F and G follow as in Phillips [6] and Hannan [4]. This is due to the fact that the matrix polynomial $I - Fz$ and $I + Gz$ have a greatest left divisor which is the identity matrix. Since the null spaces of F and G have a null intersection, Hannan's [4] theorem can now be invoked guaranteeing the identifiability of the matrices. We like to add that the question of left coprimeness of $(I - Fz, I + Gz)$ is related to the dependence of F and G on the parameters θ and μ. Notice also that the condition that (F, G) has full rank is related to the dependence on the parameters. It could be that, for example, the rows of (F, G) are linearly independent for all feasible values of θ and μ. Thus, for an actual analysis of identifiability the properties of the parameterization of (F, G) are of central importance.

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89.1.2. *Optimal Instrumental Variable Estimator of the AR Parameter of an ARMA(1.1)*—Solution, proposed by Juan J. Dolado. The model can be written as

$$y_t = \alpha y_{t-1} + u_t; \qquad u_t = e_t + \theta e_{t-1}; \qquad Euu' = \sigma^2\Omega$$

with $|\theta| < 1$, $|\alpha| < 1$, $e_t \sim \text{i.i.d.}(0, \sigma^2)$, and $E(e_t^4) = \eta\sigma^4$.

The matrix Ω is tridiagonal with $1 + \theta^2$ on the leading diagonal, except the first element which is equal to 1, and θ on the off-diagonals. Since plim n^{-1} $\Sigma y_{t-1}u_t \neq 0$, an IVE ($\hat{\alpha}_h = \Sigma y_t y_{t-h}/\Sigma y_{t-1}y_{t-h}$) has to be used.

1. $\text{plim}\,(\hat{\alpha}_1 - \alpha) = \text{plim}\; n^{-1}\,\Sigma y_{t-1}u_t/\text{plim}\; n^{-1}\Sigma y_{t-1}^2 = \theta\sigma^2/\gamma_0$ **(1)**

   where $\gamma_0 = \sigma^2(1 + 2\alpha\theta + \theta^2)/(1 - \alpha^2)$, which is the variance of an ARMA(1.1) process. Therefore, $\hat{\alpha}_1$ is consistent iff $\theta = 0$.
2. For $h \geq 2$,

   $\text{plim}(\hat{\alpha}_h - \alpha) = \text{plim}\; n^{-1}\Sigma y_{t-h}u_t/\text{plim}\; n^{-1}\Sigma y_{t-h}y_{t-1}$

   $\text{plim}\; n^{-1}\,\Sigma y_{t-h}u_t = 0$ for $h \geq 2$

$$\text{plim } n^{-1}\Sigma y_{t-h}y_{t-1} = \alpha^{h-2}\gamma_1 \text{ for } h \geq 2$$

$$\gamma_1 = \sigma^2(\alpha + \theta)(1 + \alpha\theta)/1 - \alpha^2. \tag{2}$$

In order to achieve consistency, the second plim has to be different from zero (Slutsky's theorem). Hence, α needs to differ from zero, except when $\theta \neq 0$ and $h = 2$ where the plim of the denominator in (2) corresponds to the first-order covariance of an MA(1) process ($= \theta\sigma^2$).

3. $n^{1/2}(\hat{\alpha}_h - \alpha) = n^{-1/2}\, \Sigma y_{t-h}u_t/n^{-1}\, \Sigma y_{t-h}y_{t-1}$. Under the assumptions
   (i) $E(y_{t-h}\, u_t) = 0$ (verified for $h \geq 2$)
   (ii) plim $n^{-1}\, \Sigma y_{t-h}y_{t-1} < \infty$
   (iii) $Ee_t^4 < \infty$
   (iv) $\lambda = \lim n^{-1}\, E(\Sigma y_{t-h}u_t)^2$ exists and is positive.

   Hansen [1] proves a CLT by which $n^{-1/2}\Sigma y_{t-h}u_t$ converges in distribution to $N(0,\lambda)$. Thus,

   $$n^{1/2}(\hat{\alpha}_h - \alpha) \sim N[0, (\text{plim } n^{-1}\, \Sigma y_{t-h}y_{t-1})^{-2}\lambda], \tag{3}$$

   where

   $$\lambda = \lim n^{-1}E(\Sigma y_{t-h}u_t)^2 = \sigma^2(1 + \theta^2)\,[\gamma_0 + 2\theta\gamma_1/1 + \theta^2], \qquad \forall h \geq 2$$

   and

   $$\text{plim } n^{-1}\, \Sigma y_{t-h}y_{t-1} = \alpha^{h-2}\gamma_1, \qquad \forall h \geq 2.$$

   Since the numerator of the variance in the limiting distribution is constant for all $h \geq 2$ and the denominator is $\alpha^{h-2}\gamma_1$, where $|\alpha| < 1$, $h = 2$ gives the minimum variance.
4. If $\alpha = 0$, plim $n^1\Sigma y_{t-h}y_{t-1} = 0$ for $h \geq 3$, since the covariances of an MA(1) process are zero from the second order onward. Hence, by Slutsky's theorem, $\hat{\alpha}_h$ is not consistent for α. However, if $h = 2$, the previous plim is $\theta\sigma^2$ and the numerator in (2) tends to zero, thus $\hat{\alpha}_2$ is consistent. To compute the asymptotic distribution of $\hat{\alpha}_2$ when $\alpha = 0$, we use (3) obtaining

   $$n^{1/2}\hat{\alpha}_2 \sim N\left[0, \frac{(1 + \theta^2)^2 + 2\theta^2}{\theta^2}\right]. \tag{4}$$

   When $\alpha = \theta = 0$, it is clear from (4) that the limiting distribution for $\hat{\alpha}_2$ does not have moments. In fact, $\hat{\alpha}_2 = n^{-1/2}\Sigma e_t e_{t-2}/n^{-1/2}\Sigma e_{t-1}e_{t-2}$ which tends asymptotically to the ratio of two independent $N(0,1)$ variates. Therefore, the asymptotic distribution is Cauchy (see exercise 6.24 in Brockwell and Davies [2]).
5. Within the general class of ARMA(p,q) processes, the results can be generalized as follows:

   The set $(y_{t-q-1}, \ldots, y_{t-q-p})$ is the optimal instrument set within the class of instruments where its number is equal to the number of to regressors (p). For a number of instruments larger than p, Sargan's [3] optimal GIVE method applies.

NOTE

A very good solution has been proposed by Sastri G. Pentula, the poser of the problem. For the generalization of the result, he has suggested the following paper

Stoica, P., T. Soderstrom, & B. Friedlander. Optimal instrumental variable estimates of the AR parameters of an ARMA process. *IEEE Transactions on Automatic Control, AC-30* 11 (1985): 1066–1074

and the references contained therein.

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89.1.3. *The Singular-Value Decomposition of the First-Order Difference Matrix*—Solution, proposed by N.G. Shepard. The singular-value decomposition (c.f. theorem 16, chapter 1 of Magnus and Neudecker [2]) implies

$D' = T\Lambda^{1/2}S'$, where $S'S = I_{n-1}$ and $T'T = I_{n-1}$.

Therefore

$$A = DD' = S\Lambda S'$$

$$B = D'D = T\Lambda T'$$

where Λ is an $(n-1)$th order positive diagonal matrix. S is an $n \times (n-1)$ matrix of eigenvectors of A corresponding to the $(n-1)$ nonzero eigenvalues. It is given by (c.f. Von Neumann [3] or Anderson [1, pp. 285–288])

$$S = \sqrt{\frac{2}{n}} \cdot \begin{bmatrix} \cos\frac{\pi}{2n} & \cos\frac{2\pi}{2n} & \cdots & \cos\frac{(n-1)\pi}{2n} \\ \cos\frac{3\pi}{2n} & \cos\frac{3\cdot 2\pi}{2n} & \cdots & \cos\frac{3(n-1)\pi}{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \cos\frac{(2n-1)\pi}{2n} & \cos\frac{(2n-1)2\pi}{2n} & \cdots & \cos\frac{(2n-1)(n-1)\pi}{2n} \end{bmatrix}$$