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The Hidden Quantum Group of the 8–vertex Free Fermion Model: q–Clifford Algebras

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Abstract

We prove in this paper that the elliptic R–matrix of the eight vertex free fermion model is the intertwiner R–matrix of a quantum deformed Clifford–Hopf algebra. This algebra is constructed by affinization of a quantum Hopf deformation of the Clifford algebra.

1 Introduction

The realm of two dimensional integrable models contains two important families associated to the six vertex and eight vertex solutions to the Yang-Baxter equation [1].

Whereas the family of six vertex solutions (six vertex model and their higher spin descendants) are $R-$matrix intertwiners for different finite dimensional irreducible representations of $U_q(\widehat{Sl(2)})$, the elliptic eight vertex solutions do not admit, for the time being, the interpretation as quantum group intertwiners. To find such a quantum group interpretation of the eight vertex model would provide, for instance, a natural way to extend the known hidden quantum group structure of conformal field theories [2] to $q-$conformal field theories defined by the $q-$deformed Knizhnik-Zamolodchikov equation [3].

A special class of solutions to the vertex Yang-Baxter (YB) equation are the ones satisfying the so called free fermion condition:

$$R^{00}_{00}(u)R^{11}_{11}(u) + R^{10}_{01}(u)R^{01}_{10}(u) = R^{11}_{00}(u)R^{00}_{11}(u) + R^{01}_{01}(u)R^{10}_{10}(u) \tag{1}$$

In the six vertex case, $R^{11}_{00}(u) = R^{00}_{11}(u) = 0$ the solutions to YB satisfy (1) and are given by the $R-$matrix intertwiners of the Hopf subalgebra $U_{\hat{q}}(gl\widehat{(1,1)})$ [4]. These intertwiners can be mapped into the ones of $U_q(\widehat{sl(2)})$ ($q^4 = 1$) for non classical nilpotent irreducible representations [5] with $\hat{q} = \lambda$ and λ^2 the eigenvalue of the casimir K^2. The physical interest of the free fermion six vertex solutions is their close connection with $N = 2$ integrable models. In fact we can define using the generators of $U_q(\widehat{sl(2)})$ for $q^4 = 1$ a $N=2$ supersymmetric algebra [6], and in this case the free fermion condition (1) reflects the $N=2$ invariance of the $R-$matrix. Moreover the $N=2$ piece of the solitonic $S-$matrix for the $N=2$ Ginzburg-Landau superpotential $W = X^{N+1}/(N+1) - \beta X$ [7] can be shown to be given by the intertwiners of $U_{\hat{q}}(gl\widehat{(1,1)})$ with $\hat{q}^N = 1$.

In the eight vertex case, $R^{11}_{00}(u) \neq 0$ $R^{00}_{11}(u) \neq 0$, solutions to YB satisfying (1) have been known for a long time [8]. The most general solution corresponding to imposing non-zero field [9], [10] depends on three spectral parameters u, ψ_1, ψ_2, and is given by:

$$\begin{aligned}
a &\equiv R^{00}_{00} = 1 - e(u)e(\psi_1)e(\psi_2) \\
\tilde{a} &\equiv R^{11}_{11} = e(u) - e(\psi_1)e(\psi_2) \\
b &\equiv R^{10}_{01}(u) = e(\psi_1) - e(u)e(\psi_2) \\
\tilde{b} &\equiv R^{01}_{10} = e(\psi_2) - e(u)e(\psi_1) \\
c &\equiv R^{01}_{01} = R^{10}_{10} = (e(\psi_1)sn(\psi_1))^{1/2}(e(\psi_2)sn(\psi_2))^{1/2}(1 - e(u))/sn(u/2) \\
d &\equiv R^{11}_{00} = R^{00}_{11} = -ik(e(\psi_1)sn(\psi_1))^{1/2}(e(\psi_2)sn(\psi_2))^{1/2}(1 + e(u))sn(u/2)
\end{aligned} \tag{2}$$

with $e(u)$ the elliptic exponential:

$$e(u) = cn(u) + isn(u) \tag{3}$$

and k the elliptic modulus. The Yang–Baxter equation satisfied by this R matrix is [10]:

$$\begin{aligned}&(\mathbf{1} \otimes R(u;\psi_1,\psi_2))(R(u+v;\psi_1,\psi_3)\otimes \mathbf{1})(\mathbf{1}\otimes R(v;\psi_2,\psi_3)) = \\ &(R(v;\psi_2,\psi_3)\otimes \mathbf{1})(\mathbf{1}\otimes R(u+v;\psi_1,\psi_3))(R(u;\psi_1,\psi_2)\otimes \mathbf{1})\end{aligned} \tag{4}$$

The simplest way to catch the physical meaning of solution (2) is to define the corresponding spin chain hamiltonian:

$$H = \sum_{j=1}^{N} i \frac{\partial}{\partial u} R_{j,j+1}(u;\psi,\psi)\Big|_{u=0} \tag{5}$$

which is the well known $XY-$ model in an external magnetic field [11]:

$$H = \sum_{j=1}^{N} [(1+\Gamma)\sigma_j^x \sigma_{j+1}^x + (1-\Gamma)\sigma_j^y \sigma_{j+1}^y + h(\sigma_j^z + \sigma_{j+1}^z)] \tag{6}$$

where:

$$\begin{aligned}\Gamma &= \frac{2cd}{ab+\tilde{a}\tilde{b}} = ksn(\psi) \\ h &= \frac{a^2+b^2-\tilde{a}^2-\tilde{b}^2}{2(ab+\tilde{a}\tilde{b})} = cn(\psi)\end{aligned} \tag{7}$$

In this letter and as a preliminary step of the long term process of finding the quantum group symmetry of the eight vertex model, we will define a fully fledged Hopf algebra such that its $R-$intertwiners coincide with the elliptic free fermionic eight vertex solution (2).

2 The quantum Clifford algebra

A Clifford algebra $C(\eta)$ associated to a cuadratic form η is the associative algebra generated by the elements $\{\Gamma_\mu\}_{\mu=0}^{D}$, which satisfy:

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}\mathbf{1} \quad \mu = 1,\ldots,D \tag{8}$$

Associated to $C(\eta)$ we define the Clifford–Hopf algebra CH(D) as the associative algebra generated by $\Gamma_\mu \quad (\mu = 1,\ldots,D)$, Γ_{D+1} and the central elements $E_\mu \quad (\mu = 1,\ldots,D)$ satisfying the following relations:

$$
\begin{aligned}
&\Gamma_\mu^2 = E_\mu \;\;,\;\; \Gamma_{D+1}^2 = \mathbf{1} \\
&\{\Gamma_\mu, \Gamma_\nu\} = 0, \;\; \mu \neq \nu \\
&\{\Gamma_\mu, \Gamma_{D+1}\} = 0 \\
&[E_\mu, \Gamma_\nu] = [E_\mu, \Gamma_{D+1}] = [E_\mu, E_\nu] = 0 \;\; \forall \mu, \nu
\end{aligned}
\tag{9}
$$

The algebra CH(D) is a Hopf algebra with the following comultiplication Δ, antipode S and counit ϵ:

$$
\begin{array}{lll}
\Delta(E_\mu) = E_\mu \otimes \mathbf{1} + \mathbf{1} \otimes E_\mu, & S(E_\mu) = -E_\mu, & \epsilon(E_\mu) = 0 \\
\Delta(\Gamma_\mu) = \Gamma_\mu \otimes \mathbf{1} + \Gamma_{D+1} \otimes \Gamma_\mu, & S(\Gamma_\mu) = \Gamma_\mu \Gamma_{D+1}, & \epsilon(\Gamma_\mu) = 0 \\
\Delta(\Gamma_{D+1}) = \Gamma_{D+1} \otimes \Gamma_{D+1}, & S(\Gamma_{D+1}) = \Gamma_{D+1}, & \epsilon(\Gamma_{D+1}) = 1
\end{array}
\tag{10}
$$

For D even the elements $E_\mu \quad (\mu = 1,\ldots,D)$ and the product $\Gamma_1 \cdots \Gamma_D \Gamma_{D+1}$ are casimirs of CH(D), therefore in an irreducible representation of CH(D) we get $E_\mu = \eta_{\mu\mu}$, and $\Gamma_{D+1} \sim \Gamma_1 \cdots \Gamma_D$ which means that the irreps of CH(D) are isomorphic to those of $C(\eta)$ for all possible signatures of η (there is a unique faithful representation of $C(\eta)$ of dimension 2^D). For D odd similar arguments show that the representation theory of CH(D) is related to that of $C(\eta)$ for η a quadratic form defined in one dimension higher, namely D+1.

The quantum deformation of CH(D), that we will denote CH_q(D), is defined by:

$$
\Gamma_\mu^2 = [E_\mu]_q = \frac{q^{E_\mu} - q^{-E_\mu}}{q - q^{-1}} \tag{11}
$$

with the rest of equations (9) unchanged. The comultiplication for Γ_μ is now given by:

$$
\Delta \Gamma_\mu = \Gamma_\mu \otimes q^{-E_\mu/2} + q^{E_\mu/2} \Gamma_{D+1} \otimes \Gamma_\mu \tag{12}
$$

The Hopf algebra CH_q(D) for $D = 2$ is very close to the two parameter quantum supergroup $\mathcal{U}_{\alpha,\beta}(su(1,1))$ defined in [12]. The correspondence between both algebras is given by the substitutions $q^{E_x} \longrightarrow \alpha^E$ and $q^{E_y} \longrightarrow \beta^E$. Notice that in $CH_q(2)$ we have two central elements E_x, E_y and one quantum deformation parameter, while in $\mathcal{U}_{\alpha,\beta}(su(1,1))$ there exist one central element and two parameters. This difference will be important in the representation theory. We observe that the "SUSY grading" is played in our case by Γ_3, and in general by Γ_{D+1} for $D > 2$.

Next we proceed to define a sort of affinization of the Hopf algebra $\mathrm{CH}_q(\mathrm{D})$. The generators of this new algebra that we denote $\widehat{CH_q(D)}$ are: $E_\mu^{(i)}, \Gamma_\mu^{(i)}, \Gamma_{D+1}^{(i)}, \quad i = 0, 1$ satisfying (11) and (12) for each value of i. In what follows we will consider only the case $D = 2$[1].

A two dimensional irrep π_ξ of $\widehat{CH_q(D)}$ is labelled by three complex parameters $\xi = (z, \lambda_x, \lambda_y) \in C_\times^3$ and reads:

$$\pi_\xi(\Gamma_x^{(0)}) = \left(\frac{\lambda_x^{-1} - \lambda_x}{q - q^{-1}}\right)^{1/2} \begin{pmatrix} 0 & z^{-1} \\ z & 0 \end{pmatrix} \quad , \quad \pi_\xi(\Gamma_x^{(1)}) = \left(\frac{\lambda_x - \lambda_x^{-1}}{q - q^{-1}}\right)^{1/2} \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix}$$

$$\pi_\xi(\Gamma_y^{(0)}) = \left(\frac{\lambda_y^{-1} - \lambda_y}{q - q^{-1}}\right)^{1/2} \begin{pmatrix} 0 & -iz^{-1} \\ iz & 0 \end{pmatrix} \quad , \quad \pi_\xi(\Gamma_y^{(1)}) = \left(\frac{\lambda_y - \lambda_y^{-1}}{q - q^{-1}}\right)^{1/2} \begin{pmatrix} 0 & -iz \\ iz^{-1} & 0 \end{pmatrix}$$

$$\pi_\xi(\Gamma_3^{(0)}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \pi_\xi(\Gamma_3^{(1)}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13}$$

$$\pi_\xi(q^{E_x^{(0)}}) = \lambda_x^{-1} \quad , \quad \pi_\xi(q^{E_x^{(1)}}) = \lambda_x$$

$$\pi_\xi(q^{E_y^{(0)}}) = \lambda_y^{-1} \quad , \quad \pi_\xi(q^{E_y^{(1)}}) = \lambda_y$$

The intertwiner R_{ξ_1,ξ_2} for two of these irreps is defined by the condition

$$R_{\xi_1\xi_2}\Delta_{\xi_1\xi_2}(a) = \Delta_{\xi_2\xi_1}(a)R_{\xi_1\xi_2} \quad \forall a \in \widehat{CH_q(2)} \tag{14}$$

with $\Delta_{\xi_1\xi_2} = \pi_{\xi_1} \otimes \pi_{\xi_2}(\Delta)$. Assuming that R_{ξ_1,ξ_2} is an invertible matrix, then the intertwiner equation (14) implies:

$$\mathrm{tr}\ \Delta_{\xi_1\xi_2}(a) = \mathrm{tr}\ \Delta_{\xi_2\xi_1}(a) \quad \forall a \in \widehat{CH_q(2)} \tag{15}$$

For $a = \Gamma_x^{(0)}\Gamma_y^{(1)}$ we obtain the following constraint on the labels of the irreps which admit an intertwiner:

$$\frac{2(\lambda - \mu)}{(1-\lambda^2)^{1/2}(1-\mu^2)^{1/2}(z^2 - z^{-2})} = k \tag{16}$$

[1]To define the algebra $\widehat{CH_q(D)}$ properly we should add to (11) the equivalent to Serre's relations. These will not be relevant for the discussion in this paper.

with k an arbitrary constant. Equation (16) defines a two dimensional variety embedded in $\mathbf{C}^3$ which can be uniformized in terms of elliptic functions. Identifying k in (16) with the elliptic modulus we define a new variable φ by:

$$z^2 = cn(\varphi) + isn(\varphi) \tag{17}$$

Definig now:

$$\lambda_x = \tanh x \quad , \quad \lambda_y = \tanh y \tag{18}$$

equation (16) becomes

$$e^{x-y} = dn(\varphi) + iksn(\varphi) \tag{19}$$

which means that $x+y$ is independent of φ, therefore each point in the curve (16) can be parametrized by two complex parameters (φ, ψ) with ψ defined by

$$\tanh\left(\frac{x+y}{2}\right) = cn(\psi) + isn(\psi) \tag{20}$$

The main result of this paper is that, given two irreps lying on the same curve (16), ξ_1 (φ_1, ψ_1) and ξ_2 (φ_2, ψ_2) the intertwiner R–matrix R_{ξ_1,ξ_2} coincides with the one given in (2) (up to a diagonal change of basis) provided we identify $u = \varphi_1 - \varphi_2$. Notice from (17) that the "affine" parameter z becomes the standard exponential in the trigonometric limit. The derivation of (2) is long but straighforward, and we have used the following identity among elliptic functions:

$$e(\varphi_1 - \varphi_2) = \frac{e(\varphi_1)(dn(\varphi_1)+1)(dn(\varphi_2)+1) - k^2 e(\varphi_2)sn(\varphi_1)sn(\varphi_2)}{e(\varphi_2)(dn(\varphi_1)+1)(dn(\varphi_2)+1) - k^2 e(\varphi_1)sn(\varphi_1)sn(\varphi_2)} \tag{21}$$

Summarizing our results we have proved that the intertwiner R–matrix for two dimensional irreps of the Hopf algebra $\widehat{CH_q}(2)$ is the free fermion eight vertex solution to the Yang–Baxter equation.

3 Comments

The Sklyanin algebra [13] of the eight vertex model is determined by the corresponding elliptic curve and the anisotropy γ [14]. In the free fermionic case, i.e. $\gamma = K$, the curve, for the most general case with non–zero field, is given by (16). An important question that we will address in a future publication, is the mathematical meaning, inside $\widehat{CH_q}(2)$, of the $\gamma = K$–Sklyanin algebra.

Taking into account that the trigonometric limit of the free fermion model is given by the six vertex free fermion model and that this $R-$matrix is the intertwiner of $U_{\hat{q}}(gl\widehat{(1,1)})$, it is plausible to conjecture that the Hopf algebra $C\widehat{H_q}(2)$ plays the role, in the sense of reference [3], of hidden quantum group of the $q-$WZW model defined by $U_q(gl\widehat{(1,1)})$. More precisely we expect that the connection matrices of the $q-$KZ equation for $U_q(gl\widehat{(1,1)})$ are quantum $6-j$ symbols of $C\widehat{H_q}(2)$.

Another interesting issue is the interpretation of the eight vertex free fermion $R-$matrix as an scattering $S-$matrix in the sense of Zamolodchikov [15]. From our previous results we know that the correspondent "solitons" define now irreducible representations of $C\widehat{H_q}(2)$. Even though we cannot expect a field theory limit preserving the elliptic nature of this $S-$matrix, as a consequence of the c-theorem [16], the elliptic $S-$matrix (2) may still have a good physical meaning in the lattice, maybe related to the dynamics of the cnoidal waves in a Toda lattice [17].

Finally and based on the previously mentioned close connection between $N=2$ soliton $S-$matrices and intertwiners of $U_q(gl\widehat{(1,1)})$, it is natural to wonder if some relevant information on $N=2$ integrable models is still hidden in the quantum Clifford algebra $C\widehat{H_q}(2)$.

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