UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 02-21
Economics Series 05
May 2002

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INFORMATIONAL MATCHING *

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Abstract

This paper analyzes the problem of matching heterogenous agents in a Bayesian learning model. One agent gives a noisy signal to another agent, who is responsible for learning. If production has a strong informational component, a phase of cross-matching occurs, so that agents of low knowledge catch up with those of higher one. It is shown that

(i) a greater informational component in production makes cross-matching more likely;

(ii) as the new technology is mastered, production becomes relatively more physical and less informational;

(iii) a greater dispersion of the ability to learn and transfer information makes self-matching more likely; and

(iv) self-matching leads to more self-matching, whereas croos-matching can make less productive agents overtake more productive ones.

Keywords: Diffusion of Information, Bayesian Learning, Matching, Assortative Mating.

JEL Classification: C11, D83, J12, J24, J41.

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1 Introduction

This paper analyzes the problem of matching heterogenous agents in a Bayesian learning model. The informational, 'dial-setting', or 'input-target' production function has proven appropriate to model processes in which learning-by-doing plays an important role. Proposed by Prescott (1973) and Wilson (1975), it has been the subject of recent research by Jovanovic & Nyarko (1996) and has been applied to explain firm dynamics by Mitchell (2000) and labor market dynamics by Nagypál (2000). Jovanovic & Nyarko (1995) have also used it to model the transfer of human capital from older to younger agents. However, no one has tried to analyze the implications of matching agents in an information-theoretic framework. The matching problem proposed by Becker (1993) has been typically analyzed with standard symmetric production functions.

Over the recent years, increased sorting into homogenous groups such as firms, marriages, schools, or neighborhoods has been receiving a great deal of attention, both empirically and theoretically (e.g. by Kremer (1997) and Fernández & Rogerson (2000)). The puzzle in need of an explanation is why in the past people with different productivities used to work together (cross-matching), while now groups are formed with people of similar productivities (self-matching).

In this paper, production has a physical and an informational component. The physical component is modelled as a standard production function of two inputs with a positive cross-derivative. As was shown by Becker (1993), in this context, if each agent supplies one of the inputs, it is optimal to have positive assortative matching or self-matching.

The informational component captures agents' learning of the best way to produce, modelled as a process of Bayesian updating. Agents are ranked according to their productivities; however, their roles in production are different. While one agent, the engineer, collects noisy observations on an unknown parameter that characterizes the best way to produce, the other agent, the worker, produces and improves thereby her knowledge on that parameter. To match them means to combine the engineer's ability to collect accurate information with the worker's productivity as captured in her knowledge of the unknown parameter. It is shown that the informational component unambiguously induces negative assortative matching or cross-matching.

If a production function has a strong informational component, it is optimal to have a phase of cross-matching, so that workers of low knowledge catch up with those of higher knowledge. When information is spread across agents, that is, when learning is attained, the production component becomes relatively more important and self-matching becomes optimal. The switch in sorting optimality corresponds to production becoming relatively more physical and less informational over time.

An implication of this analysis is that economies that are in early stages of development would like to mix their people to produce learning and activate the growth process. Economies that master the production process and have exhausted their possibilities of learning are more oriented towards physical than informational matching. The same applies to the introduction of a new technology: the learning stage is associated with mixing agents of different productivities, but when the new technology is mastered, self-matching becomes optimal. In a way, a phase of cross-matching corresponds to a stage of social accumulation of knowledge, whereas a phase of self-matching is more agreeable with a stage where knowledge is applied to production.

A second implication is that increased differences within each type of agent favor mating of the likes. In other words, both increased dispersion of engineers' abilities to transfer accurate information and increased dispersion of workers' productivities have the effect of making self-matching more likely. This coincides with Kremer & Maskin (1996) finding that a wider dispersion of skills explains the predominance of self-matching. A further implication is that self-matching in one period implies self-matching in the next period, whereas cross-matching can make less productive workers not only catch up with, but also overtake more productive ones.

The remainder of the paper is organized as follows. The next section describes the model; Section 3 presents two examples: one with a purely informational production function and one with both informational and physical components; conclusions are presented in Section 4.

2 Model

This section discusses the effects of matching two heterogenous agents when production has an informational and a physical component. The purpose is to build intuition on the determinants of self-matching and cross-matching in an information-theoretic framework.

2.1 Agents

There are two types of agents:

> The worker. Her productivity is x $(0 < x < \infty)$. She applies her knowledge in production and experiences learning by doing, i.e., her productivity grows over time.
>
> The engineer. His productivity is e $(0 < e < \infty)$. He supplies the worker with a noisy signal on the best way to produce. The higher the productivity of the engineer, the more precise this signal. His productivity does not change.

There is a basic asymmetry between these two agents: the worker produces and accumulates knowledge on the production process; the engineer gathers new evidence on production outcomes and transfers this information to the workers. One can also think about these agents as a student (the worker), who is ranked by her productivity, and learns and becomes more productive over time, and a teacher (the engineer), who is ranked by his ability to bring accurate observations to the student, thereby training her and improving her productivity.

2.2 Production Function

The production function is

$$q = f(x, e) - \alpha (y - z)^2 .$$

The first term corresponds to the physical component of production as a function of the agents' match. This function is a standard physical production function with $f_x > 0$, $f_e > 0$, $f_{xe} > 0$. The second term corresponds to the informational component, a quadratic loss function in which y is an unknown stochastic "target" fluctuating around θ, z is the action determined by the worker, and $\alpha > 0$ denotes the importance of the informational component in the production function. The engineer gives a noisy information on θ to the worker:

$$y = \theta + \varepsilon,$$

where ε is a white noise with distribution $N(0, e^{-1})$, that is, better engineers give more precise signals on θ to the workers. The worker has a prior distribution of θ :

$$\theta \sim N\left(\widehat{\theta}, x^{-1}\right),$$

that is, better workers have more precise prior knowledge on θ.[1] The worker chooses z to

$$\min_z E\left[(\theta + \varepsilon - z)^2\right] .$$

[1] A planner would have no way to rank workers according to their knowledge of the mean of θ if he or she also ignores true θ. In consequence, workers' productivity is only contained in the precision of their knowledge of θ, not in the location of the mean of θ.

The optimal solution coming from the first order condition is $z^* = E\theta = \widehat{\theta}$. Hence, expected production is

$$Eq = f(x,e) - \alpha\left(Var(\theta) + Var(\varepsilon)\right) = f(x,e) - \alpha\left(x^{-1} + e^{-1}\right) \tag{1}$$

In this function workers' and engineers' qualities enter in the production function both through the physical and through the informational component.

2.3 Learning

The worker is responsible for learning from realized production. She updates her beliefs on the unknown parameter θ according to Bayes's rule:

$$\begin{aligned} \widehat{\theta}' &= \frac{x}{x+e}\widehat{\theta} + \frac{e}{x+e}y, \\ x' &= x+e, \end{aligned} \tag{2}$$

where the superscript denotes next period's variables. Knowledge is accumulated and fully appropriated by the worker, i.e., when the worker changes partner, knowledge goes with her. The engineer's quality does not change.

2.4 Matching

A planner matches two workers to two engineers for two periods.[2] Let the best engineer be indexed by 1, so $\Delta e = e_1 - e_2 > 0$ always. The initial difference in workers at period 1 can be positive or negative: $\Delta x = x_1 - x_2 \lesseqgtr 0$. This model is solved backwards by finding the optimal solution at period two and then using it to find the optimal solution at period one.

[2] A private solution with transferable utilities would give the same solution as the planner's solution. If utilities are non-transferable and there are no frictions, self-matching always occurs.

2.4.1 Period 2

The value function in the second period is the sum of the expected production of the two teams:

$$V'\left(x_1', x_2'\right) = \max_{j\in\{1,2\}} \left(Eq_{1j}' + Eq_{2,3-j}'\right).$$

Define the maximand as $U_j' = Eq_{1j} + Eq_{2,3-j}$ and the gain of assigning the best engineer to worker 1 (or of choosing $j = 1$) as $\Delta U' = U_1' - U_2'$. Then using (1), we obtain

$$\begin{aligned} \Delta U' &= f\left(x_{1,}'e_1\right) + f\left(x_{2,}'e_2\right) - f\left(x_{1,}'e_2\right) - f\left(x_{2,}'e_1\right), \\ &\simeq f_{xe}\Delta x'\Delta e. \end{aligned} \tag{3}$$

Different assignments of engineers to workers make a difference only in the physical component, not in the informational component. As shown by Becker, when $f_{xe} > 0$, self-matching occurs, that is, it is optimal to assign the best engineer to the best worker in the second period, but it may not be optimal in the first period. Consequently, the optimal solution in period 2 is the following:

$$j^* = \begin{cases} 1, \text{ if } \Delta x' > 0; \\ 2, \text{ if } \Delta x' < 0, \\ \text{indifferent if } \Delta x' = 0. \end{cases}$$

The sign of $\Delta x'$ depends on the solution at period 1, denoted by i^*. Suppose that at period 1 worker 1 is the best, $\Delta x > 0$, then from (2) we know that

$$\begin{aligned} \text{if } i^* &= 1, \text{ then } \Delta x' = \Delta x + \Delta e > 0 \text{ and } j^* = 1; \\ \text{if } i^* &= 2, \text{ then } \Delta x' = \Delta x - \Delta e \begin{cases} > 0 \text{ and } j^* = 1; \\ < 0 \text{ and } j^* = 2. \end{cases} \end{aligned}$$

When $\Delta x < 0$, the solution is similar, the only difference being that the index 2 represents the best worker and 1 the worst; hence we can express the general solution as

$$j^* \begin{cases} = 1, \text{ if } |\Delta x| > \Delta e \text{ and } \Delta x > 0; \\ = 2, \text{ if } |\Delta x| > \Delta e \text{ and } \Delta x < 0; \\ = i^*, \text{ if } |\Delta x| < \Delta e. \end{cases}$$

The initial difference in workers' productivities determines then the optimal solution in the second period.

2.4.2 Period 1

The value function in the first period is

$$V(x_1, x_2) = \max_{i \in \{1,2\}} \{Eq_{1i} + Eq_{2,3-i} + \beta V'(x_1 + e_i, x_2 + e_{3-i})\}.$$

where β is the subjective discount rate. The maximand in the first period is $U_i = Eq_{1i} + Eq_{2,3-i} + \beta V'(x_1 + e_i, x_2 + e_{3-i})$. Using (1) and (2) we determine the gain of choosing $i = 1$ when $\Delta x > 0$:

$$\begin{aligned} \Delta U &\simeq \begin{cases} \left[f_{xe} + \beta f'_{xe}\frac{\Delta e}{|\Delta x|} - \beta\alpha A\right]\Delta x \Delta e, \text{ if } |\Delta x| > \Delta e; \\ \left[f_{xe} + \beta f'_{xe} - \beta\alpha A\right]\Delta x \Delta e, \text{ if } \Delta x < \Delta e; \end{cases} \\ &\simeq \left(f_{xe} + \beta f'_{xe}\min\left[\frac{\Delta e}{|\Delta x|}, 1\right] - \beta\alpha A\right)\Delta x \Delta e \end{aligned} \tag{4}$$

where $A = \frac{e_1+x_1+x_2+e_2}{(x_1+e_2)(x_2+e_1)(x_1+e_1)(x_2+e_2)} > 0$. Note that the difference in the physical component is always positive, while the difference in the informational component is always negative. The reason is that the cross-derivative of the physical component is positive, while the cross-derivative of the informational component, resulting from the quadratic loss function and Bayes's rule, is negative. The optimal solution depends on the relative importance of these two components:

$$i^* \begin{cases} = 1, \text{ if } \alpha^* = \frac{1}{\beta A}\left(f_{xe} + \beta f'_{xe} \min\left[\frac{\Delta e}{|\Delta x|}, 1\right]\right) > \alpha \text{ and } \Delta x > 0; \\ = 2, \text{ otherwise.} \end{cases}$$

Cross-matching happens when the informational component in the production function is above a certain threshold. A low informational component, on the contrary, facilitates self-matching. Notice that this result does not follow only from the particular informational production function used, but from its interaction with Bayes's rule. In the second period the informational component is present and does not play any role in the optimal assignment of engineers to workers. It is only in the first period, when the planner takes into account the evolution of workers' productivities, that rewards to learning make cross-matching desirable.

3 Examples

To illustrate the optimal solution under different specifications, I will give two examples for the physical component of the production function. If $\alpha = 0$, there is always self-matching as in other physical matching models. When $\alpha \neq 0$, it is interesting to see what happens in the two possible cases:

(i) when the production function is purely informational; and

(ii) when the production function consists of an informational and a physical component.

3.1 Purely Informational Production Function

If the physical component is not sensitive to the match, say $f(x, e) = 1$, the optimal action is determined only by the informational component. We can also assume that

$\alpha = 1$ and $\beta = 1$. In period 2 the value function is

$$V'(x_1', x_2') = Eq_{11}' + Eq_{22}' = Eq_{12}' + Eq_{21}'.$$

As shown in Equation (3), because $f_{xe} = 0$, in period 2 there is indifference in matching. In period 1 matching will be negative assortative. From (4)

$$\begin{aligned} \Delta U &\simeq -A\Delta x \Delta e, \\ &\text{hence } j^* \begin{cases} = 2 \text{ if } \Delta x > 0, \\ = 1 \text{ if } \Delta x < 0. \end{cases} \end{aligned}$$

With a purely informational production function, it is optimal to mix workers in order to spread information and give less informed workers the opportunity to catch up with more informed ones. In other words, purely informational matching means cross-matching.

3.2 Informational and Physical Production Function

Suppose now that the physical component is given by $f(x, e) = xe$. This was the production function used by Becker (1993) for analyzing positive assortative matching and by Kremer (1993) for studying connections between labor markets and development. As this physical production function has $f_{xe} = 1$, when $\beta = 1$

$$\alpha^* = \frac{1}{A}\left(1 + \min\left[\frac{\Delta e}{\Delta x}, 1\right]\right), \forall \Delta x \geq 0$$

If α is above this threshold, there is cross-matching in the first period. As shown in subsection 2.4.1., in the second period there is always self-matching. It is important to note that this threshold depends on the dispersion of the productivities of workers and of engineers; the following proposition links these dispersions with the optimal solution.

Proposition 1 *(i) A wider dispersion of engineers' productivities makes self-matching more likely, that is,* $\frac{\partial \alpha^*}{\partial \Delta e} > 0$, $\forall \Delta e, \forall \Delta x \geq 0$*;*
(ii) a wider dispersion of workers' productivities makes self-matching more likely, that is, $\frac{\partial \alpha^*}{\partial \Delta x} > 0$, $\forall \Delta x \geq 0$, *if* $\Delta e > \sqrt{0.5}\,(x_2 + e_2)$*. Proof: In Appendix.*

The optimal solution depends on the initial dispersion of engineers' and workers' productivities. A greater dispersion of engineers' productivities Δe unambiguously increases α^* and makes thereby self-matching more likely. On the other hand, if the dispersion of engineers' productivities is greater than a given threshold, an increased dispersion of workers' productivities makes self-matching more likely. For both cases, more dispersion of productivities, or more inequality within each type of agent, implies more self-matching, as in Kremer & Maskin (1996).

If $\Delta e > \sqrt{0.5}\,(x_2 + e_2)$, let $\alpha_l = \alpha^*|_{\Delta x = 0}$ and $\alpha_h = \alpha^*|_{\Delta x = \Delta e}$, then, as illustrated in Figure 1, depending on Δx and α^*, there are three possible cases for a solution:

- I: $\alpha_l > \alpha$. There is always self-matching: the physical component of the production function is predominant.

- II: $\alpha_h > \alpha > \alpha_l$. There is self-matching for a high dispersion of workers' productivities, but cross-matching when dispersion is low. Cross-matching implies overtaking, that is, the worst worker in period 1 becomes the best in period 2.

- III: $\alpha > \alpha_h$. There is self-matching for a high dispersion of workers' productivities and cross-matching for a low dispersion of workers' productivities. If the dispersion of workers' productivities is lower than the dispersion of engineers' productivities, the worst worker overtakes the best one in the first period.

If $\Delta e \leq \sqrt{0.5}\,(x_2 + e_2)$, then α^* is not monotonically increasing in $\Delta x \geq 0$ and there are more than these three baseline cases. In this model, the dispersion of workers' productivities evolves according to the following rule:

$$\text{If } \Delta x > 0,\ \Delta x' \begin{cases} = \Delta x + \Delta e, \text{ if } \alpha^* > \alpha; \\ = \Delta x - \Delta e, \text{ otherwise;} \end{cases} \quad \text{if } \Delta x < 0, \Delta x' \begin{cases} = \Delta x - \Delta e, \text{ if } \alpha^* > \alpha; \\ = \Delta x + \Delta e, \text{ otherwise.} \end{cases}$$

Figure 2 graphs this evolution for the three cases analyzed above. It shows that self-matching tends to widen up the dispersion of workers' productivities, while cross-matching reduces this dispersion and may produce overtaking of the best worker by the worst one. The importance of the informational component in the production function will crucially determine which case happens.

4 Conclusion

This paper has shown that for a class of production functions where information is important, it is optimal to have a phase of cross-matching to increase aggregate knowledge in an economy. It is by investing in increasing and spreading information over all agents that knowledge is accumulated. In a second phase, as learning is attained, the physical aspect of production becomes more important and, consequently, self-matching is observed. One can establish three basic conclusions:

(i) a greater informational component in production makes cross-matching more likely;

(ii) as a new technology is mastered, production becomes relatively more physical and less informational;

(iii) a greater dispersion of engineers' and workers' productivities makes self-matching more likely;

(iv) self-matching leads to more self-matching; while cross-matching may lead to overtaking of the best worker by the worst one.

4.1 Appendix: Proof of Proposition 1.

(i) Let $X = x_2 + e_2$, then

$$\begin{aligned}\frac{\partial \alpha^*}{\partial \Delta e} &= \begin{cases} \frac{\partial(1/A)}{\partial \Delta e}\left(1+\frac{\Delta e}{\Delta x}\right)+\frac{1}{A\Delta x}, \text{ if } \Delta x \geq \Delta e; \\ 2\frac{\partial(1/A)}{\partial \Delta e}, \text{ if } \Delta x < \Delta e; \end{cases} \\ &> 0,\end{aligned}$$

because $A > 0$ and $\frac{\partial(1/A)}{\partial \Delta e} = \frac{1}{A}\frac{X(3(X+\Delta x)+4\Delta e)+(\Delta x+\Delta e)^2}{(2X+\Delta x+\Delta e)(X+\Delta x+\Delta e)(X+\Delta x)}$.■

(ii) Similarly,

$$\frac{\partial \alpha^*}{\partial \Delta x} = \begin{cases} \frac{\partial(1/A)}{\partial \Delta x}\left(1+\frac{\Delta e}{\Delta x}\right)-\frac{1}{A}\frac{\Delta e}{\Delta^2 x}, \text{ if } \Delta x \geq \Delta e; \\ 2\frac{\partial(1/A)}{\partial \Delta x}, \text{ if } \Delta x < \Delta e. \end{cases}$$

If $\Delta x < \Delta e$, clearly $\frac{\partial \alpha^*}{\partial \Delta x} > 0$. If $\Delta x \geq \Delta e$,

$$\begin{aligned}\frac{\partial \alpha^*}{\partial \Delta x} &= \frac{\Delta x^3 + 2\Delta x^2\left(\Delta e + 2X\right) + \Delta x\left(\Delta e^2 + 3X\Delta e + 3X^2\right)}{A\left(2X+\Delta x+\Delta e\right)\left(X+\Delta x+\Delta e\right)\left(X+\Delta x\right)\Delta x} \\ &\quad - \frac{2X\Delta e\left(X+\Delta e\right) + X\frac{\Delta e}{\Delta x}\left(\Delta e + X\right)\left(\Delta e + 2X\right)}{A\left(2X+\Delta x+\Delta e\right)\left(X+\Delta x+\Delta e\right)\left(X+\Delta x\right)\Delta x}.\end{aligned}$$

Notice that $\left.\frac{\partial \alpha^*}{\partial \Delta x}\right|_{\Delta x=\Delta e} = \frac{2\Delta e^2 - X^2}{A(X+2\Delta e)(X+\Delta e)\Delta e} > 0$, if $\Delta e > \sqrt{0.5}X$. This same condition implies that $\frac{\partial \alpha^*}{\partial \Delta x} > 0$, for $\Delta x \geq \Delta e$.■

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Figure 1: α^* as a function of Δx

$(e_2 + x_2 = 1,\ \Delta e = 0.8)$.

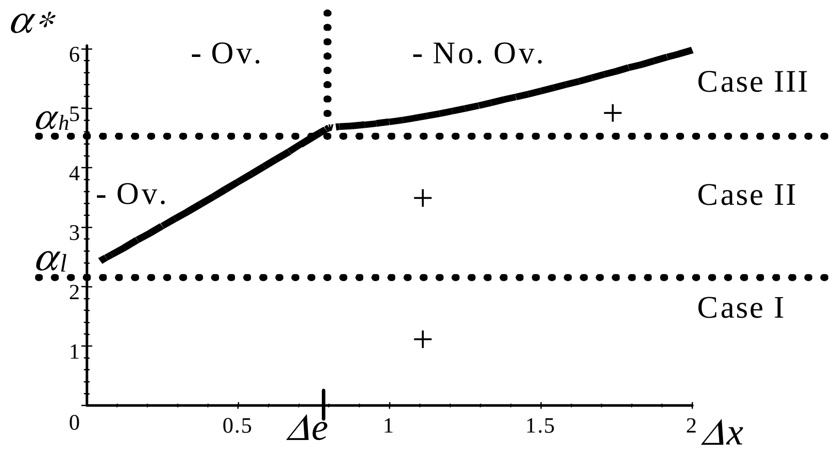
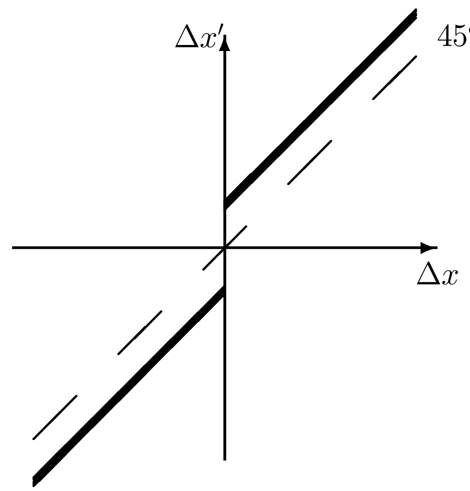


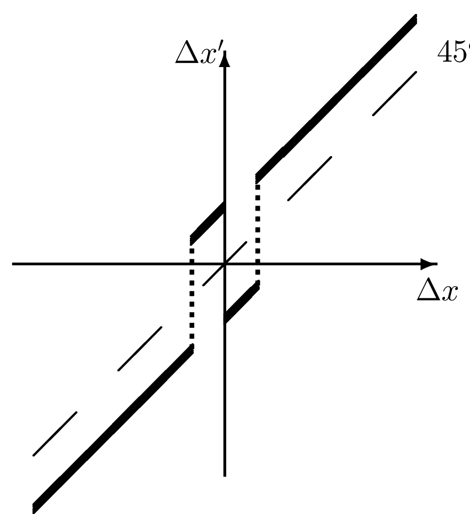
Figure 2: $\Delta x'$ as a function of Δx

Case I: $\alpha < \alpha_l$

Self-Matching always.



Case II: $\alpha_l < \alpha < \alpha_h$
Self-Matching if $\alpha^* > \alpha$
Cross-Matching with overtaking if $\alpha^* < \alpha$



Case III: $\alpha > \alpha_h$

Self-Matching if $\alpha^* > \alpha$
Cross-Matching if $\alpha^* < \alpha$:
No Overtaking $|\Delta x| > \Delta e$
Overtaking $|\Delta x| < \Delta e$