ON THE TOMOGRAPHIC PICTURE OF QUANTUM MECHANICS

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ABSTRACT. We formulate necessary and sufficient conditions for a symplectic tomogram of a quantum state to determine the density state. We establish a connection between the (re)construction by means of symplectic tomograms with the construction by means of Naimark positive-definite functions on the Weyl-Heisenberg group. This connection is used to formulate properties which guarantee that tomographic probabilities describe quantum states in the probability representation of quantum mechanics.

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1. Introduction

It has been shown recently ([1, 2, 3], see also [4]) how to describe quantum states by using a standard positive probability distribution called a symplectic probability distribution or symplectic tomogram. The symplectic tomogram $\mathcal{W}(X, \mu, \nu)$ is a nonnegative function of the random position X measured in reference frames in phase–space with rotated and scaled axes $q \to \mu q$, $p \to \nu p$ where $\mu = e^{\lambda} \cos \theta$, $\nu = e^{-\lambda} \sin \theta$, θ is the angle of rotation and e^{λ} is the scaling parameter.

The symplectic tomographic probability distribution $\mathcal{W}(X, \mu, \nu)$ contains complete information on quantum states in the sense that for a given wave function $\psi(x)$ or density operator $\hat{\rho}$ (determining the quantum state [5], [6] in the conventional formulation of quantum mechanics) the tomogram can be calculated. On the other hand for a given tomogram $\mathcal{W}(X, \mu, \nu)$ one can reconstruct explicitly the density operator $\hat{\rho}$. It means that for a given symplectic tomogram of a system with continuous variables all the properties of the quantum system can be obtained as well as for a given density operator $\hat{\rho}$. Analogous complete information on the quantum states is contained in the Wigner function [7] $W(q, p)$ which is a real function on the phase space of the system. The Wigner function is related to the symplectic tomogram by means of an integral Radon transform [8], however the Wigner function is not definite in sign, it takes negative values for some quantum states and cannot be considered as a positive probability distribution on phase space. The necessary and sufficient conditions for a real function on the phase space to describe the Wigner function of a quantum state were found in [9] where the corresponding properties of the function under consideration were associated with the so called h–positivity condition of a function on the Abelian translation group on the phase space.

As we have shown elsewhere [10], in this description plays an important role the Weyl-Heisenberg group and its group of automorphisms, along with the Abelian vector group which arises as quotient group of Weyl-Heisenberg group by its central subgroup. In this paper we would like to consider the tomographic description of quantum mechanics as another picture, on the same footing as the Schroedinger, Heisenberg or Weyl–Wigner pictures. To this aim we have to provide a characterization of symplectic tomograms which stands on its own, without relying on other pictures. In other terms, we need necessary and sufficient conditions for a function $f(X, \mu, \nu)$ to be the symplectic tomogram $\mathcal{W}(X, \mu, \nu)$ of a quantum state. The strategy to find these conditions is based on Naimark's theorem [11] that provides a characterization of positive operator valued measures and that allows to characterize functions which are elements of matrices of group representations. In particular we use the result that a function $\varphi(g)$ on a group G, $g \in G$, which is a diagonal matrix element of a unitary representation of the group G has the property of being positive definite in the sense that the matrix

$$(1) \qquad\qquad M_{jk} = \varphi(g_j g_k^{-1})$$

for any $j, k = 1, 2, ..., N$ and arbitrary N, is positive definite. Below we will show that symplectic tomograms can be associated with positive definite functions φ on the Weyl-Heisenberg group. Since Naimark's theorem for positive operator–valued measures allows to construct and determine uniquely a Hilbert space and a vector in it representing the function φ (using what today is called the Gelfand-Naimark-Segal (GNS) method) the connection established below of the symplectic tomograms with positive definite functions on the Weyl-Heisenberg group yields the necessary and sufficient condition which we are looking for.

It is worthy to note that this condition can be also studied using the necessary and sufficient condition for a function to be a Wigner function [9] but we do not use here the connection of symplectic tomogram with the Wigner function and provide the condition for the tomogram independently of any other result concerning Wigner functions.

2. Symplectic tomography

In this section we briefly review the construction of tomographic probability densities determining the quantum state of a particle in one degree of freedom [4]. Given the density operator $\hat{\rho}$ of a particle quantum state, $\hat{\rho} = \hat{\rho}^{\dagger}$, $\text{Tr}\hat{\rho} = 1$, and $\hat{\rho} \geq 0$, the symplectic tomogram of $\hat{\rho}$ is defined by:

$$(2) \qquad \mathcal{W}(X, \mu, \nu) = \text{Tr}[\hat{\rho}\, \delta(X\,\hat{1} - \mu\hat{Q} - \nu\hat{P})], \qquad X, \mu, \nu \in \mathbb{R}.$$

Here $\hat{Q}$ and $\hat{P}$ are the position and momentum operators. The Dirac delta–function with operator arguments is defined by the standard Fourier integral,

$$\delta(X\,\hat{1} - \mu\hat{Q} - \nu\hat{P}) = \int e^{-ik(X\,\hat{1} - \mu\hat{Q} - \nu\hat{P})}\frac{dk}{2\pi}.$$

The symplectic tomogram $\mathcal{W}(X, \mu, \nu)$ has the properties which follow from its definition by using the known properties of delta-function, namely:

      i. Nonnegativity:

$$(3) \qquad\qquad\qquad\qquad \mathcal{W}(X, \mu, \nu) \geq 0$$

        (this holds by observing that the trace of the product of two positive operators is a positive number).

     ii. Normalization:

$$(4) \qquad\qquad\qquad\qquad \int \mathcal{W}(X, \mu, \nu)dX = 1.$$

     iii. Homogeneity:

$$(5) \qquad\qquad\qquad \mathcal{W}(\lambda X, \lambda\mu, \lambda\nu) = \frac{1}{|\lambda|}\mathcal{W}(X, \mu, \nu).$$

However, the three above properties are by no means sufficient to determine the quantum character of a tomographic function $f(X, \mu, \nu)$. For instance, consider

$$(6) \qquad f(X, \mu, \nu) = \exp\left(-\frac{X^2}{2\left(\mu^2 + \nu^2\right)}\right)\frac{5\left(\mu^2 + \nu^2\right) - X^2}{\sqrt{2\left(\mu^2 + \nu^2\right)^3}}.$$

Despite the uncertainty relations are satisfied by such a function, f is not a quantum tomogram because $\langle P^2 \rangle = \langle Q^2 \rangle = -1/2$, as it can be checked using

$$(7) \qquad \langle P^2 \rangle = \int X^2\, f(X, \mu, \nu)|_{\mu=0, \nu=1}\, dX$$

and analogously for $\langle Q^2 \rangle$.

On the other hand, it is easy to see that formula (2) has an inverse [12]:

$$(8) \qquad \hat{\rho} = \frac{1}{2\pi} \int \mathcal{W}(X, \mu, \nu) e^{i(X\,\hat{1} - \mu\hat{Q} - \nu\hat{P})}\, dX\, d\mu\, d\nu.$$

Thus the knowledge of the symplectic tomogram $\mathcal{W}(X, \mu, \nu)$ means that the density operator $\hat{\rho}$ is also known, more precisely, can be reconstructed. This correspondence between symplectic tomograms $\mathcal{W}(X, \mu, \nu)$ and density operators $\hat{\rho}$ gives the possibility to formulate the notion of quantum state using tomograms as the primary notion. However to make this idea precise, we need to formulate additional conditions to be satisfied by the function $\mathcal{W}(X, \mu, \nu)$ which are extra to the conditions (3)-(5) and which guarantee that by using the inversion formula (8) we get an operator with all the necessary properties of a density state. The general recipe to formulate these demands can be given by checking the nonnegativity condition of the integral (see [4]):

$$(9) \qquad \int \mathcal{W}(X, \mu, \nu) e^{i(X\,\hat{1} - \mu\hat{Q} - \nu\hat{P})}\, dX\, d\mu\, d\nu \geq 0.$$

It means that for a given function $\mathcal{W}(X, \mu, \nu)$ satisfying the conditions (3)-(5) one has to check the nonnegativity of the operator (9), thus if the inequality (9) holds the function $\mathcal{W}(X, \mu, \nu)$ is the symplectic tomogram of a quantum state, however it must be realized that this is not an operative procedure.

Below we formulate the conditions for a function $\mathcal{W}(X, \mu, \nu)$ to be a tomogram of a quantum state avoiding the integrations in eq. (9). As anticipated in the introduction, to be able to use Naimark's results we have to deal with functions defined on a group. Thus, we have to show how symplectic tomograms may be associated with the Weyl-Heisenberg group. In doing this we can exploit results in [11] where the theorems on properties of diagonal matrix elements of unitary representations provide the key to construct tomograms which represent quantum states.

3. Tomographic probability measures

To get a mathematical formulation of the tomographic picture we invoke the spectral theory of Hermitian operators, which moreover will provide us with a probabilistic interpretation of the symplectic tomogram. We start rewriting the formal definition, eq. (2), for a quantum tomogram:

$$(10) \qquad \mathcal{W}(X, \mu, \nu) = \mathrm{Tr}\left[\hat{\rho} \int e^{ik(X\,\hat{1} - \mu\hat{Q} - \nu\hat{P})} \frac{dk}{2\pi}\right] = \int e^{ikX} \mathrm{Tr}[\hat{\rho} e^{-ik(\mu\hat{Q} + \nu\hat{P})}] \frac{dk}{2\pi}.$$

then we observe that

$$(11) \qquad \mu\hat{Q} + \nu\hat{P} = S_{\mu\nu}\hat{Q}S_{\mu\nu}^{\dagger}$$

where

(12) $$S_{\mu\nu} = \exp\left[\frac{i\lambda}{2}\left(\hat{Q}\hat{P} + \hat{P}\hat{Q}\right)\right]\exp\left[\frac{i\theta}{2}\left(\hat{Q}^2 + \hat{P}^2\right)\right],$$

with

(13) $$\mu = e^{\lambda}\cos\theta \ , \nu = e^{-\lambda}\sin\theta \ .$$

In other words, by acting with the unitary operators $S_{\mu\nu}$ on the position operator $\hat{Q}$ we get out the iso-spectral family of hermitian operators

$$X_{\mu\nu} = \mu\hat{Q} + \nu\hat{P}.$$

This family is a symplectic tomographic set [13].

To any operator of this family is associated a projector valued measure $\Pi_{\mu\nu}$ on the σ–algebra of Borel sets on the real line:

$$\mu\hat{Q} + \nu\hat{P} = \int \lambda \, d\,\Pi_{\mu\nu}(\lambda).$$

Given any density state $\hat{\rho}$, the projector valued measure $\Pi_{\mu\nu}$ yields a normalized probability measure $m_{\rho,\mu\nu}$ on the Borel sets $E \in \mathrm{Bo}(\mathbb{R})$ of the real line:

(14) $$m_{\rho,\mu\nu}(E) = \mathrm{Tr}[\,\hat{\rho}\,\Pi_{\mu\nu}(E)]; \quad m_{\rho,\mu\nu}(\mathbb{R}) = 1.$$

We recall that $m_{\rho,\mu\nu}(E)$ is the probability that a measure of the observable $\mu\hat{Q} + \nu\hat{P}$ in the state $\hat{\rho}$ is in E. All these measures $m_{\rho,\mu\nu}$ are absolutely continuous with respect to the Lebesgue measure dX on the real line, so that densities $V_\rho(X,\mu,\nu)$ may be introduced such that

(15) $$m_{\rho,\mu\nu}(E) = \int_E V_\rho(X,\mu,\nu)\,dX.$$

We can write

(16) $$\mathrm{Tr}\left(\hat{\rho}e^{-i\lambda(\mu\hat{Q}+\nu\hat{P})}\right) = \mathrm{Tr}\left(\hat{\rho}S_{\mu\nu}e^{-i\lambda\hat{Q}}S_{\mu\nu}^{\dagger}\right) = \int e^{-i\lambda X}\,V_\rho(X,\mu,\nu)\,dX$$

so that

(17) $$\begin{aligned}
\mathcal{W}(X,\mu,\nu) &= \int e^{ikX}\,\mathrm{Tr}[\hat{\rho}e^{-ik(\mu\hat{Q}+\nu\hat{P})}]\frac{dk}{2\pi} \\
&= \int e^{ikX}\,e^{-ikX\,'}V_\rho(X',\mu,\nu)dX'\frac{dk}{2\pi} \\
&= \int \delta(X - X')V_\rho(X',\mu,\nu)dX' = V_\rho(X,\mu,\nu).
\end{aligned}$$

In other words we have shown that the symplectic tomogram $\mathcal{W}(X,\mu,\nu)$ of a given state $\hat{\rho}$ is nothing but the density $V_\rho(X,\mu,\nu)$ of the probability measure associated to the state by means of the symplectic tomographic set. The tomographic character of the family of observables $X_{\mu\nu}$ is contained in the possibility of reconstructing any state out of the corresponding probability measures by means of the previous reconstructing formula. By using eqs. (16) and (8), we get

(18) $$\hat{\rho} == \frac{1}{2\pi}\int \mathrm{Tr}[\hat{\rho}e^{i(\mu\hat{Q}+\nu\hat{P})}]\,e^{-i(\mu\hat{Q}+\nu\hat{P})}\,d\mu\,d\nu,$$

moreover

(19)
$$\frac{1}{2\pi} \int \mathrm{Tr}\big[e^{i(\mu\hat{Q}+\nu\hat{P})}\big]\, e^{-i(\mu\hat{Q}+\nu\hat{P})}\, d\mu\, d\nu = \hat{1}.$$

The presence of Weyl operators $D(\mu,\nu) = e^{i(\mu\hat{Q}+\nu\hat{P})}$ suggest that we are dealing with projective representations of the Abelian vector group. We shall take up group theoretical aspects in next section.

4. A GROUP THEORETICAL DESCRIPTION OF QUANTUM TOMOGRAMS

The probabilistic interpretation above allows to consider the tomographic description of quantum states as a picture of quantum mechanics on the same footing as other well known representations, like Schrödinger, Heisenberg and Wigner-Weyl for instance. Thus, to be an alternative picture of quantum mechanics we need criteria to recognize a function $f(X,\mu,\nu)$ as a tomogram of a quantum state. For this, the use of the reconstruction formula to check if the obtained operator is a density operator would be unsatisfactory, mainly because this check requires to switch from tomographic to Schrödinger picture. In other words, we would like to establish self-contained criteria for a function to be a quantum tomogram. More precisely, we have to address the following problem: given a tomogram-like function $f(X,\mu,\nu)$, that is a function with the above properties eqs. (3)-(5) of a tomogram, what are the necessary and sufficient conditions to recognize f as a quantum tomogram?

To this aim we begin to observe that in the characteristic tomographic function

(20)
$$\mathrm{Tr}[\hat{\rho}e^{i(\mu\hat{Q}+\nu\hat{P})}] = \mathrm{Tr}[\hat{\rho}D(\mu,\nu)]$$

a projective representation of the translation group appears. This projective representation can be lifted to a true unitary representation of the Weyl-Heisenberg group (see for instance [10] and references therein for a detailed discussion of the subject) by means of a central extension of the translation group. Such central extension defines the Weyl-Heisenberg group $WH(2)$ whose elements are denoted by (μ,ν,t) and the group law reads:

(21)
$$(\mu,\nu,t) \circ (\mu',\nu',t') = (\mu+\mu',\nu+\nu',t+t'+\tfrac{1}{2}\omega((\mu,\nu),(\mu',\nu'))),$$

where $\omega((\mu,\nu),(\mu',\nu')) = \mu\nu' - \nu\mu'$ denotes the symplectic form on $\mathbb{R}^2$. The nontrivial unitary irreducible representations of the Weyl-Heisenberg group are provided by the expression:

(22)
$$U_\gamma(\mu,\nu,t) = D_\gamma(\mu,\nu)e^{i\gamma tI}.$$

where γ is a non-vanishing real number. In what follows we will set $\gamma = 1$. Hence we immediately observe that

(23)
$$\mathrm{Tr}[\hat{\rho}D(\mu,\nu)] = e^{-it}\mathrm{Tr}[\hat{\rho}U(\mu,\nu,t)]$$

where the function $\mathrm{Tr}[\hat{\rho}U(\mu,\nu,t)]$ is of positive type [11].

For convenience we recall the definition of functions of positive type. Given a group G a function $\varphi(g)$ on G ($g \in G$) is of positive type, or

definite positive, if for any n–tuple of group elements $(g_1, g_2, ..., g_n)$ the matrix

(24) $$M_{jk} = \varphi(g_j g_k^{-1}) \qquad j, k = 1, 2, ..., n,$$

is positive semi-definite for any $n \in \mathbb{N}$, or in other words, if for any finite family of elements $g_1, g_2, \ldots, g_n \in G$ and for any family of complex numbers $\xi_1, \ldots, \xi_n$, we have $\sum_{j,k=1}^n \bar{\xi}_j \xi_k \varphi(g_j g_k^{-1}) \geq 0$, for any n. Moreover, a simple computation shows that given any unitary representation $U(g)$ of G and a state ρ, $\text{Tr}[\hat\rho U(g)]$ is a group function of positive type. Viceversa any positive type group function $\varphi(g)$ can be written in the form

(25) $$\text{Tr}[\hat\rho_\xi U(g)] = \langle \xi, U(g)\xi \rangle,$$

where $U(g)$ is a unitary representation and ξ is a cyclic vector in a suitable Hilbert space, obtained for instance by means of a GNS construction [11]. Thus the condition on the matrix introduced in (24) is a way to affirm that φ is associated with a state without making recourse to a representation.

Thus we can state the required condition: a tomogram–like function $f(X, \mu, \nu)$ is a quantum tomogram, i.e., there exists a quantum state $\hat\rho$ such that $f(X, \mu, \nu) = \text{Tr}[\hat\rho\, \delta(X\,\hat 1 - \mu\hat Q - \nu\hat P)]$, if and only if its Fourier transform evaluated at 1 may be written in the form

(26) $$\int f(X, \mu, \nu)e^{iX} dX = e^{-it}\varphi_f(\mu, \nu, t)$$

where $\varphi_f(\mu, \nu, t)$ is a positive definite function on the Weyl-Heisenberg group. In fact if $\mathcal{W}$ is a quantum tomogram, then because of eqs. (10) and (23) we have,

(27)
$$\int \mathcal{W}(X, \mu, \nu)e^{iX}\, dX = \text{Tr}[\hat\rho D(\mu, \nu)] = e^{-it}\text{Tr}[\hat\rho U(\mu, \nu, t)] = e^{-it}\varphi(\mu, \nu, t),$$

where $\varphi(\mu, \nu, t)$ is a positive definite function on the Weyl-Heisenberg group. Moreover if we define $\psi(\mu, \nu) = \text{Tr}(\hat\rho D(\mu, \nu))$, then $\psi(\mu, \nu)$ is a function on the translation group considered as a quotient of the Weyl-Heisenberg group by the central element. It means that we are dealing with a projective representation and not a unitary representation like in Naimark's theorem eq. (25). Then we could ask about the properties enjoyed by the matrix $\tilde M_{jk}$ constructed using ψ instead of φ. If we denote as above by ω the 2–cocycle defining the projective representation, then we will say that $\tilde M_{jk}$ is of ω–positive type, i.e.

(28) $$\tilde M_{jk} = \psi((\mu_j, \nu_j)^{-1} \circ (\mu_k, \nu_k))e^{i\frac{1}{2}\omega((\mu_k,\nu_k),(\mu_j,\nu_j))}$$

is positive semidefinite. This yields the corresponding condition: a tomogram–like function $f(X.\mu, \nu)$ is a quantum tomogram if and only if its Fourier transform evaluated at 1 may be written in the form

(29) $$\int f(X, \mu, \nu)e^{iX} dX = \psi_f(\mu, \nu)$$

where $\psi_f(\mu, \nu)$ is a function of the translation group of ω–positive type.

We observe that $\psi_f(\mu, \nu)$ may be at same time of positive and ω–positive type on the translation group. Then by Bochner theorem $\psi_f(\mu, \nu)$ is the Fourier transform of a probability measure on the phase space. In other

words $f(X, \mu, \nu)$ is the (classical) Radon transform of such a probability measure i.e. a classical tomogram. The tomogram of the ground state of the harmonic oscillator provides an example of the above situation. In that case, the GNS construction yields a Hilbert space of square integrable functions on phase space with respect to the measure provided by the Bochner theorem.

To finish this analysis let us notice that if ψ is a function of ω–positive type on the translation group, then the function $\varphi(\mu, \nu, t) = e^{it}\psi(\mu, \nu)$ will be a positive definite function on the Weyl-Heisenberg group $\mathcal{WH}(2)$ and, by Naimark's theorem, there will exist a unitary representation U of $\mathcal{WH}(2)$ and a cyclic state vector $|\xi\rangle$ such that $\varphi(\mu, \nu, t) = \langle\xi, U(\mu, \nu, t)\xi\rangle$. On the other hand, $\psi(\mu, \nu)$ is obtained by $f(X, \mu, \nu)$, which is a tomogram of a quantum state $\hat{\rho}$. Up to a unitary transformation $\hat{\rho}$ will coincide with $\hat{\rho}_\xi$ iff it is a pure state. Notably, the purity of $\hat{\rho}$ can be expressed as:

$$(30) \quad \operatorname{tr}\hat{\rho}^2 = \frac{1}{2\pi}\int \mathcal{W}(X, v)\mathcal{W}(Y, -v)e^{i(X+Y)}dX\,dY\,dv = \frac{1}{2\pi}\int_{\mathbb{R}^2}|\psi(v)|^2\,dv$$

so that the above condition can be stated as:

$$(31) \qquad\qquad \int_{\mathbb{R}^2}|\psi(v)|^2\,dv = 1$$

Here vector $v(\mu, \nu)$. The case of a mixed density state ρ, when eq.(31) does not hold, will be discussed elsewhere.

5. Conclusions and outlooks

To conclude we resume the main results of our work. The symplectic tomographic probability distribution considered as the primary concept of a particle quantum state alternative to the wave function or density matrix, we have shown to be associated with a unitary representation of the Weyl-Heisenberg group. This connection was used to formulate the autonomous conditions for the symplectic tomogram to describe quantum states using the positivity properties of the matrix M_{jk} of eq.(24) introduced in [11] and connected with diagonal elements of the unitary representation (positive-type function $\varphi(g)$ on the group). The function $f(X, \mu, \nu)$, satisfying the necessary properties of tomographic probability distribution, i.e. non-negativity, homogeneity and normalization, was shown to be a quantum tomogram iff its Fourier transform in the quadrature variable X can be written in the form of eq.(26) as the product of a positive type function on the Weyl-Heisenberg group and a phase factor associated with a central element of the group. By using the quantum Radon anti-transform eq.(8), this condition guarantees that the function $f(X, \mu, \nu)$ provides a density state, so that f is the symplectic tomogram of a quantum state. The criterion, formulated in terms of positivity properties of a group function obtained from the tomographic function, is not easy to implement operatively.Nevertheless, it is simpler than the criterion based on checking the non negativity of the operator given by the quantum Radon anti transform. Also we have shown that the purity of the quantum state can be expressed as the square of the L_2- norm of that positive group function, which is obtained by tomograms measured directly in optical experiments, without considering density matrices or Wigner functions. As a spin-off we have shown that the notion of $h-$

positivity may be subsumed under the notion of positivity for a centrally extended group. In this paper we have considered tomograms associated with the Weyl-Heisenberg group. In a forthcoming paper we will show how to deal with the tomographic picture for general Lie groups and for finite groups. In this connection we shall also elaborate more on the C^*-algebraic approach to quantum mechanics and its counterpart in terms of tomograms.

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