–(iii) *have a unique solution*. It is easier to prove this result and the convergence results in Lemma 3 using this new notation for conditions (i)–(iii):

(i) $y_1 = r_1$.
(ii) $\Lambda_j^{G_J}(y_{j-1}, y_j, \ldots, y_R) = 0$ if $y_j < 1$ $(j \neq 1)$.
(iii) $\Lambda_j^{G_J}(y_{j-1}, y_j, \ldots, y_R) \leqslant 0$ if $y_j = 1$ $(j \neq 1)$.

Where $\Lambda_j^{G_J}(y_{j-1}, y_j, \ldots, y_R) \equiv \int_{-\infty}^{+\infty} x\, d\mu_{j-1}(x|\vec{y}, G_J) - r_j$, $(j \in \{2, 3, \ldots, R\})$ and,

$$\mu_{j-1}(x|\vec{y}, G_J) \equiv \begin{cases} 0 & \text{if } x \in (-\infty, r_{j-1}), \\ \frac{z(y_{j-1};\vec{y},G_J)^{kJ-1}}{z(y_j;\vec{y},G_J)^{kJ-1}} & \text{if } x \in [r_{j-1}, y_{j-1}), \\ \frac{z(x;\vec{y},G_J)^{kJ-1}}{z(y_j;\vec{y},G_J)^{kJ-1}} & \text{if } x \in [y_{j-1}, y_j), \\ 1 & \text{if } x \in [y_j, \infty). \end{cases} \tag{A.1}$$

Lemma A.1. *The function* $\Lambda_j^{G_J}(y_{j-1}, y_j, \ldots, y_R)$ $(j \in \{2, 3, \ldots, R\})$ *is continuous, weakly decreasing in* y_{j-1}, *strictly increasing in* y_j *and weakly increasing in* $y_{j+1}, y_{j+2}, \ldots, y_R$.

Proof. Since F is continuous, $\mu_{j-1}(x|\vec{y}, G_J)$ is continuous in $\vec{y}$ for any x. Then for any sequence $\{\vec{y}_n\} \to \vec{y}$, $\mu_{j-1}(x|\vec{y}_n, G_J) \to \mu_{j-1}(x|\vec{y}, G_J)$ and in particular this is true at each point of continuity of $\mu_{j-1}(.|\vec{y}, G_J)$. Thus, $\mu_{j-1}(x|\vec{y}_n, G_J)$ converges weakly to $\mu_{j-1}(x|\vec{y}, G_J)$ by Billingsley [1, Theorem 25.8, p. 335]. Then, by the definition of weak convergence $\int_{-\infty}^{+\infty} x\, d\mu_{j-1}(x|\vec{y}_n, G_J) \to \int_{-\infty}^{+\infty} x\, d\mu_{j-1}(x|\vec{y}, G_J)$ that proves the continuity of $\Lambda_j^{G_J}$.

It is straightforward from the definition of z that a decrease in y_{j-1} or an increase in y_j shifts $\mu_{j-1}(x|\vec{y}, G_J)$ in the sense of first order stochastic dominance downwards. An increase in y_l, $l \geqslant j$ decreases the ratio $z(\tilde{x}; \vec{y}, G_J)/z(x; \vec{y}, G_J)$, as one can verify through differentiation, and hence it also shifts downwards the distribution function $\mu_{j-1}(.|x; \vec{y}, G_J)$ in the sense of first order stochastic dominance downwards. $\square$

We can now apply an induction argument to conditions (i)–(iii).

Lemma A.2. *For $R \neq 1$, there exists a unique function $\psi_R^{G_J} : [r_{R-1}, 1] \to [r_R, 1]$ that satisfies*:

(ii) $\Lambda_R^{G_J}(y_{R-1}, \psi_R^{G_J}(y_{R-1})) = 0$ *if* $\psi_R^{G_J}(y_{R-1}) < 1$.
(iii) $\Lambda_R^{G_J}(y_{R-1}, \psi_R^{G_J}(y_{R-1})) \leqslant 0$ *if* $\psi_R^{G_J}(y_{R-1}) = 1$.

This function is continuous, weakly increasing and $\psi_R^{G_J}(y_{R-1}) \geqslant y_{R-1}$.

Proof. Trivially, $\Lambda_R^{G_J}(y_{R-1}, \max\{r_R, y_{R-1}\}) < 0$. Then, distinguish two cases for any $y_{R-1} \in [r_{R-1}, 1]$, either: $(*)$ $\Lambda_R^{G_J}(y_{R-1}, 1) > 0$, or $(**)$ $\Lambda_R^{G_J}(y_{R-1}, 1) \leqslant 0$. In case $(*)$ condition (iii) is not satisfied, and Lemma A.1 implies that there is a unique value for y_R that solves condition (ii). In case $(**)$ there is no value that solves condition (ii), but condition (iii) is satisfied. Hence, we have a unique function $\psi_R^{G_J}$ such that $y_R = \psi_R^{G_J}(y_{R-1})$ solves conditions (ii)–(iii) for any given value of y_{R-1}. Finally, Lemma A.1 implies that $\psi_R^{G_J}$ is a continuous and weakly increasing function. $\square$

Suppose that there exist some functions $\{\psi_i^{G_J} : [r_{l-1}, 1] \to [r_l, 1]\}_{l=j+1}^{R}$ continuous and weakly increasing, and $\psi_l^{G_J}(y_{l-1}) \geqslant y_{l-1}$ for all $l \in \{j+1, \ldots, R\}$. Let $\omega_l^{G_J} : [r_j, 1] \to [r_l, 1]$ where $\omega_l^{G_J}(y_j) \equiv \psi_l^{G_J} \circ \psi_{l-1}^{G_J} \circ \cdots \circ \psi_{j+1}^{G_J}(y_j)$ for $l = j+1, j+2, \ldots, R$ which are obviously continuous and weakly increasing functions.

Lemma A.3. *For $j \neq 1$, there exists a unique function $\psi_j^{G_J} : [r_{j-1}, 1] \to [r_j, 1]$ that satisfies*:

(ii) $\Lambda_{j-1}^{G_J}(y_{j-1}, y_j, \omega_{j+1}^{G_J}(y_j), \ldots, \omega_R^{G_J}(y_j)) = 0$ *if* $y_j < 1$.
(iii) $\Lambda_{j-1}^{G_J}(y_{j-1}, y_j, \omega_{j+1}^{G_J}(y_j), \ldots, \omega_R^{G_J}(y_j)) \leqslant 0$ *if* $y_j = 1$.

This function is continuous, weakly increasing and $\psi_j^{G_J}(y_{j-1}) \geqslant y_{j-1}$.

Proof. Similar to the proof of Lemma A.2. $\square$

Hence, by induction there exists a unique solution for conditions (i)–(iii) and this is such that $y_1 = r_1$ and $y_j = \psi_j^{G_J} \circ \psi_{j-1}^{G_J} \circ \cdots \circ \psi_2^{G_J}(r_1)$ for $j \in \{2, 3, \ldots, R\}$.

A.2. Proof of Lemma 2

One property of the unique equilibrium of the entry game is that bidders pay the same expected price in every auction they enter with positive probability. This is a consequence of the indifference condition that must hold in a mixed strategy equilibrium. Thus, a bidder's expected payment conditional on winning an auction with reserve price r_j when her type is x (obviously for $x \geqslant y_j$) equals $\Psi(x; \vec{y}^*(G_J), G_J)$ in equilibrium.

Then, the expected payoffs of an auctioneer that fixes a reserve price r_j equals the probability of selling times the expected value conditional on selling of the following difference: the expected price that the winner of the auction pays conditional on winning minus the auctioneer's production cost w.

To complete the proof of the lemma note that the probability of selling equals the probability that at least one bidder enters, this is $1 - z(y_j^*(G_J); \vec{y}^*(G_J), G_J)^{kJ}$. Conditional on the former event, the probability that the bidder that wins has a type below x, for $x \in [y_j^*(G_J), 1]$, equals $z(x; \vec{y}^*(G_J), G_J)^{kJ}/(1 - z(y_j^*(G_J); \vec{y}^*(G_J), G_J)^{kJ})$.

A.3. *Proof of Lemma 3*

For cut-offs associated to reserve prices less or equal than $r_{\underline{j}(G)}$ the claim is straightforward. For the other cut-offs, we can apply a similar proof to that of step 2 in Lemma 1.

A.4. *Proof of Lemma 4*

We divide the proof in three steps. First, we rewrite our equilibrium conditions (i)–(iii) so that along any sequence of entry games the set of equations and unknowns remains the same. In the second step, we compute the limit of these new conditions and we show that the limit of the solutions converge to the solution of the limit conditions. Finally, we use the former result to compute the convergence of the auctioneers' payoff function.

Step 1: *Rewriting the equilibrium conditions*. From now on, we shall describe a cut-off strategy with an *extended* vector of cut-offs, $\vec{y} \in [0, 1]^R$. This differs from the original vector of cut-offs in that we associate a cut-off value to each reserve price in Π, and not only to reserve prices in the support of G_J/J. Note that applying the definition of a cut-off strategy, changes in cut-off values associated to reserve prices that are not announced by any auctioneer do not change the entry strategy of the bidder.

We next adapt conditions (i)–(iii) and the corresponding functions to the extended vector of cut-offs. The new conditions, that we refer as the *extended conditions* (i)–(iii), are such that the solution cut-offs associated to reserve prices in the support of G_J/J are also solutions to the original conditions (i)–(iii). Now, we refer to the minimum reserve price in the support of G_J/J as $r_{\underline{j}(G_J)}$ and to its associated cut-off as $y_{\underline{j}(G_J)}$.

We start noting that the original definition of the function z can be directly applied to the extended vector of cut-offs for $x \geqslant y_{\underline{j}(G_J)}$. Clearly, z keeps the same meaning as before. Something similar happens with $\Lambda_j^{G_J}$ (and $\mu_{j-1}(.|\vec{y}, G_J)$), for $j > \underline{j}(G_J)$. We also define $\Lambda_j^{G_J}(y_{j-1}, y_j, \ldots, y_R) \equiv y_j - r_j$ for $j \leqslant \underline{j}(G_J)$. This means that cut-offs equal reserve prices for those reserve prices weakly below the minimum reserve price announced by the auctioneers.

We redefine the extended conditions (i)–(iii) according to the original conditions (i)–(iii) but for the extended vector of equilibrium cut-offs and with the new definition of $\Lambda_j^{G_J}$. Clearly, the new conditions also verify Lemma A.1. Thus, we can adapt step 2 in the proof of Lemma 1 to prove that there exists a unique extended vector of cut-offs that satisfies our conditions.

We next show that the cut-offs that solve the extended conditions (i)–(iii) are also solution to the original conditions (i)–(iii). The proof is direct for the cut-off associated to the minimum reserve price in the support of G_J/J. Consider next the extended condition (ii) associated to other reserve prices r_j that belong to the support of G_J/J. If r_{j-1} also belongs to the support of G_J/J then the extended condition (ii) is exactly the same as the original condition (ii). Suppose now that r_{j-1} does not belong to the support of G_J. The extended condition (ii) for r_j can be written as the following equation:

$$\int_{y_{j-1}}^{y_j} \tilde{x}\frac{dz(\tilde{x};\vec{y},G_J)^{kJ-1}}{z(y_j;\vec{y},G_J)^{kJ-1}} + r_{j-1}\frac{z(y_{j-1};\vec{y},G_J)^{kJ-1}}{z(y_j;\vec{y},G_J)^{kJ-1}} = r_j.$$

The extended condition (ii) also implies that a similar equation must hold for r_{j-1}. We can combine both equations substituting r_{j-1} to get:

$$\int_{y_{j-2}}^{y_j} \tilde{x}\frac{dz(\tilde{x};\vec{y},G_J)^{kJ-1}}{z(y_j;\vec{y},G_J)^{kJ-1}} + r_{j-2}\frac{z(y_{j-2};\vec{y},G_J)^{kJ-1}}{z(y_j;\vec{y},G_J)^{kJ-1}} = r_j.$$

If r_{j-2} belongs to the support of G_J/J, the above condition is basically the original condition (ii) for r_j. Otherwise, we can continue substituting recursively until we get a reserve price in the support of G_J/J.

A similar procedure also works for extended conditions (iii). Hence, any solution of the extended conditions (i)–(iii) must also be a solution of conditions (i)–(iii).

Step 2: *Convergence of the equilibrium cut-offs*. We assume along step 2 that there exists an infinite sequence $\{G_J\}_J$, where $G_J \in \mathcal{G}_J$, such that G_J converges weakly (and thus point-wise) to $G \in \mathcal{G}$ when J tends to infinity. We start by showing the convergence of the functions $\Lambda_j^{G_J}$.

Lemma A.4. *For any* $j \leqslant \underline{j}(G)$,

$$\Lambda_j^{G_J}(y_{j-1}, y_j, \ldots, y_R) \xrightarrow{J\to\infty} \bar{\Lambda}_j^G(y_{j-1}, y_j, \ldots, y_R)$$

point-wise, and where $\bar{\Lambda}_j^G(y_{j-1}, y_j, \ldots, y_R) \equiv y_j - r_j$.

Proof. We split the sequence $\{G_J\}_J$ into two: a subsequence that includes distribution functions such that $j \leqslant \underline{j}(G_J)$, and a subsequence that includes the other distribution functions. For the first subsequence the claim follows by definition of $\Lambda_j^{G_J}$, see its definition in step 1. For the second subsequence, note first that $G(r_{j-1})/J$ tends to zero for $j \leqslant \underline{j}(G)$ as G_J/J converges weakly (and thus point-wise) to G.

Then, for $x \in [y_{j-1}, y_j)$,

$$0 \leqslant \left(\frac{z(x;\vec{y},G_J)}{z(y_j;\vec{y},G_J)}\right)^{kJ-1} = \left(1 - \frac{1}{z(y_j;\vec{y})}\frac{F(y_j)-F(x)}{G(r_{j-1})}\right)^{kJ-1}$$
$$\leqslant \left(1 - \frac{F(y_j)-F(x)}{G(r_{j-1})}\right)^{kJ-1} = \left(1 - \frac{\frac{F(y_j)-F(x)}{G(r_{j-1})/J}}{J}\right)^{kJ-1} \xrightarrow{J\to\infty} 0,$$

where we have used that following mathematical result: for any sequence $a_J \xrightarrow{J\to\infty} +\infty$, and such that $a_J \in (0, J)$ then $(1 - a_J/J)^J \xrightarrow{J\to\infty} 0$.

This means that $\mu_{j-1}(.|\vec{y}, G_J)$ converges everywhere to a probability measure with a single mass point at y_j. This implies convergence in all the continuity points and thus, weak convergence of the probability measures by Billingsley [1, Theorem 25.8, p. 335]. Then, the lemma follows by definition of weak convergence. □

Lemma A.5. *For any $j > \underline{j}(G)$,*

$$\Lambda_j^{G_J}(y_{j-1}, y_j, \ldots, y_R) \xrightarrow{J\to\infty} \bar{\Lambda}_j^G(y_{j-1}, y_j, \ldots, y_R)$$

point-wise, and where $\bar{\Lambda}_j^G(y_{j-1}, y_j, \ldots, y_R) \equiv \int_{-\infty}^{+\infty} x\, d\bar{\mu}_{j-1}(x|\vec{y}, G) - r_j$, with

$$\bar{\mu}_{j-1}(x|\vec{y}, G) \equiv \begin{cases} 0 & \textit{if } x \in (-\infty, r_{j-1}), \\ \frac{\bar{z}(y_{j-1};\vec{y},G)}{\bar{z}(x;\vec{y},G)} & \textit{if } \tilde{x} \in [r_{j-1}, y_{j-1}), \\ \frac{z(x;\vec{y},G)}{\bar{z}(y_j;\vec{y},G)} & \textit{if } x \in [y_{j-1}, y_j), \\ 1 & \textit{if } x \in [y_j, \infty). \end{cases} \tag{A.2}$$

Proof. Since G_J/J converges weakly to G (and thus point-wise) $y_{\underline{j}(G_J)} \leqslant y_{\underline{j}(G)}$ but for finitely many elements in the sequence $\{G_J\}_J$. Thus, we can disregard them to compute the limit. Then for $x \in [y_{j-1}, y_j]$, with $y_{j-1} \geqslant y_{\underline{j}(G)}$, and so $y_{j-1} \geqslant y_{\underline{j}(G_J)}$,

$$z(x;\vec{y},G_J)^{kJ-1} = \left(1 - \frac{\frac{F(y_j)-F(x)}{G_J(r_{j-1})/J} + \sum_{q=j}^R \frac{F(y_{q+1})-F(y_q)}{G_J(r_q)/J}}{J}\right)^{kJ-1}$$
$$\xrightarrow{J\to\infty} e^{-k\left[\frac{F(y_j)-F(x)}{G(r_{j-1})} + \sum_{q=j}^R \frac{F(y_{q+1})-F(y_q)}{G(r_q)}\right]} = \bar{z}(x;\vec{y},G), \tag{A.3}$$

where we have used to compute this limit the following mathematical result: for any sequence $a_J \xrightarrow{J\to\infty} a$, it is verified that $(1 + a_J/J)^J \xrightarrow{J\to\infty} e^a$.

As a consequence, $\mu_{j-1}(.|\vec{y}, G_J) \xrightarrow{J\to\infty} \bar{\mu}_{j-1}(.|\vec{y}, G)$ everywhere. Again, this completes the proof. □

Clearly, conditions (i′)–(iii′) can be stated as follows:

(i′) $y_1 = r_1$.
(ii′) $\bar{\Lambda}_j^{G_J}(y_{j-1}, y_j, \ldots, y_R) = 0$ if $y_j < 1$ $(j \neq 1)$.
(iii′) $\bar{\Lambda}_j^{G}(y_{j-1}, y_j, \ldots, y_R) \leqslant 0$ if $y_j = 1$ $(j \neq 1)$.

We can use the arguments in the proof of Lemma 1 to show that conditions (i′)–(iii′) define implicitly some functions ψ_j^G. These are such that the jth entry of the R dimensional solution of conditions (i′)–(iii′) equals $\psi_j^G \circ \psi_{j-1}^G \circ \cdots \psi_2^G(r_1)$ for all $j \in \{2, 3, \ldots, R\}$. Lemmas A.4 and A.5 shows that the equations in conditions (i)–(iii) converge point-wise to the equations in conditions (i′)–(iii′). Restrict for simplicity to the case in which all the equilibrium cut-offs are interior, i.e. strictly less than one. Then we can apply recursively Lemma A.8 (see at the end of step 3) to show that $\psi_j^{G_J}$ converges uniformly to ψ_j^G for any $j = 2, 3, \ldots, R$. This implies the following lemma.

Lemma A.6. *The extended vector of equilibrium cut-offs $\vec{y}^*(G_J)$ converges to the limit vector of equilibrium cut-offs $\vec{y}^*(G)$ when J tends to infinity.*

Step 3: *Convergence of the auctioneers' payoff function.*

Lemma A.7. *Take an infinite sequence of distributions of reserve prices $\{G_J\}_J$ $(G_J \in \mathcal{G}_J)$ such that G_J/J converges weakly to $G \in \mathcal{G}$ when J tends to infinity. Then*:[13]

(a) *If $x \in [\sup\{y^*_{\underline{j}(G_J)}\}_J, 1)$ and $x \neq y_j^*(G)$ $(j \in \{1, 2, \ldots, R\})$, then*:

$$z(x; \vec{y}^*(G_J), G_J)^{kJ} \xrightarrow{J \to \infty} \begin{cases} 0 & \text{if } x < y_{\underline{j}(G)}, \\ \bar{z}(x; \vec{y}^*(G), G) & \text{if } x > y_{\underline{j}(G)}, \end{cases}$$

$$\Psi(x; \vec{y}^*(G_J), G_J) \xrightarrow{J \to \infty} \bar{\Psi}(x; \vec{y}^*(G), G) \quad \text{for } x > y_{\underline{j}(G)}.$$

(b) *If $j \geqslant \max\{\underline{j}(G_J)\}_J$, then*:

$$z(y_j^*(G_J); \vec{y}^*(G_J), G_J)^{kJ} \xrightarrow{J \to \infty} \begin{cases} 0 & \text{if } j < \underline{j}(G), \\ \bar{z}(y_j^*(G); \vec{y}^*(G), G) & \text{if } j \geqslant \underline{j}(G). \end{cases}$$

(c) *If $x \in [\sup\{y^*_{\underline{j}(G_J)}\}_J, y_{\underline{j}(G)})$, then*:

$$kJ\, z(x; \vec{y}^*(G_J), G_J)^{kJ-1} \xrightarrow{J \to \infty} 0.$$

[13] We restrict x in (a) and (c) to be greater than the supremum of $\{y^*_{\underline{j}(G_J)}\}_J$ and j in (b) to be greater than the maximum of $\{\underline{j}(G_J)\}_J$ as the function z is defined only for x greater or equal than the cut-off associated to the minimum reserve price announced by at least one auctioneer.

Proof. It can be deduced from conditions (i′) and (ii′) that $y_j^*(G) < y_{j+1}^*(G)$ (where recall that $y_{R+1}^*(G) \equiv 1$) for all $y_j^*(G) < 1$. Let j be such that $x \in (y_j^*(G), y_{j+1}^*(G))$. Lemma A.6 implies that for J sufficiently large $x \in (y_l^*(G_J), y_{l+1}^*(G_J))$. Next, we can use an adaptation of the proofs of Lemmas A.4 and A.5 in step 2 together with Lemma A.6 to prove point (a). Point (b) can also be proved using an adaptation of the proof of Lemma A.5 in step 2 together with Lemma A.6. In order to prove (c) note that for $x < y_{\underline{j}(G)}$ the function $z(x; \vec{y}^*(G_J), G_J)$ can be written as $1 - a_J/J$ with $a_J \xrightarrow{J\to\infty} \infty$ because $G_J(r_j)/J \to 0$ for $r_j < r_{\underline{j}(G)}$. Hence,

$$
\begin{aligned}
0 &\leqslant \lim_{J\to\infty} kJ(1 - z(x; \vec{y}^*(G_J), G_J))z(x; \vec{y}^*(G_J), G_J)^{kJ-1} \\
&= \lim_{J\to\infty} kJ\frac{a_J}{J}\left(1 - \frac{a_J}{J}\right)^{kJ-1} \leqslant \lim_{J\to\infty} ka_J\, e^{-(kJ-1)\frac{a_J}{J}} = \lim_{J\to\infty} \frac{ka_J}{e^{a_J(k-\frac{1}{J})}} = 0,
\end{aligned}
$$

where we have used $(1-a) \leqslant e^{-a}$ in the third step. □

We can now conclude the proof of the convergence of the auctioneers' payoffs. Note that if $\tilde{G}_{J-1}/(J-1)$ converges weakly to G, then G_J (the distribution of reserve prices that describes r_j and $\tilde{G}_{J-1}$ together) also converges weakly to G. We start with the case $r_j \geqslant r_{\underline{j}(G)}$. Lemmas A.6, A.7(a), and the Lebesgue bounded convergence theorem [9, Theorem 16, p. 91] in the third step below imply that:

$$
\begin{aligned}
\lim_{J\to\infty} \Phi(r_j, w, \tilde{G}_{J-1}) &= \lim_{J\to\infty} \left\{ \int_{y_j^*(G_J)}^{1} (\Psi(x;\ \vec{y}^*(G_J), G_J) - w)\, dz(x; \vec{y}^*(G_J), G_J)^{kJ} \right\} \\
&= \lim_{J\to\infty} \left\{ \sum_{l=j}^{R} \int_{y_l^*(G_J)}^{y_{l+1}^*(G_J)} (\Psi(x;\ \vec{y}^*(G_J), G_J) - w) \right. \\
&\qquad \left. \times\, z(x;\ \vec{y}^*(G_J), G_J)^{kJ-1} k \frac{f(x)}{G_J(r_l)/J}\, dx \right\} \\
&= \sum_{l=j}^{R} \int_{y_l^*(G)}^{y_{l+1}^*(G)} (\bar{\Psi}(x; \vec{y}^*(G), G) - w)\bar{z}(x; \vec{y}^*(G), G) k \frac{f(x)}{G(r_l)}\, dx \\
&= \int_{y_j^*(G)}^{1} (\bar{\Psi}(x; \vec{y}^*(G), G) - w)\, d\bar{z}(x; \vec{y}^*(G), G) = \bar{\Phi}(r_j, w, G).
\end{aligned}
$$

Consider now the case $r_j < r_{\underline{j}(G)}$, then we split the integral that defines $\Phi(r_j, w, \tilde{G}_{J-1})$ into the following two halves:

$$
\begin{aligned}
&\int_{y_j^*(G_J)}^{y_{\underline{j}(G)}^*(G_J)} (\Psi(x; \vec{y}^*(G_J), G_J) - w)\, dz(x; \vec{y}^*(G_J), G^J) \\
&\quad + \int_{y_{\underline{j}(G_J)}^*(G_J)}^{1} (\Psi(x; \vec{y}^*(G_J), G^J) - w)\, dz(x; \vec{y}^*(G_J), G^J),
\end{aligned}
$$

where $y_{\underline{j}(G)}(G_J)$ is the equilibrium cut-off associated to the minimum reserve price in the support of G, when the distribution of reserve prices is G_J.

Note that we can compute the limit of the second part of the above integral following exactly the same steps as in the case $r_j \geqslant r_{\underline{j}(G)}$. For the second part note the following algebraic transformations:[14]

$$
\begin{aligned}
&\int_{y_j^*}^{y_l^*} (\Psi(x) - w)\, dz(x)^{kJ} \\
&\quad = \int_{y_j^*}^{y_l^*} \left(\int_{y_j^*}^{x} \tilde{x} d \frac{z(\tilde{x})^{kJ-1}}{z(x)^{kJ-1}} + r_j \frac{z(y_j^*)^{kJ-1}}{z(x)^{kJ-1}} - w \right) dz(x)^{kJ} \\
&\quad = \int_{y_j^*}^{y_l^*} \left[x - w - (y_j^* - r_j) \frac{z(y_j^*)^{kJ-1}}{z(x)^{kJ-1}} - \int_{y_j^*}^{x} \frac{z(\tilde{x})^{kJ-1}}{z(x)^{kJ-1}} d\tilde{x} \right] dz(x)^{kJ} \\
&\quad = \int_{y_j^*}^{y_l^*} (x - w)\, dz(x)^{kJ} - \int_{y_j^*}^{y_l^*} kJ \left[(y_j^* - r_j) z(y_j^*)^{kJ-1} + \int_{y_j^*}^{x} z(\tilde{x})^{kJ-1}\, d\tilde{x} \right] dz(x) \\
&\quad = (y_l^* - w) z(y_l^*)^{kJ} - (y_j^* - w) z(y_j^*)^{kJ} - \int_{y_j^*}^{y_l^*} z(x)^{kJ}\, dx \\
&\qquad - (y_j^* - r_j)\, kJ\, z(y_j^*)^{kJ-1} [z(y_l^*) - z(y_j^*)] \\
&\qquad - kJ \int_{y_j^*}^{y_l^*} \int_{x}^{y_l^*} dz(\tilde{x})\, z(x)^{kJ-1}\, dx \\
&\quad = (y_l^* - w) z(y_l^*)^{kJ} - (y_j^* - w) z(y_j^*)^{kJ} - \int_{y_j^*}^{y_l^*} z(x)^{kJ}\, dx \\
&\qquad - (y_j^* - r_j)\, kJ\, z(y_j^*)^{kJ-1} [z(y_l^*) - z(y_j^*)] \\
&\qquad - \int_{y_j^*}^{y_l^*} kJ\, z(x)^{kJ-1} [z(y_l^*) - z(x)]\, dx.
\end{aligned}
$$

Hence, we can apply Lemma A.8 to prove using the Lebesgue bounded convergence theorem [9, Theorem 16, p. 91]:

$$
\begin{aligned}
&\lim_{J \to \infty} \int_{y_j^*(G_J)}^{y^*_{\underline{j}(G)}(G_J)} (\Psi(x; \vec{y}^*(G_J), G^J) - w)\, dz(x; \vec{y}^*(G_J), G^J)^{kJ} \\
&\quad = (r_{\underline{j}(G)} - w) \bar{z}(r_{\underline{j}(G)}; \vec{y}^*(G_J), G).
\end{aligned}
$$

This last result completes the proof of Lemma 4.

Lemma A.8. *Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of continuous functions with compact domain in $\mathbb{R}^2$ that converges point-wise to a function Y. Suppose that each of the functions Y_n*

[14] To simplify the notation we write $\Psi(x)$, y_l^*, and $z(x)$ for $\Psi(x; \vec{y}^*(G_J), G_J)$, $y_l^*(G_J)$, and $z(x; \vec{y}^*(G_J), G_J)$, respectively.

and Y are strictly increasing in the first argument and weakly decreasing in the second argument, and are such that for any x in the domain, there exists a $\tilde{y}_n$ such that $Y_n(\tilde{y}_n, x) = 0$ and a $\tilde{y}$ such that $Y(\tilde{y}, x) = 0$. Then the sequence of functions y_n defined by $Y_n(y_n(x), x) = 0$ converges uniformly to the function y defined by $Y(y(x), x) = 0$.

Proof. We start taking an $\varepsilon > 0$. Note next that the monotonic properties and continuity of Y imply that y must be continuous. Hence, for each x in the domain of y, there exists a $\delta(x) > 0$ such that if $x' \in (x - \delta(x), x + \delta(x))$, then $y(x') \in (y(x) - \frac{\varepsilon}{4}, y(x) + \frac{\varepsilon}{4})$. We denote by $J(x)$ the set of such x', i.e. $J(x) \equiv (x - \delta(x), x + \delta(x))$. Since by definition $Y(y(x'), x') = 0$, and $y(x) - \frac{\varepsilon}{2} < y(x) - \frac{\varepsilon}{4} < y(x')$ and $y(x) + \frac{\varepsilon}{2} > y(x) - \frac{\varepsilon}{4} > y(x')$, the monotonic properties of Y imply that for all $x' \in J(x)$, $Y(y(x) - \frac{\varepsilon}{2}, x') < 0$, and $Y(y(x) + \frac{\varepsilon}{2}, x') > 0$.

Point-wise convergence of Y_n to Y implies that there exists a $n_0(x) \in \mathbb{N}$ such that if $n \geqslant n_0(x)$, then $Y_n(y(x) - \frac{\varepsilon}{2}, x') < 0$, and $Y_n(y(x) + \frac{\varepsilon}{2}, x') > 0$, for all $x' \in J(x)$. Hence, the continuity of Y_n implies that for all $x' \in J(x)$ and $n \geqslant n_0(x)$,

$$y_n(x') \in \left(y(x) - \frac{\varepsilon}{2}, y(x) + \frac{\varepsilon}{2}\right) \subset (y(x') - \varepsilon, y(x') + \varepsilon).$$

Note that $x \in J(x)$, thus the domain of y, say D, is a subset of $\cup_{x \in D} J(x)$. Since D is compact, the Heine–Borel theorem [9, Theorem 15, p. 44] implies that there exists a finite collection of sets in $\{J(x)\}_{x \in D}$ that covers D, i.e. $D \subset \cup_{m=1}^{M} J(x_m)$, for M finite. Take $n_0 = \max\{n_0(x_1), n_0(x_2), \ldots, n_0(x_M)\}$, then for all $n \geqslant n_0$,

$$y_n(x') \in (y(x') - \varepsilon, y(x') + \varepsilon),$$

for all $x' \in D$, this is, for all x' in the domain of y. This proves uniform convergence of y_n to y. □

A.5. End of the Proof of Proposition 1

In a symmetric equilibrium in pure strategies all the auctioneers use the same map from the set of production costs to the set of reserve prices. Since both sets are finite, the set of auctioneers' pure strategies, say Θ, is finite.

Denote by $\tilde{G}_{J-1}(.|\theta, w)$ the distribution of reserve prices that an auctioneer with production cost w faces when all the other auctioneers use the pure strategy θ in a game with J auctioneers. Clearly, the sequence $\tilde{G}_{J-1}(.|\theta, w)/(J-1)$ converges weakly to a distribution function with support in Π when J tends to infinity. We denote by $G(.|\theta)$ this limit distribution.

Let $\Delta(w, r_j, \theta) \equiv |\bar{\Phi}(w, w, G(.|\theta)) - \bar{\Phi}(r_j, w, G(.|\theta))|$, and

$$\delta \equiv \min\{\Delta(w, r_j, \theta) : \Delta(w, r_j, \theta) > 0, (w, r_j, \theta) \in \Pi_W \times \Pi \times \Theta\}.$$

Since the set is finite, then $\delta > 0$. Lemma 4 implies that there exists a $\bar{J}$ such that if $J \geqslant \bar{J}$, then $|\bar{\Phi}(r_j, w, G(.|\theta)) - \Phi(r_j, w, \tilde{G}_{J-1}(.|\theta, w))| < \delta/2$ for all $(w, r_j, \theta) \in \Pi_W \times \Pi \times \Theta$.

Putting together this result and Lemma 5 we can prove the following:

Lemma A.9. *For any* $(w, r_j, \theta) \in \Pi_W \times \Pi \times \Theta$ *there exists a* $\bar{J}$ *such that if* $J \geqslant \bar{J}$, *then for* $r_j \neq w$,

$$\Phi(w, w, \tilde{G}_{J-1}(.|\theta, w)) > \Phi(r_j, w, \tilde{G}_{J-1}(.|\theta, w))$$

in the following cases:

(A) $r_{\underline{j}(G(.|\theta)} < w < r_{\bar{j}(G(.|\theta))}$.
(B) $w \leqslant r_{\underline{j}(G(.|\theta))}$ *and* $r_j \geqslant r_{\underline{j}(G(.|\theta))}$.
(C) $w \geqslant r_{\bar{j}(G(.|\theta))}$ *and* $r_j < r_{\bar{j}(G(.|\theta))}$.

The strategy in which all the auctioneers announce reserve price equal to production cost is $\theta(w) = w$ for any $w \in \Pi_W$, and clearly, it implies that $G(.|\theta) = H$. Now, recall that by assumption (see the last paragraph in Section 2): $r_1 = r_{\underline{j}(H)}$, and $w < r_{\bar{j}(H)}$ for any $w \in \Pi_W$. Thus, Lemma A.9(A) and (B) imply that for $J > \bar{J}$ auctioneers do not have incentives to individually deviate from $\theta(w) = w$, and thus, that it is an equilibrium strategy.

To prove uniqueness we assume that all the auctioneers use the strategy $\theta \in \Theta$. Then we show that if θ is an equilibrium strategy, then $\theta(w) = w$ for any $w \in \Pi_W$.

Suppose $\theta(r_1) \neq r_1$. Clearly, $r_1 \leqslant r_{\underline{j}(G^\theta)}$ and $\theta(r_1) \geqslant r_{\underline{j}(G(.|\theta))}$, thus, Lemma A.9(B) implies that auctioneers with production cost r_1 have a profitable deviation which is a contradiction. Similarly, suppose that $\theta(w) = r_1$ for $w \neq r_1$. Clearly, $r_1 < r_{\bar{j}(G(.|\theta))}$, so Lemma A.9(A) and (C) implies that $\Phi(w, w, \tilde{G}_{J-1}(.|\theta, w)) > \Phi(r_1, w, \tilde{G}_{J-1}(.|\theta, w))$. This means that auctioneers with production cost $w \neq 1$ have a profitable deviation which is a contradiction.

We now proceed by induction. Suppose that there exists a $r_j \leqslant \max\{\Pi_W\}$ such that $\theta(w) = w$ for $w < r_j$, and such that $\theta(w) \geqslant r_j$ for $w \geqslant r_j$, this is $H(x) = G(x)$ for all $x < r_j$. Clearly, $r_j > r_{\underline{j}(G(.|\theta))}$, and moreover, the arguments below show that $r_j < r_{\bar{j}(G(.|\theta))}$. Thus, Lemma A.9(A) imply that if $r_j \in \Pi_W$ then either $\theta(r_j) = r_j$ or there is a profitable deviation. Moreover, Lemma A.9(A) and (C) imply that if $w > r_j$, then $\theta(w) > r_j$, otherwise there is a profitable deviation.

It only remains to be shown that if $r_j < \max\{\Pi_W\}$ and $H(x) = G(x)$ for all $x < r_j$, then $r_j < r_{\bar{j}(G)}$. Under our assumption that $w < r_{\bar{j}(H)}$ for any $w \in \Pi_W$, it is sufficient to show that $y_j^*(G) = y_j^*(H)$ if $H(x) = G(x)$ for $x < r_j$.

If $r_j \leqslant r_{\underline{j}(G)}$ the claim is direct from condition (i′). Suppose now that $r_j > r_{\underline{j}(G)}$. Then condition (ii′), and something similarly could be done for condition (iii′), can be rewritten as follows:

$$\int_{y_{j-1}}^{y_j} xk \frac{f(x)}{G(r_{j-1})} e^{-k \frac{F(y_j)-F(x)}{G(r_{j-1})}} dx + r_{j-1} e^{-k \frac{F(y_j)-F(y_{j-1})}{G(r_{j-1})}} = r_j.$$

If $r_j = r_{\underline{j}(G)+1}$, then y_j^* depends on G only up to the value of $G(r_{\underline{j}(G)})$. Consequently the claim follows for this reserve price. Note that in general for $r_j > r_{\underline{j}(G)}$ we can apply recursively the last argument to show that y_j^* depends on G only up to the value of $G(r_{\underline{j}(G)}), G(r_{\underline{j}(G)+1}), \ldots, G(r_{j-1})$. This completes the proof of our claim.

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