PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 8, Pages 2309–2317
S 0002-9939(00)05535-0
Article electronically published on February 25, 2000

THE SOBOLEV-TYPE MOMENT PROBLEM

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(Communicated by David R. Larson)

Abstract. We propose necessary and sufficient conditions for a bisequence of complex numbers to be a moment one of Sobolev type over the real line, the unit circle and the complex plane. We achieve this through converting the moment problem in question into a matrix one and utilizing some techniques coming from operator theory. This allows us to consider the Sobolev type moment problem in its full generality, not necessarily in the diagonal case and even of infinite order.

Orthogonality of polynomials with respect to a Sobolev inner product has been studied extensively (see [4] for a kind of overview and also [8] for a slightly more updated one though with more emphasis on coherent pair approach) for last ten years or so. It started with the somehow simpler, diagonal case and then, while developing, gradually has involved more complicated ones which take into consideration off-diagonal ingredients of the inner product as well. All this has been done for concrete families of polynomials; more general questions as well about this kind of orthogonality have been addressed. Here we would like to mention a study of Sobolev orthogonality from the numerical point of view [3], asymptotic behaviour of orthogonal polynomials [7], differential properties [5] of weighted orthogonality, involving [6] higher order derivatives in the inner product in question and finally the recurrence relation approach [2], the latter with potentially broader applications. On the other hand, orthogonality of polynomials is closely related to moment problems and this is what has turned our interest towards the moment problem of Sobolev type.

The Sobolev type moment problem

Assuming $N \in \{1, 2, \dots\} \cup \{+\infty\}$ we introduce the following shorthand notation:

$$\mathbb{N}_N \overset{\mathrm{df}}{=} \begin{cases} \{0, 1, \dots, N\} & \text{if } N \text{ is finite,} \\ \mathbb{N} & \text{otherwise,} \end{cases}$$

Received by the editors September 10, 1998.

2000 *Mathematics Subject Classification.* Primary 44A99; Secondary 47B15, 47B20, 47B25.

This research carried out within the framework of scientific and technical cooperation between Spain and Poland was supported by the Ministry of Foreign Affairs of Spain and the Committee of Scientific Research (KBN) of Poland, grand 07/R98.

The work of Francisco Marcellán was also partially supported by Dirección General de Enseñanza Superior (DGES) of Spain, grant PB96-0120C03-01 and INTAS project INTAS-93-219Ext.

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and

$$\ell^2_N \overset{\mathrm{df}}{=} \ell^2(\mathbb{N}_N) = \begin{cases} \mathbb{C}^{N+1} & \text{if } N \text{ is finite,} \\ \ell^2 & \text{otherwise.} \end{cases}$$

Moreover, denote by $\mathcal{F}_N$ the linear space of all sequences $\xi = \{\xi_n\}_{n=0}^N \in \ell^2_N$ such that $\xi_n = 0$ except for a finite number of n's. If $N < +\infty$, $\mathcal{F}_N$ is apparently equal to $\mathbb{C}^{N+1}$.

In what follows Σ stands for one of the sets $\mathbb{R}$, $\mathbb{T}$ or $\mathbb{C}$.

Given a (bi)sequence $\{s_{m,n}\}_{m,n=0}^\infty$, we say it is a *Sobolev type moment sequence of order N on Σ*, if

1° there is a collection $\{\mu_{i,j}\}_{i,j=0}^N$ of complex Borel measures on Σ such that

$$s_{m,n} = \sum_{i,j=0}^N \int_{\mathbb{R}} (x^m)^{(i)} (x^n)^{(j)} \mu_{i,j}(\mathrm{d}x), \quad m,n = 0,1,\ldots, \tag{1}$$

if $\Sigma = \mathbb{R}$ or

$$s_{m,n} = \sum_{i,j=0}^N \int_{\Sigma} \frac{\mathrm{d}^i}{\mathrm{d}z^i} z^m \frac{\mathrm{d}^j}{\mathrm{d}\bar{z}^j} \bar{z}^n \mu_{i,j}(\mathrm{d}x\,\mathrm{d}y), \quad z = x + \mathrm{i}\,y, \quad m,n = 0,1,\ldots, \tag{2}$$

if Σ is either $\mathbb{T}$ or $\mathbb{C}$;

2° for any Borel set $\sigma \subset \Sigma$ the matrix $(\mu_{i,j}(\sigma))_{i,j=0}^N$ is a positive operator in ℓ^2_N in the sense that

$$\sum_{i,j=0}^N \mu_{i,j}(\sigma)\xi_i\bar{\xi}_j \geq 0, \quad \{\xi_i\}_{i=0}^N \in \mathcal{F}_N. \tag{3}$$

Notice that even if $N = +\infty$ the sums in (1) and (2) always reduce to *finite* ones.

Condition (3) means precisely that, if $N < +\infty$, the scalar matrix $(\mu_{i,j}(\sigma))_{i,j=0}^N$ is positive definite in the usual matrix sense and, if $N = +\infty$, all the finite principal matrices of the infinite matrix $(\mu_{i,j}(\sigma))_{i,j=0}^\infty$ are positive definite scalar matrices. Consequently, in any case,

$$\mu_{i,i}(\sigma) \geq 0,\ \mu_{i,j}(\sigma) = \overline{\mu_{j,i}(\sigma)} \text{ for } i,j \in \mathbb{N}_N,\ \sigma \text{ a Borel subset of } \Sigma. \tag{4}$$

Moreover, by the Cauchy-Schwarz inequality,

$$|\mu_{i,j}(\sigma)| \leq \sqrt{\mu_{i,i}(\sigma)}\sqrt{\mu_{j,j}(\sigma)}. \tag{5}$$

This implies that if $\{\sigma_k\}_k$ is any finite Borel partition of σ, then (again the Cauchy-Schwarz inequality)

$$\begin{aligned} \sum_k |\mu_{i,j}(\sigma_k)| \leq \sum_k \sqrt{\mu_{i,i}(\sigma_k)}\sqrt{\mu_{j,j}(\sigma_k)} &\leq \sqrt{\sum_k \mu_{i,i}(\sigma_k)}\sqrt{\sum_k \mu_{j,j}(\sigma_k)} \\ &= \sqrt{\mu_{i,i}(\sigma)}\sqrt{\mu_{j,j}(\sigma)}. \end{aligned}$$

This gives

$$|\mu_{i,j}(\sigma)| \leq |\mu_{i,j}|(\sigma) \leq \sqrt{\mu_{i,i}(\sigma)}\sqrt{\mu_{j,j}(\sigma)} \text{ for } i,j \in \mathbb{N}_N,\ \sigma \text{ a Borel subset of } \Sigma, \tag{6}$$

where $|\mu_{i,j}|$ stands for the variation measure of $\mu_{i,j}$. Inequality (6) forces that

$$\mu_{i,j}(\sigma) = 0 \text{ if } \mu_{i,i}(\sigma) = 0, \text{ for } i,j \in \mathbb{N}_N,\ \sigma \text{ a Borel subset of } \Sigma \tag{7}$$

and

$$\operatorname{supp}|\mu_{i,j}| \subset \operatorname{supp}\mu_{i,i} \cap \operatorname{supp}\mu_{j,j} \text{ for } i,j \in \mathbb{N}_N. \tag{8}$$

Remark 1. If there is a positive measure ν such that every $\mu_{i,i}$ is absolutely continuous with respect to it (which always happens when N is finite; then one may take simply $\nu \stackrel{\text{df}}{=} \sum_{i=0}^{N} \mu_{i,i}$), then, by (7), there is an $(N+1)\times(N+1)$-matrix $(h_{i,j})_{i,j=0}^{N}$ of $\mathcal{L}^1(\nu)$ functions such that $\mu_{i,j} = h_{i,j}\nu$, $i,j \in \mathbb{N}_N$.

Remark 2. Condition 2° guarantees that the scalar (inner) product defined by $\{s_{m,n}\}_{m,n=0}^{\infty}$ is positive definite. More precisely, in case $\Sigma = \mathbb{R}$, if one sets

$$\langle p, q\rangle \stackrel{\text{df}}{=} \sum_{k,m} p_k q_m s_{k,m},$$

where $p = \sum_k p_k X^k$ and $q = \sum_m q_m X^m$ are in $\mathbb{C}[X]$, then, because

$$\langle p, q\rangle = \sum_{i,j=0}^{N} \int_{\mathbb{R}} p^{(i)} \bar{q}^{(j)} \mu_{i,j}(\mathrm{d}x),$$

the inner product so defined on $\mathbb{C}[X]$ is apparently positive definite.

When Σ is either $\mathbb{T}$ or $\mathbb{C}$ we can define the inner product for polynomials in $\mathbb{C}[Z,\bar{Z}]$ by extending it linearly from

$$\langle Z^k \bar{Z}^l, Z^m \bar{Z}^n\rangle \stackrel{\text{df}}{=} s_{k+n,l+m}, \quad k,l,m,n = 0,1,\ldots.$$

The tool

Given a sequence of $(N+1)\times(N+1)$-matrices $(a_{m,n}^{(i,j)})_{i,j=0}^{N}$, $m,n = 0,1,\ldots$, we say it is a $(N+1)\times(N+1)$-*matrix moment sequence over* Σ if

1° there exists a matrix $(\mu_{i,j})_{i,j=0}^{N}$ of complex measures on Σ such that

$$a_{m,n}^{i,j} = \int_{\mathbb{R}} x^{m+n} \mu_{i,j}(\mathrm{d}x), \quad i,j = 0,1,\ldots,N, \quad m,n = 0,1,\ldots, \tag{9}$$

if $\Sigma = \mathbb{R}$, or

$$a_{m,n}^{i,j} = \int_{\Sigma} z^m \bar{z}^n \mu_{i,j}(\mathrm{d}x\,\mathrm{d}y), \quad i,j = 0,1,\ldots,N, \quad m,n = 0,1,\ldots, \tag{10}$$

if Σ is either $\mathbb{T}$ or $\mathbb{C}$;

2° for any Borel set $\sigma \subset \Sigma$ the matrix $(\mu_{i,j}(\sigma))_{i,j=0}^{N}$ is a positive operator in ℓ_N^2 in the sense that

$$\sum_{i,j}^{N} \mu_{i,j}(\sigma)\xi_i\bar{\xi}_j \geq 0, \quad \{\xi_n\}_{n=0}^{N} \in \mathcal{F}_N. \tag{11}$$

Remark 3. Because of (5), if $(a_{m,n}^{i,j})_{i,j=0}^{N}$, $m,n = 0,1,\ldots$, is an $(N+1)\times(N+1)$-matrix moment sequence, then

$$|a_{0,0}^{i,j}| \leq \sqrt{a_{0,0}^{i,i}}\sqrt{a_{0,0}^{j,j}} \quad \text{for} \quad i,j \in \mathbb{N}_N.$$

Our basic observation is in the following

Proposition 4. *$\{s_{m,n}\}_{m,n=0}^{\infty}$ is a Sobolev type moment sequence of order N on Σ if and only if there exists an $(N+1)\times(N+1)$-matrix moment sequence $(a_{m,n}^{i,j})_{i,j=0}^{N}$, $m,n=0,1,\ldots,$ on Σ such that, after redefining,*

$$b_{m,n}^{i,j} \stackrel{\mathrm{df}}{=} a_{m-i,n-j}^{i,j} \text{ for } m \geq i,\ n \geq j, \tag{12}$$

one has the following decomposition[1]*:*

$$s_{m,n} = \sum_{i,j=0}^{N} i\,!j\,!\binom{m}{i}\binom{n}{j} b_{m,n}^{i,j} \text{ for } m,n=0,1,\ldots. \tag{13}$$

Proof. Suppose $s_{m,n}$ is of the form (1). Then

$$s_{m,n} = \sum_{i,j=0}^{N} i\,!j\,!\binom{m}{i}\binom{n}{j}\int_{\mathbb{R}} x^{m-i}x^{n-j}\mu_{i,j}(\mathrm{d}x).$$

Taking

$$b_{k,l}^{i,j} \stackrel{\mathrm{df}}{=} \begin{cases} \int_{\mathbb{R}} x^{k-i}x^{l-j}\mu_{i,j}(\mathrm{d}x), & k \geq i \text{ and } l \geq j, \\ \text{whatsoever} & k < i \text{ or } l < j, \end{cases}$$

we get an $(N+1)\times(N+1)$-matrix moment sequence $(a_{m,n}^{(i,j)})_{i,j=0}^{N}$, $m,n=0,1,\ldots,$ after defining $a_{m,n}^{i,j} \stackrel{\mathrm{df}}{=} b_{m+i,n+j}^{i,j}$, $m,n \geq 0$.

Conversely, suppose $s_{m,n}$ is of the form (13) with appropriate $b_{m,n}^{i,j}$ and $a_{m,n}^{i,j}$. Then

$$\begin{aligned} s_{m,n} &= \sum_{i,j=0}^{N} i\,!j\,!\binom{m}{i}\binom{n}{j} b_{m,n}^{i,j} = \sum_{i,j=0}^{N} i\,!j\,!\binom{m}{i}\binom{n}{j}\int_{\mathbb{R}} x^{m-i}x^{n-j}\mu_{i,j}(\mathrm{d}x) \\ &= \sum_{i,j=0}^{N}\int_{\mathbb{R}} (x^m)^{(i)}(x^n)^{(j)}\mu_{i,j}(\mathrm{d}x). \end{aligned}$$

This means that $\{s_{m,n}\}_{m,n=0}^{\infty}$ is a Sobolev type moment sequence of order N on Σ.

The cases $\Sigma=\mathbb{C}$ and $\Sigma=\mathbb{T}$ fit in with the same formal pattern. □

The real line case

Here $\Sigma=\mathbb{R}$. If the sequences $\{\xi_i\}_{i=0}^{N}, \{\eta_k\}_{k=0}^{N}$ and $\{\xi_j^{(0)}\}_{i=0}^{N},\ldots,\{\xi_i^{(p)}\}_{i=0}^{N}$ are in $\mathcal{F}_N$, we set for short

$$\xi \stackrel{\mathrm{df}}{=} \{\xi_i\}_{i=0}^{N},\ \eta \stackrel{\mathrm{df}}{=} \{\eta_k\}_{k=0}^{N},\ \xi^{(m)} \stackrel{\mathrm{df}}{=} \{\xi_i^{(m)}\}_{i=0}^{N},\ m=0,1,\ldots. \tag{14}$$

In what follows $\{\varepsilon^{(i)}\}_{i=0}^{N}$ stands for the canonical zero-one basis in ℓ_N^2 defined as $\varepsilon^{(i)} \stackrel{\mathrm{df}}{=} \{\delta_{i,j}\}_{i,k=0}^{N}$.

We have the following

Theorem 5. *A sequence $\{s_{m,n}\}_{m,n=0}^{\infty}$ is a Sobolev type moment sequence of order N on $\mathbb{R}$ if and only if there is a sequence $(a_{m,n}^{(i,j)})_{i,j=0}^{N}$, $m,n=0,1,\ldots$ of $(N+1)\times(N+1)$-matrices satisfying* (12) *and* (13) *and such that*

$$a_{m+k,n}^{i,j} = a_{m,n+k}^{i,j} \text{ for } i,j \in \mathbb{N}_N,\ m,n,k=0,1,\ldots, \tag{15}$$

[1] Always $\binom{k}{l} \stackrel{\mathrm{df}}{=} 0$ if $l > k$.

and

$$\sum_{m,n=0}^{p}\sum_{i,j=0}^{N} a_{m,n}^{i,j}\xi_i^{(m)}\overline{\xi_j^{(n)}} \geq 0,\ \xi^{(0)},\dots,\xi^{(p)} \in \mathcal{F}_N,\ p=0,1,\dots. \tag{16}$$

Proof. Set $a_m^{i,j} \stackrel{\mathrm{df}}{=} a_{m,0}^{i,j}$, $m=0,1\dots$, $i,j\in\mathbb{N}_N$. Then, because $a_{m+n}^{i,j}=a_{m,n}^{i,j}$, we check directly that

$$\mathbb{N}\times\mathcal{F}_N\times\mathcal{F}_N \ni (m,\xi,\eta)\mapsto \sum_{i,k=0}^{N} a_m^{i,k}\xi_i\bar{\eta}_k \in \mathbb{C}$$

is a positive definite form over the involution semigroup $\mathbb{N}$ (with involution $m^* \stackrel{\mathrm{df}}{=} m$, $m\in\mathbb{N}$) in the sense of [10]. It can be deduced from [10][2] that there are a Hilbert space $\mathcal{H}$ containing ℓ_N^2 and a selfadjoint operator A in $\mathcal{H}$ such that the domain $\mathcal{D}(A)$ of A contains $\mathcal{F}_N$ and such that

$$\sum_{i,k=0}^{N} a_m^{i,k}\xi_i\bar{\eta}_k = \langle A^m V\xi, V\eta\rangle,\ \xi,\eta\in\mathcal{F}_N,\ m\in\mathbb{N},$$

where $V:\mathcal{F}_N\mapsto\mathcal{H}$ is a linear operator. Let E be the spectral measure of A. Set

$$\mu_{i,j}(\sigma)\stackrel{\mathrm{df}}{=}\langle E(\sigma)V\varepsilon^{(i)}, V\varepsilon^{(j)}\rangle,\ i,j\in\mathbb{N}_N,\ \sigma \text{ is a Borel set in } \Sigma. \tag{17}$$

It is easy to check that for any σ the matrix $(\mu_{i,j})_{i,j=0}^N$ is positive definite in the sense of (11) which completes the proof. □

The unit circle case

Here $\Sigma=\mathbb{T}$ and this seems to be the simplest case. Keeping notation of (14) we have

Theorem 6. *A sequence $\{s_{m,n}\}_{m,n=0}^\infty$ is a Sobolev type moment sequence of order N on $\mathbb{T}$ if and only if there is a sequence $(a_{m,n}^{i,j})_{i,j=0}^N$, $m,n=0,1,\dots$ of $(N+1)\times(N+1)$-matrices satisfying* (12) *and* (13) *and such that*

$$a_{m+k,n+k}^{i,j}=a_{m,n}^{i,j} \text{ for } i,j\in\mathbb{N}_N,\ m,n,k=0,1,\dots, \tag{18}$$

and

$$\sum_{i,j=0}^{N}\Big(\sum_{\substack{m,n=0\\ m\geq n}}^{p} a_{m,n}^{i,j}+\sum_{\substack{m,n=0\\ m<n}}^{p} a_{m,n}^{i,j}\Big)\xi_i^{(m)}\overline{\xi_j^{(n)}}\geq 0,\ \xi^{(0)},\dots,\xi^{(p)}\in\mathcal{F}_N,\qquad p=0,1,\dots. \tag{19}$$

Proof. Set $t_m^{i,j}\stackrel{\mathrm{df}}{=}a_{m,0}^{i,j}$ if $m\geq 0$ and $t_m^{i,j}\stackrel{\mathrm{df}}{=}\overline{a_{0,m}^{i,j}}$ if $m<0$, $i,j=0,1,\dots$. Then, by (18) and (19), the function

$$\mathbb{Z}\times\mathcal{F}_N\times\mathcal{F}_N\ni(m,\xi,\eta)\mapsto\sum_{i,k=0}^{N} t_m^{i,k}\xi_i\bar{\eta}_k\in\mathbb{C}$$

[2] When N is finite, the form in question is represented by bounded operators; the relevant result can be found in [11] , provided the form is isometric.

is a positive definite form over the group $\mathbb{Z}$ in the sense of [10]. Using the technique of [10][3] we get a Hilbert space $\mathcal{H}$ contaning ℓ^2_N and a unitary operator U in $\mathcal{H}$ such that

$$\sum_{i,k=0}^{N} t_m^{i,k}\xi_i\bar{\eta}_k = \langle U^m V\xi, V\eta\rangle,\ \xi,\eta\in\mathcal{F}_N,\ m\in\mathbb{Z},$$

where $V:\mathcal{F}_N\mapsto\mathcal{H}$ is a linear operator. Let E be the spectral measure of U. As before for any σ the matrix $(\mu_{i,j})_{i,j=0}^N$ defined by (17) is positive definite in the sense of (11) which completes the proof. □

The complex plane case

Here $\Sigma=\mathbb{C}$. This is the most involved case. Fortunately very recent results, though they are pretty complicated both to state and to prove, help us to settle this case as well; unfortunately, they are the only available results so far. In fact we have two results of different nature. In order to state the first one we need some specific terminology (introduced in [10]) which we adapt to our circumstances.

Let $\mathcal{S}$ be a subset of $\mathbb{Z}\times\mathbb{Z}$ which contains $(0,0)$ and which is closed under the following operations: if $(m,n),(p,q)\in\mathcal{S}$, then $(m+n,p+q)$ and (n,m) are both in $\mathcal{S}$. Call a mapping $\omega:\mathcal{S}\times\mathcal{F}_N\times\mathcal{F}_N\mapsto\mathbb{C}$ a *form over* $(\mathcal{S},\mathcal{F}_N)$ if for any $(m,n)\in\mathcal{S}$ the mapping $\omega(m,n,\cdot,-)$ is a sesquilinear form on $\mathcal{F}_N$. The form ω is said to be *positive definite* if for any finite set $\{\xi^{m,n}\}_{m,n=0}^{\alpha}\subset\mathcal{F}_N$

$$\sum_{m,n,p,q=0}^{\alpha}\omega((m+q,n+p),\xi^{(p,q)},\xi^{(p,q)})\geq 0.$$

We are going to deal with two particular sets

$$\mathcal{S}:\mathcal{N}\stackrel{\mathrm{df}}{=}\mathbb{N}\times\mathbb{N}\quad\text{and}\quad\mathcal{N}_+\stackrel{\mathrm{df}}{=}\{(m,n)\in\mathbb{Z}\times\mathbb{Z};\ m+n\geq 0\}.$$

Given a sequence $(a_{m,n}^{i,j})_{i,j=0}^N$, $m,n=0,1,\dots$ of $(N+1)\times(N+1)$-matrices, define a form ω over $(\mathcal{N},\mathcal{F}_N)$ as follows:

$$\omega((m,n),\xi,\eta)\stackrel{\mathrm{df}}{=}\sum_{i,j=0}^{N}a_{m,n}^{i,j}\xi_i\bar{\eta}_j,\ \xi,\eta\in\mathcal{F}_N. \tag{20}$$

On the other hand, if ω_+ is a form over $(\mathcal{N}_+,\mathcal{F}_N)$, then positive definiteness of ω_+ can be restated as

(21)

$$\sum_{\substack{m+n\geq 0\\ p+q\geq 0}}\sum_{i,j=0}^{N}\omega_+(m+q,n+p,\varepsilon^{(i)},\varepsilon^{(j)})\xi_i^{(m,n)}\overline{\xi_j^{(p,q)}}\geq 0,\ \{\xi^{(m,n)}\}_{m+n\geq 0}\subset\mathcal{F}_N.$$

[3]Again, in the case of N finite the operators involved therein are bounded and an application of [11] can yield the result provided the form is isometric.

Our first result in this direction is as follows:

Theorem 7. *A sequence* $\{s_{m,n}\}_{m,n=0}^{\infty}$ *is a Sobolev type moment sequence of order* N *on* $\mathbb{C}$ *if and only if there is a sequence* $(a_{m,n}^{i,j})_{i,j=0}^{N}$, $m,n=0,1,\ldots,$ *of* $(N+1)\times(N+1)$*-matrices satisfying* (12) *and* (13) *for which there is a positive definite form* ω_+ *over* $(\mathcal{N}_+,\mathcal{F}_N)$ *such that*

$$\omega_+((m,n),\xi,\eta)=\sum_{i,j=0}^{N} a_{m,n}^{i,j}\xi_i\bar{\eta}_j,\ (m,n)\in\mathcal{N},\ \xi,\eta\in\mathcal{F}_N.$$

Proof. According to Theorem 19 of [9] existence of the extending form ω_+ is equivalent to existence of a semispectral measure F in ℓ_N^2 such that

$$\sum_{i,j=0}^{N} a_{m,n}^{i,j}\xi_i\bar{\xi}_j=\int_{\mathbb{C}} z^m\bar{z}^n\langle F(\mathrm{d}z)V\xi,V\eta\rangle,\ \xi,\eta\in\mathcal{F}_N,$$

where V is a linear operator from $\mathcal{F}_N$ to $\mathcal{H}$. We get the required measure simply by setting

$$\mu_{i,j}(\sigma)\stackrel{\mathrm{df}}{=}\langle F(\sigma)\varepsilon^{(i)},\varepsilon^{(j)}\rangle,\ i,j\in\mathbb{N}_N,\ \sigma \text{ is a Borel set in } \Sigma. \tag{22}$$

The next result looks much more complicated; nevertheless, in contrast to the previous one, it is, like Theorems 5 and 6, a kind of test for a given sequence to be a Sobolev-type moment one. ☐

Theorem 8. *A sequence* $\{s_{m,n}\}_{m,n=0}^{\infty}$ *is a Sobolev type moment sequence of order* N *on* $\mathbb{C}$ *if and only if there is a sequence* $(a_{m,n}^{i,j})_{i,j=0}^{N}$, $m,n=0,1,\ldots,$ *of* $(N+1)\times(N+1)$*-matrices satisfying* (12) *and* (13) *and such that the following condition holds:*

- $(\bullet)$ *for all positive integers* $N\geq M$, *for all finite sequences* $\{\lambda_{m,n}^{(1)}\}_{m+n\geq 0},\ldots,\{\lambda_{m,n}^{(M)}\}_{m+n\geq 0}$ *in* $\mathbb{C}^N$ *such that*

$$\sum_{\substack{m+n,p+q\geq 0\\ m+q=i,n+p=j}}\langle\lambda_{m,n}^{(k)},\lambda_{p,q}^{(l)}\rangle_{\mathbb{C}^N}=0,\quad i+j\geq 0,\ i\cdot j<0,\ \ k,l=1,\ldots,M,$$

and for every sequence $\{e_j\}_{j=1}^{M}$ *of orthonormal vectors in* $\mathcal{F}_N$, *the following inequality holds:*

$$\sum_{k,l=1}^{M}\ \sum_{\substack{m+n,p+q\geq 0\\ m+q,n+p\geq 0}}\ \sum_{i,j=0}^{N} a_{m+q,n+p}^{i,j}\xi_i^{(k)}\overline{\xi_j^{(l)}}\,\langle\lambda_{m,n}^{(k)},\lambda_{p,q}^{(l)}\rangle_{\mathbb{C}^N}\geq 0.$$

Proof. Since, due to Theorem 28 of [9], condition $(\bullet)$ is equivalent to the existence of a semispectral measure F in ℓ_N^2 such that

$$\sum_{i,j=0}^{N} a_{m,n}^{i,j}\xi_i\bar{\xi}_j=\int_{\mathbb{C}} z^m\bar{z}^n\langle F(\mathrm{d}z)\xi,\eta\rangle,\ \xi,\eta\in\mathcal{F}_N.$$

Again define the required measure as in (22). ☐

Some comments

The problem which has been left somehow apart is whether, given a sequence $\{s_{m,n}\}_{m,n=0}^{\infty}$, the equations (13) can be solved in $b_{m,n}^{i,j}$, $m,n=0,1,\ldots,\ i,j\in\mathbb{N}_N$. A little bit more precisely, the questions are:

(α) can (13) be solved at all?
(β) can (13) be solved so as to get a candidate for an appropriate matrix moment sequence?
(γ) how many solutions may one get?

Let us contribute to the discussion. First a definition: call a solution of (12) and (13) in $a_{m,n}^{i,j}$, $m,n=0,1,\ldots,\ i,j\in\mathbb{N}_N$, satisfying (15) and (16) in case $\Sigma=\mathbb{R}$, (18) and (19) in case $\Sigma=\mathbb{T}$ or ($\bullet$) in case $\Sigma=\mathbb{C}$ *diagonal* of order N on σ if it is of the form $a_{m,n}^{i,j}=\delta_{i,j}a_{m,n}^{i}$. Searching diagonal solutions is certainly easier (as well as it is more comfortable to deal with) from one hand; from the other it seems to be more restrictive.

Suppose now that we are looking for a diagonal solution of order 1 in case $\Sigma=\mathbb{R}$. Because in this case for a sequence $(a_{m,n}^{i,j})_{i,j=0}^{N}$ of $(N+1)\times(N+1)$-matrices to be a matrix moment one on $\mathbb{R}$ it is necessary that the entries be of the form $a_{m+n}^{i}\overset{\mathrm{df}}{=}a_{m,n}^{i}$. Then the very first of the equations (13) take the form

$$s_{0,0}=a_0^0,\ s_{1,0}=s_{1,0}=a_1^0,\ s_{1,1}=a_2^0+a_0^1,$$
$$s_{2,1}=s_{1,2}=a_3^0+2a_1^1,\ s_{2,2}=a_4^0+4a_1^2+4a_0^2,\ \ldots\ .$$

Because of (15), it forces $s_{1,0}=s_{0,1}$ and starts a recurrence procedure to determine $a_{m,n}^{i,j}$'s. Then the only thing which remains to be checked is if the so-obtained sequence $\{a_{m,n}^{i,j}\}_{m,n=0}^{\infty}$ satisfies (16) and also how many solutions one may get in this way.[4] In general, referring to question (β) the point is how one can express conditions required for $\{a_{m,n}^{i,j}\}_{m,n=0}^{\infty}$ to be an appropriate matrix moment sequence in terms of the sequence $\{s_{m,n}\}_{m,n=0}^{\infty}$ itself.

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[4] We notify here that there is a parallel work [1] dealing with question (γ) in the diagonal case of finite order on the real line.

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