Option pricing with Lévy-Stable processes generated by Lévy-Stable integrated variance

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We show how to calculate European-style option prices when the log-stock price process follows a Lévy-Stable process with index parameter $1 \leq \alpha \leq 2$ and skewness parameter $-1 \leq \beta \leq 1$. Key to our result is to model integrated variance $\int_t^T \sigma_s^2 \, ds$ as an increasing Lévy-Stable process with continuous paths in T.

Keywords: Commodity markets; Commodity prices; Lévy process; Hedging techniques

1. Introduction

Up until the early 1990s most of the underlying stochastic processes used in the financial literature were based on Brownian motion, modelling in continuous time a large number of independent 'microscopic' price changes, with finite total variance; and Poisson processes, modelling occasional large changes. These two processes are the canonical models for continuous sample paths and those with a finite number of jumps, respectively. More generally, dropping the assumption of finite variance, the sum of many iid events always has, after appropriate scaling and shifting, a limiting distribution termed a Lévy-Stable law; this is the generalized version of the Central Limit Theorem (GCLT) (Samorodnitsky and Taqqu 1994), and the Gaussian distribution is one example. Based on this fundamental result, it is plausible to generalize the assumption of Gaussian price increments by modelling the formation of prices in the market by the sum of many stochastic events with a Lévy-Stable limiting distribution.

An important property of Lévy-Stable distributions is that of stability under addition: when two independent copies of a Lévy-Stable random variable are added then, up to scaling and shift, the resulting random variable is again Lévy-Stable with the same shape. This property is very desirable in models used in finance and particularly in portfolio analysis and risk management; see, for example, Fama (1971), Ziemba (1974) and the more recent work by Tokat and Schwartz (2002), Ortobelli *et al.* (2002) and Mittnik *et al.* (2002). Only for Lévy-Stable distributed returns do we have the property that linear combinations of different return series, for example portfolios, again have a Lévy-Stable distribution (Feller 1966).

Based on the GCLT we have, in general terms, two ways of modelling stock prices or stock returns. If it is believed that *stock returns* are at least approximately governed by a Lévy-Stable distribution the accumulation of the random events is additive. On the other hand, if it is believed that the *logarithm of stock prices* is approximately governed by a Lévy-Stable distribution then the accumulation is multiplicative. In the literature, most models have assumed that log-prices, instead of returns, follow a Lévy-Stable process. Mc-Culloch (1996) assumes that assets are log Lévy-Stable and prices options using a utility maximization argument; more recently, Carr and Wu (2003) priced European options when the log-stock price follows a maximally skewed Lévy-Stable process.

Finally, based on Mandelbrot (1997), Hurst *et al.* (1999) provide a model to price European options when returns follow a (symmetric) Lévy-Stable process. In their models the Brownian motion that drives the stochastic shocks to the stock process is subordinated to an intrinsic time process that represents 'operational time' on which the market operates. Option pricing can be done within the Black–Scholes framework and one can show that the subordinated Brownian motion is a symmetric Lévy-Stable motion.

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The motivation of this paper is as follows. It is standard to take as a starting point a model for the risk-neutral evolution of the asset price in the form

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma_t \mathrm{d}W_t^Q,$$

where W_t^Q is the underlying Brownian motion, r is the (constant) interest rate and σ_t is the volatility process; the case when σ_t is constant is the usual Black–Scholes (BS) model. It is then standard to specify a stochastic process for σ_t, resulting in one of a number of standard stochastic-volatility models.

When σ_t and W_t^Q are independent for all $0 \le t \le T$ (as is often approximately the case for FX markets), we have

$$S_T = S_t e^{r(T-t)-(1/2)\int_t^T \sigma_s^2 \mathrm{d}s + \int_t^T \sigma_s \mathrm{d}W_s^Q}, \tag{1}$$

and then the value of a European vanilla option written on the underlying stock price S_t is given by

$$V(S,t) = \mathbb{E}^Q\left[V_{\mathrm{BS}}\left(S_t, t, K, \left(\frac{1}{T-t}\int_t^T \sigma_s^2 \mathrm{d}s\right)^{1/2}, T\right)\right], \tag{2}$$

where the expected value is with respect to the random variable $\int_t^T \sigma_s^2 \mathrm{d}s$, the integrated variance, under the risk-neutral measure Q, and V_{BS} is the usual Black–Scholes value for a European option. In general, the distribution or characteristic function of the integrated variance is not known, so evaluating (2) is not straightforward, although given the characteristic function of the integrated variance we can use standard transform methods to evaluate $V(S,t)$ given by equation (2).

Notwithstanding these difficulties, the integrated variance is an important quantity, representing a measure of the total uncertainty in the evolution of the asset price, and we use it as the starting point for our model. We investigate the properties of a two-factor model in which the integrated variance follows a Lévy-Stable process, while the shocks to the stock process are conditionally Gaussian, i.e. Brownian motion, with a volatility consistent with the integrated variance process. We then show that the resulting distribution of the log-stock prices is Lévy-Stable. We also provide a characterization of the most general possible model within our class of integrated variance processes, which is an interesting result in its own right. In addition to pricing options when the integrated variance process and the stock process are independent (as above), we also show how to incorporate a 'leverage' effect, restoring a degree of 'correlation' between the two.

The paper is structured as follows. Section 2 presents definitions and properties of Lévy-Stable processes. In particular, we show how symmetric Lévy-Stable random variables may be 'built' as a combination of two independent Lévy-Stable random variables and define Lévy-Stable processes as in Samorodnitsky and Taqqu (1994). Section 3 discusses the path properties required to model integrated variance as a totally skewed to the right Lévy-Stable process. Section 4 describes the dynamics of the stock process under both the physical and risk-neutral measure and shows how option prices are calculated when the stock returns or log-stock process follows a Lévy-Stable process. Finally, section 5 shows numerical results and section 6 concludes.

2. Lévy-Stable random variables and processes

In this section we show how to obtain *any* symmetric Lévy-Stable process as a stochastic process whose innovations are the product of two independent Lévy-Stable random variables. The only conditions we require (stated precisely in proposition 2.2) are that one of the independent random variables is symmetric and the other is totally skewed to the right. This is a simple, yet very important, result since we can choose a Gaussian random variable as one of the building blocks together with any other totally skewed random variable to 'produce' symmetric Lévy-Stable random variables. Furthermore, choosing a Gaussian random variable as one of the building blocks of a symmetric random variable will be very convenient since we will be able to reformulate any symmetric Lévy-Stable process as a conditional Brownian motion, conditioned on the other building block, the totally skewed Lévy-Stable random variable, which in our case will be the model for integrated variance.

2.1. *Lévy-Stable random variables*

The characteristic function of a Lévy-Stable random variable X is given by

$$\begin{aligned}\log\mathbb{E}[\mathrm{e}^{i\theta X}] &\equiv \Psi(\theta)\\ &= \begin{cases} -\kappa^\alpha|\theta|^\alpha\{1 - i\beta \operatorname{sign}(\theta)\tan(\alpha\pi/2)\} + im\theta, & \text{for } \alpha \neq 1,\\ -\kappa|\theta|\{1 + (2i\beta/\pi)\operatorname{sign}(\theta)\log|\theta|\} + im\theta, & \text{for } \alpha = 1,\end{cases}\end{aligned} \tag{3}$$

where the parameter $\alpha \in (0,2]$ is known as the stability index, $\kappa > 0$ is a scaling parameter, $\beta \in [-1,1]$ is a skewness parameter and m is a location parameter (Samorodnitsky and Taqqu 1994). If the random variable X has a Lévy-Stable distribution with parameters α, κ, β and m we write $X \sim S_\alpha(\kappa, \beta, m)$.

It is straightforward to see that, for the case $0 < \alpha \le 1$, the random variable X does not have any moments, and for the case $1 < \alpha < 2$ only the first moment exists (the case $\alpha = 2$ is Gaussian); however, fractional moments $\mathbb{E}[|X|^p]$ do exist for $p < \alpha$ (Samorodnitsky and Taqqu 1994). Moreover, given the asymptotic behaviour of the tails of the distribution of a Lévy-Stable random variable it can be shown that the Laplace transform $\mathbb{E}[\mathrm{e}^{-\tau X}]$ of X exists only when its distribution is totally skewed to the right, that is $\beta = 1$, which we state in the following proposition which we use later.

Proposition 2.1: (*the Laplace transform; Samorodnitsky and Taqqu (1994)*) *The Laplace transform* $\mathbb{E}[e^{-\tau X}]$ *with* $\tau \ge 0$ *of the Lévy-Stable variable* $X \sim S_\alpha(\kappa, 1, 0)$ *with* $0 < \alpha \le 2$ *and scale parameter* $\kappa > 0$ *satisfies*

$$\log\mathbb{E}[e^{-\tau X}] = \begin{cases} -\kappa^\alpha\tau^\alpha \sec(\alpha\pi/2), & \text{for } \alpha \neq 1,\\ (2\kappa/\pi)\tau\log\tau, & \text{for } \alpha = 1.\end{cases} \tag{4}$$

The existence of the Laplace transform of a totally skewed to the right Lévy-Stable random variable will enable us to show how to price options as a weighted average of the classical Black–Scholes price when the shocks to the stock process follow a Lévy-Stable process. First we see that any symmetric Lévy-Stable random variable can be represented as the product of a totally skewed with a symmetric Lévy-Stable variable as shown by the following proposition.

Proposition 2.2: (*constructing symmetric variables; p. 20 of Samorodnitsky and Taqqu (1994)*) *Let* $X \sim S_{\alpha'}(\kappa, 0, 0)$, $Y \sim S_{\alpha/\alpha'}((\cos(\pi\alpha/2\alpha'))^{\alpha'/\alpha}, 1, 0)$, *with* $0 < \alpha < \alpha' \le 2$, *be independent. Then the random variable*

$$Z = Y^{1/\alpha'}X \sim S_\alpha(\kappa, 0, 0).$$

2.2. *Lévy-Stable processes*

A stochastic process $\{L_t, t \in T\}$ is Lévy-Stable if all its finite-dimensional distributions are Lévy-Stable. A particular case of a Lévy-Stable process, which will be denoted by $\{L_t^{\alpha,\beta}, t \ge 0\}$, is the Lévy-Stable motion (Samorodnitsky and Taqqu 1994).

Definition 2.3: (Lévy-Stable motion) A Lévy-Stable process $L_t^{\alpha,\beta}$ is called a Lévy-Stable motion if $L_0^{\alpha,\beta} = 0$, $L_t^{\alpha,\beta}$ has independent increments, and $L_t^{\alpha,\beta} - L_s^{\alpha,\beta} \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$ for any $0 \le s < t < \infty$ and for some $0 < \alpha \le 2$ and $-1 \le \beta \le 1$ (time-homogeneity of the increments). Observe that when $\alpha = 2$ and $\beta = 0$ it is Brownian motion, while when $\alpha < 1$ and $\beta = -1$ (respectively $\beta = 1$) the process $L_t^{\alpha,\beta}$ has support on the negative (respectively positive) line.

The log-characteristic function of a Lévy-Stable motion $L_t^{\alpha,\beta}$ is given by (Samorodnitsky and Taqqu 1994)

$$\begin{aligned}\log\mathbb{E}[\mathrm{e}^{i\theta L_t}] &\equiv \Psi_t(\theta)\\ &= \begin{cases} -t\kappa^\alpha|\theta|^\alpha\{1 - i\beta\operatorname{sign}(\theta)\tan(\alpha\pi/2)\}\\ \qquad + tim\theta, & \text{for } \alpha \neq 1,\\ -t\kappa|\theta|\{1 + (2i\beta/\pi)\operatorname{sign}(\theta)\log|\theta|\}\\ \qquad + tim\theta, & \text{for } \alpha = 1.\end{cases}\end{aligned} \tag{5}$$

Proposition 2.2 can be extended to processes; hence we may use Brownian motion as one of the building blocks to obtain symmetric Lévy-Stable processes (see proposition 3.8.1, p. 143, of Samorodnitsky and Taqqu (1994)).

3. Stochastic volatility with Lévy-Stable shocks

In modelling integrated variance as a building block there are two properties that integrated variance $Y_{t,T} = \int_t^T \sigma_s^2 \mathrm{d}s$ should have:

- it should be continuous and increasing in T; and
- it should be time-consistent in that

$$Y_{t,T} = \int_t^T \sigma_s^2 \mathrm{d}s = \int_t^\tau \sigma_s^2 \mathrm{d}s + \int_\tau^T \sigma_s^2 \mathrm{d}s = Y_{t,\tau} + Y_{\tau,T}, \tag{6}$$

for all $t \le \tau \le T$.

As motivated in the Introduction, we seek a model in which the shocks to the stock process are Lévy-Stable. If we assume that the returns process is given by

$$\frac{\mathrm{d}S_t}{S_t} = \mu\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t,$$

$$\text{so that } S_T = \mathrm{e}^{\mu(T-t)-(1/2)\int_t^T \sigma_s^2 \mathrm{d}s + \int_t^T \sigma_s \mathrm{d}W_s},$$

where μ is a constant and $\mathrm{d}W_t$ the increment of Brownian motion, we might be tempted, based on proposition 2.2, to model volatility by assuming that the integrated variance is given by

$$Y_{t,T} = \int_t^T \sigma_s^2 \mathrm{d}s = \int_t^T \mathrm{d}L_s^{\alpha/2,1}. \tag{7}$$

Note that $\mathrm{d}L_t^{\alpha/2,1}$ is the increment of a positive Lévy-Stable motion (because $\alpha/2 < 1$ so that (7) is an increasing process). This seems a reasonable choice, since

$$\mathbb{E}\left[\mathrm{e}^{i\theta\int_t^T \sigma_s \mathrm{d}W_s}\right] = \mathrm{e}^{-(1/2^{\alpha/2})\sec(\alpha\pi/4)(T-t)|\theta|^\alpha},$$

hence the shocks to the process would be symmetric Lévy-Stable by proposition 2.2.

Unfortunately, this model for integrated variance is inconsistent since on the left-hand side of (7) we have the integrated variance $\int_t^T \sigma_s^2 \mathrm{d}s$ which is, by construction, a continuous process. However, on the right-hand side, we have the non-negative Lévy-Stable motion $\int_t^T \mathrm{d}L_s^{\alpha/2,1}$ which is, by construction, a purely discontinuous process. Despite these difficulties, we do not abandon the idea of integrating against a Lévy-Stable motion. Instead, we discuss a way of constructing a process for the integrated variance that is Lévy-Stable but with continuous paths in T.

If the purely discontinuous process $\int_t^T \mathrm{d}L_s^{\alpha/2,1}$ can be modified to

$$\int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1},$$

for a suitable deterministic function $f(s,T)$, the jumps can be 'damped' and the resulting process made continuous and increasing in T. Specifically, we require that $f(s,T) > 0$ for $s < T$ and that $f(s,T) \to 0$ as $s \uparrow T$, so the 'last' jumps of the process become smoothed out. (For a general discussion of the path behaviour of processes of the type $\int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1}$, and more general Lévy-Stable stochastic integrals, see Samorodnitsky and Taqqu 1994). We now give conditions under which the stochastic integral on the right-hand side of equation (8), given by $\int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1}$, is continuous in T, denoting the class of functions $f(s,T)$ for which this is true by $\mathbb{F}$.

Proposition 3.1: *Let $f(s,T)$ be continuous in T with $f(T,T)=0$, and assume in addition that, for each T, $\partial f(s,T)/\partial s := f_1(s,T)$ is continuous on an interval $0 \le s < T^* < \infty$. Then the process $X_{t,T} = \int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1}$ is continuous in T for any T belonging to $(s,T^*]$.*

Proof: Integrating by parts (Protter 1992), and using $f(T,T)=0$,

$$\int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1} = f(t,T)L_t^{\alpha/2,1} - \int_t^T f_1(s,T)L_s^{\alpha/2,1}\mathrm{d}s.$$

The first term is continuous in T by assumption on $f(t,T)$, as t is fixed. Evaluating the second term at $T+\epsilon$ and T and subtracting gives

$$\begin{aligned}\int_t^{T+\epsilon} f_1(s,T+\epsilon)L_s^{\alpha/2,1}\mathrm{d}s - \int_t^T f_1(s,T)L_s^{\alpha/2,1}\mathrm{d}s &= \int_t^T (f_1(s,T+\epsilon)-f_1(s,T))L_s^{\alpha/2,1}\mathrm{d}s \\ &\quad + \int_T^{T+\epsilon} f_1(s,T+\epsilon)L_s^{\alpha/2,1}\mathrm{d}s.\end{aligned}$$

Both terms on the right clearly tend to zero with ϵ. $\square$

Since we are interested in pricing options where the underlying stochastic component is driven by a symmetric Lévy-Stable process we would like to specify a kernel $f(s,T)$ so the finite-dimensional distribution of integrated variance is totally skewed to the right Lévy-Stable. We propose as a model for integrated variance

$$Y_{t,T} = \int_t^T \sigma_s^2\mathrm{d}s = h(t,T)\sigma_t^2 + \int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1} \qquad (8)$$

for suitable positive functions $h(t,T)$ and $f(s,T)$. We assume that $f(T,T)=0$ for all t to damp the Lévy-Stable jumps, and that $h(t,t)=0$ for consistency when $T=t$, and for the same reason we also need to take $\partial h(t,T)/\partial T|_{T=t}=1$; this is shown below. For $t<T$ (respectively $s<T$) we require that $h(t,T)>0$ (respectively $f(s,T)>0$) to ensure that $Y_{t,T}$ is strictly positive and properly random. Further conditions on f and h which specify their general form are given in proposition 3.2. For example, in our model we may choose

$$h(t,T)=\frac{1}{\gamma}(1-\mathrm{e}^{-\gamma(T-t)}) \quad\text{and}\quad f(s,T)=\frac{1}{\gamma}(1-\mathrm{e}^{-\gamma(T-s)}), \qquad (9)$$

for $\gamma>0$ in (8) to obtain, as a particular case, the OU-type model for integrated variance first introduced by Barndorff-Nielsen and Shephard (2001) where the increments in (8) are driven by a general non-negative Lévy process L_t. (Note, however, that, in general, the functions $h(t,T)$ and $f(s,T)$ do not depend only on the lag $T-t$ (respectively $T-s$) as one might expect. Their most general form is given below.)

Before proceeding, we note an important point concerning units. The integrated variance is dimensionless (that is, a pure number it has no units). Hence the function $h(t,T)$ must have the dimensions of time, and since the Lévy process $L_t^{\alpha/2,1}$ scales as time to the power $2/\alpha$, the function $f(s,T)$ must have dimensions of time to the power $-2/\alpha$. This distinction only matters, of course, if we change the unit of time: in (9), $f(s,T)$ contains an implicit dimensional constant, equal to 1 in the time units of the model, to make the dimensions correct.

Proposition 3.2: *Suppose that the functions $f(s,T)$ and $h(t,T)$ are twice differentiable in their second argument and once differentiable in their first argument, with $f(s,T)>0$ for all $s<T$, while $f(T,T)=0$, and $h(t,T)>0$ for all $t<T$, while $h(t,t)=0$. Then the process*

$$Y_{t,T} = \int_t^T \sigma_s^2\mathrm{d}s = h(t,T)\sigma_t^2 + \int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1} \qquad (10)$$

is non-negative, continuous and increasing in T, and satisfies the consistency condition $Y_{t,T} = Y_{t,\tau} + Y_{\tau,T}$ if and only if $f(s,T)$ and $h(t,T)$ are non-negative and take the form

$$h(t,T)=\frac{H(T)-H(t)}{H'(t)}, \quad f(s,T)=F(s)(H(T)-H(s)), \qquad (11)$$

where $H(\cdot)$ is a strictly monotonic, differentiable function with derivative H', and $F(\cdot)$ is continuous and positive (respectively negative) if $H(\cdot)$ is increasing (respectively decreasing).

Proof: We use subscripts 1 (respectively 2) on $h(\cdot,\cdot)$ and $f(\cdot,\cdot)$ to denote differentiation with respect to (wrt) the first (respectively second) argument, with an obvious extension to higher derivatives.

Suppose that, for $\tau>t$,

$$\int_t^\tau \sigma_s^2\mathrm{d}s = h(t,\tau)\sigma_t^2 + \int_t^\tau f(s,\tau)\mathrm{d}L_s, \qquad (12)$$

where L_t denotes a non-negative Lévy process (including $L_t^{\alpha/2,1}$ as a special case). This is clearly a positive process with our assumptions.

Differentiating wrt τ and using $f(\tau,\tau)=0$,

$$\sigma_\tau^2 = h_2(t,\tau)\sigma_t^2 + \int_t^\tau f_2(s,\tau)\mathrm{d}L_s. \qquad (13)$$

Note that this immediately implies that

$$h_2(t,t)=1,$$

as stated above.

Since

$$\int_\tau^T \sigma_s^2\mathrm{d}s = h(\tau,T)\sigma_\tau^2 + \int_\tau^T f(s,T)\mathrm{d}L_s, \qquad (14)$$

we have

$$\begin{aligned}\int_t^\tau \sigma_s^2\mathrm{d}s + \int_\tau^T \sigma_s^2\mathrm{d}s &= h(t,\tau)\sigma_t^2 + h(\tau,T)\sigma_\tau^2 + \int_t^\tau f(s,\tau)\mathrm{d}L_s \\ &\quad + \int_\tau^T f(s,T)\mathrm{d}L_s \\ &= h(t,\tau)\sigma_t^2 + h(\tau,T)\left(h_2(t,\tau)\sigma_t^2 \right.\\ &\quad \left. + \int_t^\tau f_2(s,\tau)\mathrm{d}L_s\right) \\ &\quad + \int_t^\tau f(s,\tau)\mathrm{d}L_s + \int_\tau^T f(s,T)\mathrm{d}L_s \\ &= (h(t,\tau)+h(\tau,T)h_2(t,\tau))\sigma_t^2 \\ &\quad + \int_t^\tau (f(s,\tau)+h(\tau,T)f_2(s,\tau))\mathrm{d}L_s \\ &\quad + \int_\tau^T f(s,T)\mathrm{d}L_s.\end{aligned}$$

Writing the left-hand side as $\int_t^T \sigma_s^2\mathrm{d}s$, using (10) and noting that the path is arbitrary, the consistency condition (6) is met if and only if

$$h(t,T) = h(t,\tau) + h(\tau,T)h_2(t,\tau), \qquad (15)$$

$$f(s,T) = f(s,\tau) + h(\tau,T)f_2(s,\tau), \qquad (16)$$

for all $s,\tau \in (t,T)$.

We characterize f and h from the functional equations (15) and (16) by a 'separation of variables' technique, beginning with h. First differentiate (15) wrt τ to give

$$0 = h_2(t,\tau) + h_1(\tau,T)h_2(t,\tau) + h(\tau,T)h_{22}(t,\tau),$$

which is rearranged to

$$\frac{h_{22}(t,\tau)}{h_2(t,\tau)} = -\frac{1+h_1(\tau,T)}{h(\tau,T)}.$$

The left-hand side of this equation is a function of t and τ, the right-hand side is a function of τ and T, so both must be equal to an arbitrary function of τ alone. Setting the left-hand side equal to this function, we have an ordinary differential equation in τ for $h(t,\tau)$, whose most general solution satisfying $h(t,t)=0$ and $h_2(t,t)=1$ is indeed

$$h(t,\tau) = \frac{H(\tau)-H(t)}{H'(t)}, \qquad (17)$$

for an arbitrary non-constant function $H(\cdot)$. (The same result can be obtained by differentiating (15) with respect to T twice.)

As $h(t,\tau)>0$ and is bounded, a simple argument by contradiction shows that, for each τ, $H(t)-H(\tau)$ either increases or decreases as $\tau-t$ increases; it cannot have a turning point and $H(\cdot)$ is therefore monotonic.

Conversely, direct substitution shows that (17) satisfies (15).

The proof for f is similar: differentiation of (16) wrt τ and rearrangement leads to

$$\frac{f_{22}(s,\tau)}{f_2(s,\tau)} = -\frac{1+h_1(\tau,T)}{h(\tau,T)},$$

from which both sides are equal to an arbitrary function of τ; solving the resulting ordinary differential equation in τ for $f(s,\tau)$, with the condition $f(s,s)=0$, shows that $f(s,\tau)=F(s)(G(\tau)-G(s))$ for arbitrary $F(\cdot)$ and $G(\cdot)$, the latter being differentiable. Substitution back into (16) shows that $G(\cdot)=H(\cdot)$, as required. The sign of $F(\cdot)$ clearly follows from (11) given that h is monotonic. The converse is shown by direct substitution. $\square$

Two possible choices for $f(s,T)$ and $h(t,T)$ are†

$$f(s,T)=T-s, \quad h(t,T)=T-t, \quad s,t\le T, \qquad (18)$$

$$f(s,T)=\frac{1-\mathrm{e}^{-\gamma((T+c)^n-(s+c)^n)}}{\gamma n(s+c)^{n-1}}, \quad h(t,T)=\frac{1-\mathrm{e}^{-\gamma((T+c)^n-(t+c)^n)}}{\gamma n(t+c)^{n-1}}, \qquad (19)$$

for s, $t\le T$ and $1\le n<2$ where γ is a positive constant that can be seen as a damping factor which we can be chosen freely, and $c\ge 0$ is constant. Both choices satisfy the additivity condition (6); for example, (19) is obtained by assuming $H(T)=\mathrm{e}^{-\gamma(T+c)^n}$ and $F(s)=1/H'(s)$ in proposition 3.2.

Henceforth, we take $H(\cdot)>0$ without loss of generality, and we further assume that

$$\int_0^T \frac{1}{H'(s)}\mathrm{d}s < \infty, \quad \text{for } 0\le T<\infty, \qquad (20)$$

which is a condition we will require below to price instruments under the risk-neutral measure. It simply amounts to saying that $H'(0)>0$, namely that the time $t=0$ is not special (recall that $H(\cdot)$ cannot have turning points for $t>0$).

3.1. *Illustration*

We now illustrate the different building blocks needed to obtain the integrated variance process described above. First we simulate a totally skewed to the right Lévy-Stable motion; then we get the spot variance process, by choosing an appropriate kernel; then we produce the integrated variance process. We focus on kernels of the integrated variance of the form (19). The solid line in the two bottom graphs of figure 1 represents the case with $n=1$, $c=0.1$, $t=0$, $0\le T\le 1$, $\sigma_0^2=0$, $\gamma=25$, which is a standard OU-type process as in Barndorff-Nielsen and Shephard (2001) with a two-week mean-reversion period. In the same figure the dotted lines represent the case $n=1.2$, $T=1$ and $\gamma=25$.

†Although these functions are apparently the same, as remarked above, there is a dimensional constant multiplying them which would change if the time units were changed.

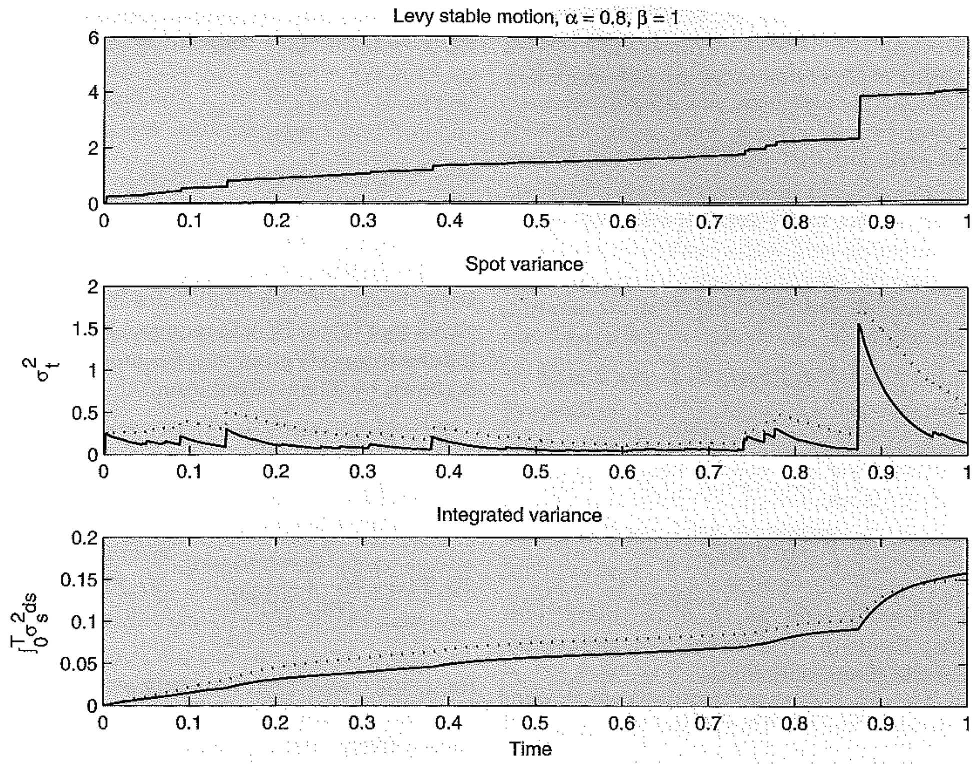

Figure 1. Simulated integrated variance with kernel $f(s,T)=25^{-1}(1-\mathrm{e}^{-25((T+c)^n-(s+c)^n)})$ with $c=0.1$, $n=1$, $T=1$ (solid line), and $c=0.1$, $n=1.2$, $T=1$ (dotted line). In both cases, $t=0$ and $\sigma_0^2=0$.

4. Model dynamics and option prices

We now turn to models of the asset price evolution and the pricing of vanilla options. Section 4.1 looks at a basic model where the shocks to the returns or log-stock process are symmetric; section 4.2 extends it to a model where shocks can also be asymmetric. Finally, section 4.3 shows how to price vanilla options when the shocks to the underlying stock process follow a Lévy-Stable process for $\alpha>1$ and $-1\le\beta\le 1$.

Given the nature of the model, there is no unique equivalent martingale measure (EMM). In line with most of the Lévy process literature we choose an EMM that is structure-preserving since, among other features (Cont and Tankov 2004), transform methods for pricing are straightforward to implement; this is discussed at the end of section 4.2.

4.1. Modelling returns

As pointed out in the Introduction we can model either returns or log-stock prices; when shocks are symmetric we can take either route. For example, if we believe that the shocks to the returns process follow a Lévy-Stable distribution, we assume that, in the physical measure P,

$$\frac{\mathrm{d}S_t}{S_t}=\mu\,\mathrm{d}t+\sigma_t\,\mathrm{d}W_t, \tag{21}$$

$$\int_0^t \sigma_s^2\mathrm{d}s = h(0,t)\sigma_0^2+\int_0^t f(s,t)\mathrm{d}L_s^{\alpha/2,1}, \tag{22}$$

where $\mathrm{d}W_t$ denotes the increment of the standard Brownian motion, $h(\cdot,\cdot)$ and $f(\cdot,\cdot)$ satisfy the conditions in proposition 3.2, $f(\cdot,\cdot)\in\mathbb{F}$ and μ is a constant. In the following proposition we show the distribution of the stock process.

Proposition 4.1: *Let the stock process follow* (21) *and the integrated variance process follow* (22). *Assume further that* W_t *and* $L^{\alpha/2,1}$ *are independent, then the log-stock process* (21) *is the sum of two independent processes: a symmetric Lévy-Stable process and a Gaussian process.*

Proof: First note that the stochastic component of the log-stock process is given by

$$U_{0,t}=\int_0^t \sigma_s\,\mathrm{d}W_s. \tag{23}$$

Now we calculate the characteristic function of the random process $U_{0,t}$. We have

$$\mathbb{E}[\mathrm{e}^{i\theta U_{0,t}}]=\mathbb{E}\left[\mathrm{e}^{i\theta\int_0^t\sigma_s\mathrm{d}W_s}\right],$$

and by independence of σ_t and W_t, $\int_0^t\sigma_s\,\mathrm{d}W_s$ is a zero-mean Normal variable whose variance is the random variable $\int_0^t\sigma_s^2\mathrm{d}s$. Thus the characteristic function of $\int_0^t\sigma_s\,\mathrm{d}W_s$ is given by

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{i\theta U_{0,t}}]&=\mathbb{E}\left[\exp\left[-\frac{1}{2}\theta^2\int_0^t\sigma_s^2\mathrm{d}s\right]\right]\\&=\exp\left[-\frac{1}{2}h(0,t)\sigma_0^2\theta^2\right]\\&\quad\times\mathbb{E}\left[\exp\left[-\frac{1}{2}\theta^2\int_0^t f(s,t)\mathrm{d}L_s^{\alpha/2,1}\right]\right].\end{aligned}$$

Further, using (5) we have that

$$\int_0^t f(s,t)\mathrm{d}L_s^{\alpha/2,1}\sim S_{\alpha/2}\left(\left(\int_0^t f(s,t)^{\alpha/2}\,\mathrm{d}s\right)^{2/\alpha},1,0\right),$$

and using proposition 2.1 we write

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{i\theta U_{0,t}}]&=\exp\left[-\frac{1}{2}h(0,t)\sigma_0^2\theta^2\right]\\&\quad\times\mathbb{E}\left[\exp\left[-\frac{1}{2}\theta^2\int_0^t f(s,t)\mathrm{d}L_s^{\alpha/2,1}\right]\right]\\&=\exp\left[-\frac{1}{2}h(0,t)\sigma_0^2\theta^2-\frac{1}{2^{\alpha/2}}\sec\frac{\alpha\pi}{4}\int_0^t f(s,t)^{\alpha/2}\,\mathrm{d}s|\theta|^\alpha\right].\end{aligned}$$

This is clearly the characteristic function of the sum of a Gaussian process and an independent symmetric Lévy-Stable process with index α. □

Note that we might also stipulate that our departure point is the risk-neutral dynamics for the stock process and that our model is as above with μ replaced with r:

$$\frac{\mathrm{d}S_t}{S_t}=r\mathrm{d}t+\sigma_t\,\mathrm{d}W_t^Q, \tag{24}$$

with $\int_t^T\sigma_s^2\mathrm{d}s$ as in (22), $\mathrm{d}W_t^Q$ being the increments of the standard Brownian motion. However, we need not specify the risk-neutral dynamics as a starting point since it is possible to postulate the physical dynamics and then choose an EMM. We discuss the relation between the measures P and Q below for a model that also allows for asymmetric Lévy-Stable shocks and the symmetric case then becomes a particular case.

We also note that the stochastic integral $\int_t^T\sigma_s\,\mathrm{d}W_s$ can be seen as a time-changed Brownian motion (Kallsen and Shiryaev 2002). In this case the integrated variance $\int_t^T\sigma_s^2\mathrm{d}s$ represents the time change and it is straightforward to show that

$$\int_t^T\sigma_s\,\mathrm{d}W_s\stackrel{d}{=}W_{\hat{T}_{t,T}}$$

where $\hat{T}_{t,T}=\int_t^T\sigma_s^2\mathrm{d}s$.

4.2. Modelling log-stock prices

Financial data suggest that returns are skewed rather than symmetric (see, for example, Kraus and Litzenberger (1976), Campbell *et al.* (1997) and Carr and Wu (2003)). For instance, since the stock market crash of 1987, the US stock index options market has shown a pronounced skewed implied volatility (volatility smirk) which indicates that, under the risk-neutral measure, log-returns have a negatively skewed distribution.

The symmetric model above can be extended to allow the dynamics of the log-stock process to follow an asymmetric Lévy-Stable process. In stochastic volatility models, one way to introduce skewness in the log-stock process is to correlate the random shocks of the volatility process to the shocks of the stock process. It is typical in the literature to assume that the Brownian motion of the stock process, say $\mathrm{d}W_t$, is correlated with the Brownian motion of the volatility process, say $\mathrm{d}Z_t$. Thus $\mathrm{d}W_t\mathrm{d}Z_t=\rho\,\mathrm{d}t$ and we can write $Z_t=\rho W_t+\sqrt{1-\rho^2}\tilde{Z}_t$, where $\tilde{Z}_t$ is independent of W_t. The correlation parameter ρ is also known in the literature as the leverage effect and empirical studies suggest that $\rho<0$ (Fouque *et al.* 2000). In our case the notion of 'correlation' does not apply because, for Lévy-Stable random variables, it is given that moments of second and higher order do not exist, nor do correlations. However, we may also include a leverage effect via a parameter ℓ to produce skewness in the stock returns.

Hence to allow for asymmetric Lévy-Stable shocks, under the physical measure we assume that

$$\log(S_T/S_t)=\mu(T-t)+\int_t^T\sigma_s\,\mathrm{d}W_s+\ell\tilde{\sigma}\int_t^T\mathrm{d}\tilde{L}_s^{\alpha,-1}, \tag{25}$$

$$\int_t^T\sigma_s^2\mathrm{d}s=h(t,T)\sigma_t^2+\int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1}. \tag{26}$$

Here $\mathrm{d}W_t$ denotes the increment of the standard Brownian motion independent of both $\mathrm{d}\tilde{L}_t^{\alpha,-1}$ and $\mathrm{d}L_t^{\alpha/2,1}$ and we note that $\mathrm{d}\tilde{L}_t^{\alpha,-1}$, independent of $\mathrm{d}L_t^{\alpha/2,1}$, is totally skewed to the left and that $1\le\alpha<2$. Moreover, μ and $\tilde{\sigma}\ge 0$ are constants, $f(t,T)$ and $h(t,T)$ satisfy the conditions in proposition 3.2 with $f(t,T)\in\mathbb{F}$ and the leverage parameter $\ell\ge 0$. In appendix A we show that the shocks to the price process are asymmetric Lévy-Stable.†

Before proceeding we discuss the connection in this model between the dynamics of the stock price under the physical measure P and the risk-neutral measure Q. Recall that a probability measure Q is called an EMM if it is equivalent to the physical probability P and the discounted price process is a martingale. It is straightforward to see that in the model proposed here the set of EMMs is not unique, hence we must motivate the choice of a particular EMM.

Let us focus on the model with no leverage (i.e. $\ell=0$). Based on Girsanov's theorem (Karatzas and Shreve 1988), we assume that the risk-neutral dynamics of

†Note that here we model log-stock prices since we *cannot* include a similar leverage effect in equation (21) because this allows negative prices due to the jumps of the increments of the Lévy-Stable motion $\mathrm{d}\tilde{L}_t^{\alpha,-1}$.

the model are obtained via the Radon–Nikodym derivative

$$Z_t = \mathrm{e}^{\int_0^t [r-\mu-(1/2)\sigma_s^2](1/\sigma_s)\mathrm{d}W_s - (1/2)\int_0^t [r-\mu-(1/2)\sigma_s^2]^2(1/\sigma_s^2)\mathrm{d}s}. \tag{27}$$

To be able to apply Girsanov's theorem we need to check two conditions.† First, we must verify that

$$P\left[\int_0^T \left(r-\mu-\frac{1}{2}\sigma_s^2\right)^2 \frac{1}{\sigma_s^2}\,\mathrm{d}s < \infty\right] = 1, \quad \text{for } 0 \le T < \infty, \tag{28}$$

and second, that Z_t is a martingale and

$$\mathbb{E}[Z_t] = 1. \tag{29}$$

Since $r-\mu$ is a constant the first condition is satisfied if $P[\int_0^T \sigma_s^2 \mathrm{d}s < \infty] = 1$ and $P[\int_0^T (1/\sigma_s^2)\,\mathrm{d}s < \infty] = 1$ for $0 \le T < \infty$. To show the first, note that $X_{0,T} := \int_0^T f(s,T)\mathrm{d}L_s^{\alpha/2,1} \sim S_{\alpha/2}((\int_0^T f(s,T)^{\alpha/2}\,\mathrm{d}s)^{2/\alpha},1,0)$; therefore, $P[X_{0,T} < \infty] = 1$ for all T because the cdf of $X(T)$ integrates to 1. To show the second, we use (13) and (20) to show that $\int_0^T (1/\sigma_s^2)\mathrm{d}s$ is bounded above:

$$\begin{aligned}\int_0^T \frac{1}{\sigma_s^2}\,\mathrm{d}s &\le \frac{1}{\sigma_0^2}\int_0^T \frac{1}{h_2(0,s)}\,\mathrm{d}s \\ &= \frac{H'(0)}{\sigma_0^2}\int_0^T \frac{1}{H'(s)}\,\mathrm{d}s < \infty, \quad \text{for } 0 \le T < \infty;\end{aligned}$$

thus, $P[\int_0^T (1/\sigma_s^2)\mathrm{d}s < \infty] = 1$ for $0 \le T < \infty$.

To verify the martingale condition it is straightforward to check, using the independence between $L_t^{\alpha/2,1}$ and W_t, that

$$\begin{aligned}\mathbb{E}[Z_t] &= \mathbb{E}[\mathbb{E}[\mathrm{e}^{\int_0^t [r-\mu-(1/2)\sigma_s^2](1/\sigma_s)\mathrm{d}W_s - (1/2)} \\ &\quad \times \mathrm{e}^{\int_0^t [r-\mu-(1/2)\sigma_s^2]^2(1/\sigma_s^2)\mathrm{d}s} \mid \sigma_s^2, 0 \le s \le t]] \\ &= 1.\end{aligned}$$

Moreover, it is simple to calculate $\mathbb{E}[Z_t \mid \mathcal{F}_u] = Z_u$ (for $0 < u \le t$) and using the Radon–Nikodym derivative, $\mathbb{E}[S_t Z_t] = S_0\,\mathrm{e}^{rt}$.

Therefore, by Girsanov's theorem,

$$W_t^Q = W_t - \int_0^t \left(r-\mu-\frac{1}{2}\sigma_s^2\right)\frac{1}{\sigma_s}\,\mathrm{d}s,$$

and the risk-neutral dynamics of the stock, with $\ell = 0$, satisfy

$$\frac{\mathrm{d}S}{S} = r\mathrm{d}t + \sigma_t\,\mathrm{d}W_t^Q,$$

$$\int_t^T \sigma_s^2 \mathrm{d}s = \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda(T-t)})\sigma_t^2 + \int_t^T \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda(T-t)})\mathrm{d}L_s.$$

The inclusion of the leverage is straightforward in this setting, hence the risk-neutral dynamics of the model (25) and (26) follows

$$\begin{aligned}\log(S_T/S_t) &= r(T-t) - \frac{1}{2}\int_t^T \sigma_s^2 \mathrm{d}s + \ell^\alpha \tilde{\sigma}^\alpha \sec\frac{\alpha\pi}{2}(T-t) \\ &\quad + \int_t^T \sigma_s\,\mathrm{d}W_s^Q + \ell\tilde{\sigma}\int_t^T \mathrm{d}\hat{L}_s^{\alpha,-1},\end{aligned} \tag{30}$$

$$\int_t^T \sigma_s^2 \mathrm{d}s = h(t,T)\sigma_t^2 + \int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1}, \tag{31}$$

where W_t^Q is the standard Brownian motion independent of the Lévy-Stable motions $\hat{L}_t^{\alpha,-1}$ and $L_t^{\alpha/2,1}$ (also independent from each other) and r is the (constant) risk-free rate. This is the most general model that we consider; note that, if $\ell = 0$, we obtain the risk-neutral dynamics for the case when the returns or log-stock process follows a symmetric Lévy-Stable process under P.

4.3. *Option pricing with Lévy-Stable volatility*

As motivated in the Introduction by equations (1) and (2), the price of a vanilla option, using the EMM Q, is given by the iterated expectations

$$\begin{aligned}V(S,t) = \mathbb{E}^Q_{\hat{L}_t^{\alpha,-1}}\Bigg[\mathbb{E}^Q_{\sigma_t}\Bigg[\mathbb{E}^Q\Bigg[V_{\mathrm{BS}}\Bigg(S_t \mathrm{e}^{\ell\tilde{\sigma}\int_t^T \mathrm{d}\hat{L}_s^{\alpha,-1}}, \\ t, K, (\overline{Y}_{t,T})^{1/2}, T\Bigg)\Bigg]\hat{L}_t^{\alpha,-1}, \sigma_t \mid \hat{L}_t^{\alpha,-1}\Bigg]\Bigg],\end{aligned} \tag{32}$$

where $\overline{Y}_{t,T} = [1/(T-t)]\int_t^T \sigma_s^2 \mathrm{d}s$ and V_{BS} is the Black–Scholes value for a European option. Note that, if we let $h(t,T) = f(t,T) = 0$ for all t, $\ell = 1$ and $1 < \alpha < 2$, then the model reduces to

$$\log(S_T/S_t) = \mu(T-t) + \tilde{\sigma}\int_t^T \mathrm{d}\hat{L}_s^{\alpha,-1},$$

which is the Finite Moment Log-Stable (FMLS) model of Carr and Wu (2003).

Proposition 4.2: *It is possible to extend the results above to price European call and put options when the skewness coefficient $\beta \in [0,1]$.*

Proof: Using put–call inversion (McCulloch 1996), we have by no-arbitrage that European call and put options are related by‡

$$C(S,t;K,T,\alpha,\beta) = SKP(S^{-1},t;K^{-1},T,\alpha,-\beta).$$

□

As an example, we can use the approach above to derive closed-form solutions for option prices when the random shocks to the price process are distributed according to a Cauchy Lévy-Stable process, $\alpha = 1$ and $\beta = 0$ in (30) and (31), so that option prices are given by

$$\begin{aligned}V(S,t) &= \frac{\int_t^T f(s,T)^{1/2}\,\mathrm{d}s}{(T-t)\sqrt{2\pi}}\int_0^\infty V_{\mathrm{BS}}(S_t,t,K,(\overline{Y}_{t,T})^{1/2},T)\frac{1}{\bar{y}^{3/2}} \\ &\quad \times \mathrm{e}^{-[\int_t^T f(s,T)^{1/2}\,\mathrm{d}s/(T-t)]^2/2\bar{y}}\,\mathrm{d}\bar{y},\end{aligned}$$

where $\overline{Y}_{t,T} = [1/(T-t)]\int_t^T \sigma_s^2 \mathrm{d}s$. To see this, first we note that the combination of a Gaussian random variable, the Brownian motion in (30), and a Lévy–Smirnov $S_{1/2}(\kappa,1,0)$ random variable, the process followed by the integrated variance in (31), results in a Cauchy random variable $S_1(\kappa,0,0)$. This can be seen by calculating the convolution of their respective pdfs. Now, recall that the pdf for a Lévy–Smirnov random variable $S_{1/2}(\kappa,1,0)$ is given by $(\kappa/2\pi)^{1/2}x^{-3/2}\,\mathrm{e}^{-\kappa/2x}$ with support $(0,\infty)$; hence, the distribution of the average integrated variance is given by

$$\begin{aligned}\overline{Y}_{t,T} &\equiv \frac{1}{T-t}\int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1} \\ &\sim S_{1/2}\left(\frac{1}{(T-t)^2}\left(\int_t^T f(s,T)^{1/2}\,\mathrm{d}s\right)^2, 1, 0\right),\end{aligned}$$

and the value of the option is as required.

5. Numerical illustration: Lévy-Stable option prices

In this section we show how vanilla option prices can be calculated according to the above derivations. One route is to calculate the expected value of the Black–Scholes formula weighted by the stochastic volatility component and the leverage effect. Another route to price vanilla options for stock prices that follow a geometric Lévy-Stable process is to compute the option value as an integral in Fourier space, using Complex Fourier Transform techniques (Carr and Madan 1999, Lewis 2001).

We use the Black–Scholes model as a benchmark to compare the option prices obtained when the returns follow a Lévy-Stable process. Our results are consistent with the findings of Hull and White (1987) where the Black–Scholes model underprices in- and out-of-the-money call option prices and overprices at-the-money options.

5.1. *Option prices for symmetric Lévy-Stable log-stock prices*

We first obtain option prices and implied volatilities when the log-stock prices follow a symmetric Lévy-Stable process. Recall that, under the risk-neutral measure Q, and assuming, for simplicity, that $\sigma_t^2 = 0$, the stock price and variance process are given by

$$S_T = S_t \mathrm{e}^{r(T-t)-(1/2)\int_t^T \sigma_s^2 \mathrm{d}s + \int_t^T \sigma_s \mathrm{d}W_s^Q},$$

$$\int_t^T \sigma_s^2 \mathrm{d}s = \int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1}.$$

The first step we take is to calculate the characteristic function of the process

$$Z_{t,T} = -\frac{1}{2}\int_t^T \sigma_s^2 \mathrm{d}s + \int_t^T \sigma_s\,\mathrm{d}W_s^Q.$$

Proposition 5.1: *The characteristic function of $Z_{t,T}$ is given by*

$$\mathbb{E}^Q[\mathrm{e}^{i\xi Z_{t,T}}] = \exp\left[-\frac{1}{2^{\alpha/2}}\sec\left(\frac{\alpha\pi}{4}\right)(i\xi+\xi^2)^{\alpha/2}\int_t^T f(s,T)^{2/\alpha}\,\mathrm{d}s\right], \tag{33}$$

where $\xi = \xi_r + i\xi_i$ and $-1 \le \xi_i \le 0$. Moreover, the characteristic function is analytic in the strip $-1 < \xi_i < 0$.

Proof: The characteristic function is given by

$$\begin{aligned}\mathbb{E}^Q[\mathrm{e}^{i\xi Z_{t,T}}] &= \mathbb{E}^Q\left[\mathbb{E}^Q\left[\exp\left[-\frac{1}{2}i\xi\int_t^T \sigma_s^2 \mathrm{d}s + i\xi\int_t^T \sigma_s \mathrm{d}W_s^Q\right]\right.\right. \\ &\qquad \left.\left. \times |\sigma_s^2, 0 \le s \le t\right]\right] \\ &= \mathbb{E}^Q\left[\exp\left[-\frac{1}{2}i\xi\int_t^T \sigma_s^2 \mathrm{d}s - \frac{1}{2}\xi^2\int_t^T \sigma_s^2 \mathrm{d}s\right]\right] \\ &= \mathbb{E}^Q\left[\exp\left[-\frac{1}{2}(i\xi+\xi^2)\int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1}\right]\right] \\ &= \exp\left[-\frac{1}{2^{\alpha/2}}\sec\left(\frac{\alpha\pi}{4}\right)(i\xi+\xi^2)^{\alpha/2}\int_t^T f(s,T)^{\alpha/2}\,\mathrm{d}s\right].\end{aligned}$$

The last step is possible since the expected value exists if ξ is restricted so that $\xi_r^2 - \xi_i^2 + \xi_i \ge 0$, by consideration of the penultimate line. The region where this is true contains the strip $-1 \le \xi_i \le 0$. Finally, it is straightforward to observe that the characteristic function is analytic in this strip. □

To price call options we use the Fourier inversion formula:

$$C(x,t) = \mathrm{e}^{x_t} - \frac{1}{2\pi}\mathrm{e}^{-r(T-t)}K\int_{i\xi_i-\infty}^{i\xi_i+\infty} \mathrm{e}^{-i\xi x_t}\frac{K^{i\xi}}{\xi^2 - i\xi}\mathrm{e}^{(T-t)\Psi(-\xi)}\,\mathrm{d}\xi, \tag{34}$$

where $x_t = \log S_t$, $0 < \xi_i < 1$, and $\Psi(\xi)$ is the logarithm of the characteristic function of the process $\log S_T$. In comparing these prices with Black–Scholes prices, we have to decide how to choose the relevant parameters of the two models. In fact, the only parameter that we must examine carefully is the scaling parameter of the Lévy-Stable process; we opt for one that can be related to the standard deviation used when the classical Black–Scholes model is used. One approach, as in Hurst *et al.* (1999), is to match a given percentile of the Normal and a symmetric Lévy-Stable distribution. For example, if we want to match the first and third quartile of a Brownian motion with standard deviation $\sigma_{\mathrm{BS}} = 0.20$ to a symmetric Lévy-Stable motion $\kappa\,\mathrm{d}L_t^{\alpha,0}$ with characteristic exponent $\alpha = 1.7$, we would require the scaling parameter $\kappa = 0.1401$. We have chosen these parameters so that for options with 3 months to expiry these quartiles match. Moreover, in the examples below, we use the

†See section 3.5 of Karatzas and Shreve (1988).
‡Note that using put–call inversion allows us to obtain put prices when the log-stock price follows a positively skewed Lévy-Stable process, based on call prices where the underlying log-stock price follows a negatively skewed Lévy-Stable process. Furthermore, put–call-parity allows us to obtain call prices when the skewness parameter $-1 \le \beta \le 0$.

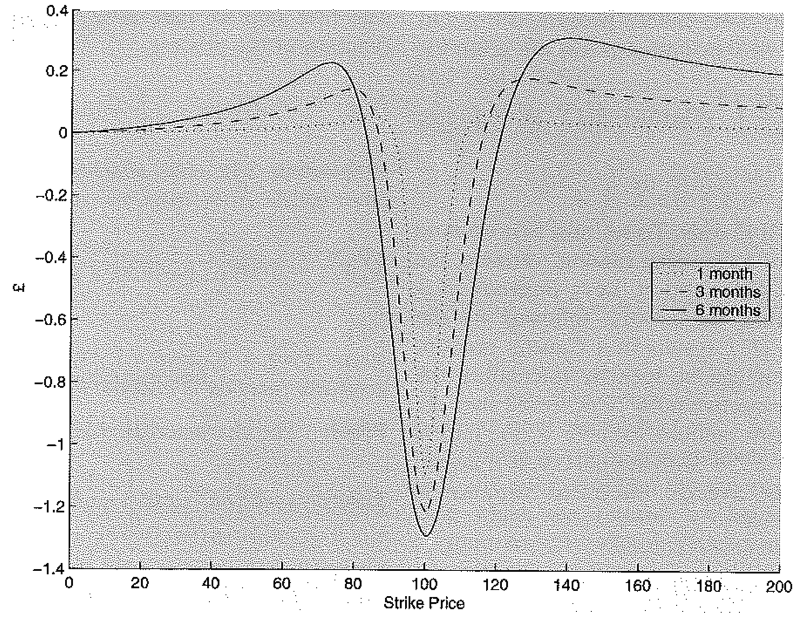

Figure 2. Difference between Lévy-Stable and Black–Scholes call option prices for different expiry dates: one, three and six months. In the Black–Scholes, annual volatility is $\sigma_{BS}=0.20$ and $\alpha=1.7$.

kernel $f(s,T)=\frac{1}{25}(1-e^{-25(T-s)})$, which is as in (19) with $n=1$, where for illustrative purposes we have assumed mean-reversion over a two-week period, i.e. $\gamma=25$.

Figure 2 shows the difference between European call options when the stock returns are distributed according to a symmetric Lévy-Stable motion with $\alpha=1.7$ and when returns follow a Brownian motion with annual volatility $\sigma_{BS}=0.20$. For out-of-the-money call options the Lévy-Stable call prices are higher than the Black–Scholes and for at-the-money options Black–Scholes delivers higher prices. These results are a direct consequence of the heavier tails under the Lévy-Stable case.

5.2. *Option prices for asymmetric Lévy-Stable log-stock prices*

We now obtain option prices and implied volatilities when there is a negative leverage effect, i.e. log-stock prices follow an asymmetric Lévy-Stable process (figure 3). Recall that, under the risk-neutral measure Q, the stock price and variance process are given by

$$S_T = S_t e^{r(T-t)-(1/2)\int_t^T \sigma_s^2 ds+(T-t)\ell^\alpha\tilde{\sigma}^\alpha \sec(\alpha\pi/2)+\int_t^T \sigma_s\, dW_s^Q+\ell\tilde{\sigma}\int_t^T d\hat{L}_s^{\alpha,-1}},$$

$$\int_t^T \sigma_s^2 ds = \int_t^T f(s,T)\, dL_s^{\alpha/2,1}.$$

where for simplicity we have assumed $\sigma_t^2=0$ in (26).

We proceed as above and calculate the characteristic function of the process

$$Z_{t,T}^\ell = -\frac{1}{2}\int_t^T \sigma_s^2 ds + \int_t^T \sigma_s\, dW_s^Q + \ell\tilde{\sigma}\int_t^T d\hat{L}_s^{\alpha,-1}.$$

Proposition 5.2: *The characteristic function of $Z_{t,T}^\ell$ is given by*

$$\mathbb{E}^Q[e^{Z_{t,T}^\ell}] = \exp\left[-\frac{1}{2^{\alpha/2}}\sec\left(\frac{\alpha\pi}{4}\right)(i\xi+\xi^2)^{\alpha/2} \times \int_t^T f(s,T)^{\alpha/2}\, ds + (T-t)(i\xi\ell\tilde{\sigma})^\alpha \sec\frac{\pi\alpha}{2}\right], \tag{35}$$

where $-1\le\xi_i\le 0$, $\xi=\xi_r+i\xi_r$, and is analytic in the strip $-1<\xi_i<0$.

Proof: The proof is very similar to the one above. It suffices to note that, for $\xi_i\le 0$,

$$\begin{aligned}|\mathbb{E}^Q[e^{i\xi\int_t^T d\hat{L}_s^{\alpha,-1}}]| &\le \mathbb{E}^Q\left[|e^{i\xi\int_t^T d\hat{L}_s^{\alpha,-1}}|\right]\\ &= \mathbb{E}^Q\left[e^{-\xi_i\int_t^T d\hat{L}_s^{\alpha,-1}}\right]\\ &< \infty.\end{aligned}$$

Moreover, for $\xi_i<0$ we have that $\mathbb{E}^Q[e^{i\xi\int_t^T d\hat{L}_s^{\alpha,-1}}]$ is analytic, i.e.

$$\left|\frac{d}{d\xi}\mathbb{E}^Q[e^{i\xi\int_t^T d\hat{L}_s^{\alpha,-1}}]\right| = \left|\mathbb{E}^Q\left[i\int_t^T d\hat{L}_s^{\alpha,-1} e^{i\xi\int_t^T d\hat{L}_s^{\alpha,-1}}\right]\right| < \infty.$$

Putting these results together with the results from proposition 5.1 we get the desired result. The requirement $-1<\xi_i<0$ arises because $d\hat{L}_t^{\alpha,-1}$ is totally skewed to the left, so we need $-\xi_i>0$. □

We use the same $f(s,T)$ as above and include a leverage parameter $\ell=1$ and $\tilde{\sigma}=0.15$ so that returns follow a negatively skewed process with $\beta(t,T)=-0.5$ when there is 3 months to expiry. Figure 4 shows the difference

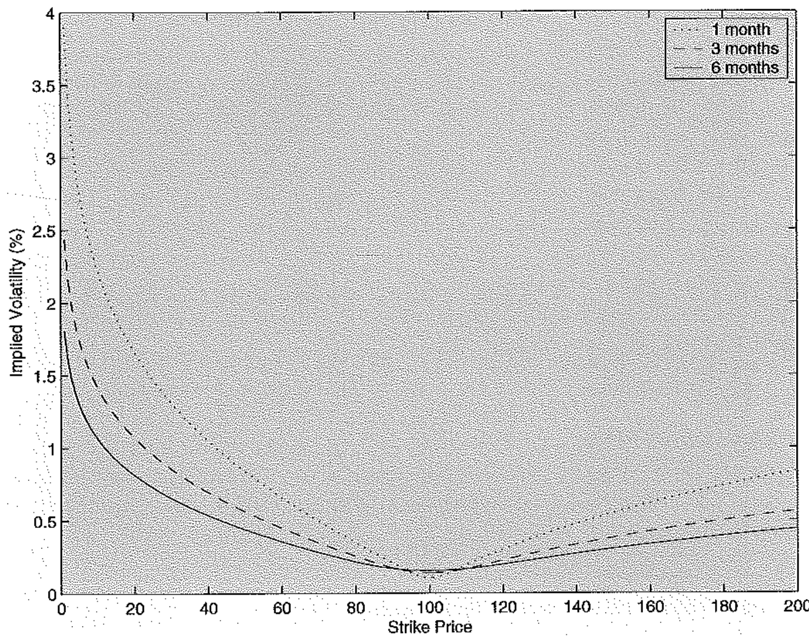

Figure 3. Black–Scholes implied volatility for the Lévy-Stable call option prices when returns follow a symmetric Lévy-Stable motion with $\alpha=1.7$, $\beta=0$ and three expiry dates: one, three and six months.

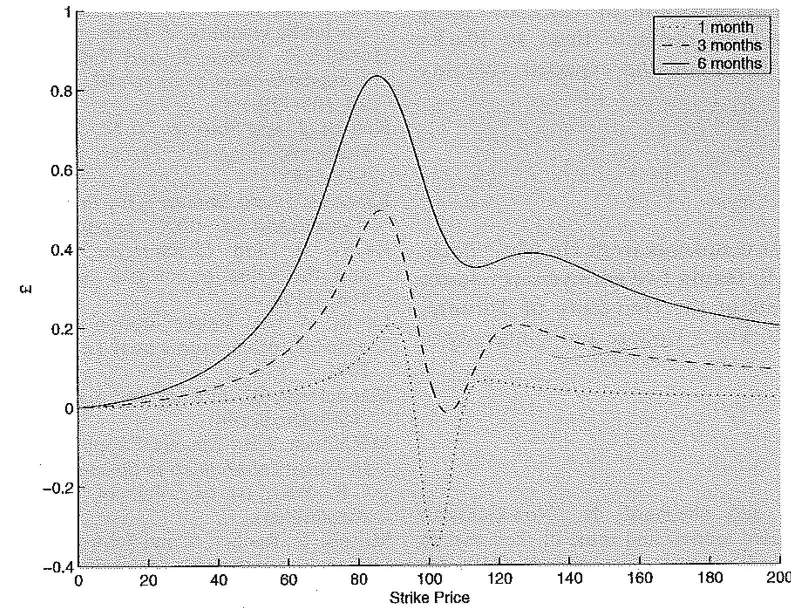

Figure 4. Difference between Lévy-Stable and Black–Scholes call option prices for different expiry dates: one, three and six months. In the Black–Scholes, annual volatility is $\sigma_{BS}=0.20$, $\alpha=1.7$ and $\tilde{\sigma}=0.15$.

between Lévy-Stable and Black–Scholes call option prices for different expiry dates. In the Black–Scholes case, annual volatility is $\sigma_{BS}=0.20$. Finally, figure 5 shows the corresponding implied volatility. The negative skewness introduced produces a 'hump' for call prices with strike below 100. This is financially intuitive since relative to the Black–Scholes the risk-neutral probability of the call option ending out-of-the-money is substantially higher in the Lévy-Stable case.

6. Conclusion

The GCLT provides a very strong theoretical foundation to argue that the limiting distribution of stock returns or log-stock prices follows a Lévy-Stable process. We have shown how to model stock returns and log-stock prices where the stochastic component is Lévy-Stable distributed covering the whole range of skewness $\beta\in[-1,1]$. We showed that European-style

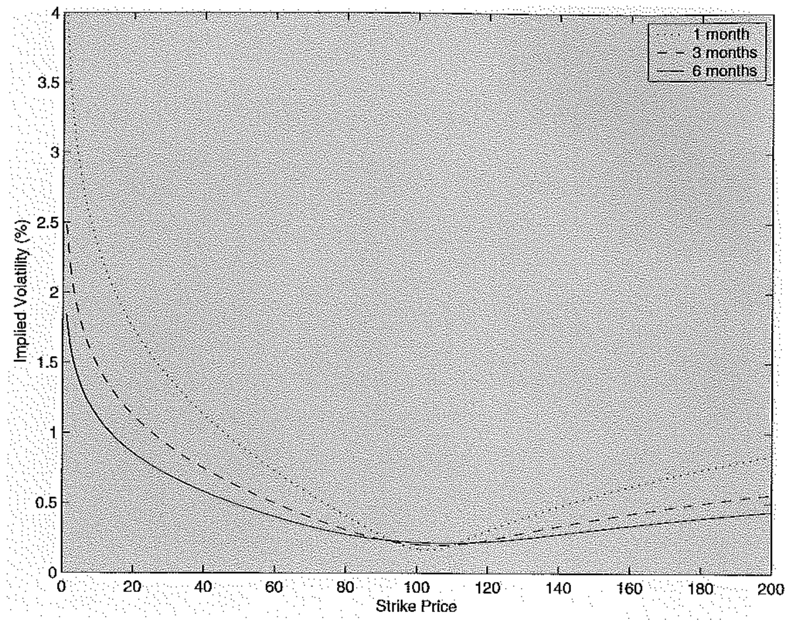

Figure 5. Black–Scholes implied volatility for the Lévy-Stable call option prices when returns follow a symmetric Lévy-Stable motion with $\alpha = 1.7$ and $\tilde{\sigma} = 0.15$ and three expiry dates: one, three and six months.

option prices are straightforward to calculate using transform methods and we compare them to Black–Scholes prices where we obtain the expected volatility smile.

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Appendix A

Suppose that the stock process, as assumed above in section 4.2, follows

$$\log(S_T/S_t) = \mu(T-t) + \int_t^T \sigma_s \, \mathrm{d}W_s + \ell\tilde{\sigma} \int_t^T \mathrm{d}\tilde{L}_s^{\alpha,-1},$$

$$\int_t^T \sigma_s^2 \mathrm{d}s = h(t,T)\sigma_t^2 + \int_t^T f(s,T)\mathrm{d}L_s^{\alpha/2,1},$$

under P where $\mathrm{d}W_t$ denotes the increment of the standard Brownian motion independent of both $\mathrm{d}\tilde{L}_t^{\alpha,-1}$ and $\mathrm{d}L_t^{\alpha/2,1}$. Then it is straightforward to verify that the shocks to the above log-stock process under the measure P are the sum of two independent processes: those of a Gaussian component and those of a Lévy-Stable process with negative skewness $\beta \in (-1,0]$. Let $G(t,T) = \int_t^T f(s,T)^{\alpha/2} \mathrm{d}s$ and, for simplicity in the calculations, assume that $\sigma_t^2 = 0$ (so we focus only on the asymmetric Lévy process).

Now consider the process

$$U_{t,T}^{\ell} = \int_t^T \sigma_s \, \mathrm{d}W_s + \ell\tilde{\sigma} \int_t^T \mathrm{d}\tilde{L}_s^{\alpha,-1}.$$

The log-characteristic function of $U_{t,T}^{\ell}$ is given by

$$\begin{aligned}
\log \mathbb{E}[\mathrm{e}^{i\theta U_{t,T}^{\ell}}] &= \log \mathbb{E}\left[\exp\left[i\theta\left(\int_t^T \sigma_s \, \mathrm{d}W_s + \ell\tilde{\sigma}\int_t^T \mathrm{d}\tilde{L}_s^{\alpha,-1}\right)\right]\right] \\
&= -\frac{1}{2^{\alpha/2}}\sec\left(\frac{\alpha\pi}{4}\right)G(T,t)|\theta|^{\alpha} + (\tilde{\sigma}\ell)^{\alpha}(T-t)|\theta|^{\alpha} \\
&\quad \times \left\{1 + i\mathrm{sign}(\theta)\tan\left(\frac{\alpha\pi}{2}\right)\right\}. \\
&= \left(\frac{1}{2^{\alpha/2}}\sec\left(\frac{\alpha\pi}{4}\right)G(t,T) + (T-t)\ell^{\alpha}\tilde{\sigma}^{\alpha}\right)|\theta|^{\alpha} \\
&\quad \times \left\{1 - \frac{-(T-t)\ell^{\alpha}\tilde{\sigma}^{\alpha}}{(1/2^{\alpha/2})\sec(\alpha\pi/4)G(t,T) + (T-t)\ell^{\alpha}\tilde{\sigma}^{\alpha}}\right. \\
&\quad \left. \times\, i\mathrm{sign}(\theta)\tan\left(\frac{\alpha\pi}{2}\right)\right\}.
\end{aligned}$$

This is obviously the characteristic function of a skewed Lévy-Stable process with (time-dependent) skewness parameter

$$\beta(t,T) = \frac{-(T-t)\ell^{\alpha}\tilde{\sigma}^{\alpha}}{(1/2^{\alpha/2})\sec(\alpha\pi/4)G(t,T) + (T-t)\ell^{\alpha}\tilde{\sigma}^{\alpha}} \in (-1,0].$$

Moreover, when $\ell = 0$ we obtain $\beta = 0$ and $\beta \to -1$ as $\ell \to \infty$.

Note that the integrated variance does not have a finite first moment since $\alpha/2 < 1$. However, in the case of the leverage effect $\int_t^T \mathrm{d}\tilde{L}_s^{\alpha,-1}$ its first moment exists, i.e. $\mathbb{E}[\int_t^T \mathrm{d}\tilde{L}_s^{\alpha,-1}] < \infty$ since $1 < \alpha < 2$.