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WHEN CAN YOU IMMUNIZE A BOND PORTFOLIO?

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Abstract

The object of this paper is to give conditions under which it is possible to immunize a bond portfolio. Maxmin strategies are also studied, as well as their relations with immunized ones. Some special shocks on the interest rate are analyzed, and general conditions about immunization are obtained. When immunization is not possible, capital losses are measured.

Key Words and Phrases

Immunized portfolio; Maxmin portfolio; Weak immunization condition; The set of worst shocks.

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Introduction

The object of this paper is to present a general framework including and homogenizing the main results in standard literature related to financial immunization, as well as to give a new necessary and sufficient condition (the so called weak immunization condition) to guarantee the existence of a immunized portfolio.

Paper's outline is as follows. First section establishes a minimal set of hypotheses which are common to most of models, and from them, the existence of maxmin portfolios and the weak immunization condition are proved.

Once we have found conditions under which immunized portfolios exist, we will devote the second section to characterize them. As in previous literature, we will prove that immunized portfolios are thus for which the (so called) worst shock is the null shock and then, a special kind of differential must be zero. This permits to obtain the classical duration measures and some new ones, depending of the possible shocks.

We will introduce the set of worst shocks in section three and will find general expressions which give us how much money we could lose if immunization were not possible.

All the obtained results are applied in fourth section to analyze classical results about immunization. To be precise, we will study the results of Bierwag and Khang (1979) Prisman and Shores (1988) Fong and Vasicek (1984) and others authors, and will find new expressions about financial immunization.

Finally, we point up the most important conclusions of the paper in section five.

I The weak condition and maxmin portfolios.

Let $[0,T]$ be the time interval being $t=0$ the present moment. Let us consider n default free and option free bonds with maturity less or equal than T, and with prices $P_1, P_2, \ldots, P_n$ respectively. We will represent by K the set of the admissible shocks over the interest rate, and therefore, K will be a subset of the vector space of real valued functions defined on $[0,T]$. If the elements of K are only constant functions we will be working with additive shocks. If these elements are polynomials we will have polynomial shocks like the ones considered by Prisman and Shores (1988) or Chambers et al (1988) between others, and if these elements are continuously differentiable functions we are under the hypothesis of Fong and Vasicek (1984). Clearly, more situations about the functions in K may be considered.

Let m $(0<m<T)$ be the investor planning period and consider n real valued functionals

$$V_i : K \longrightarrow R \qquad i=1,2,\ldots,n$$

such that $V_i(k)$ (where $k \in K$ is any admissible shock) is the i-th bond value at time m if the shock k takes place.

We will assume the following hypotheses

H1 K is a convex set which contains the zero shock (denoted by $k=0$).

H2 V_i is a convex functional for $i=1,2,\dots,n$.

H3 There exists a constant $R>0$ such that

$$V_i(0) = RP_i \qquad i=1,2,\dots,n$$

H4 $V_i(k)>0$ for $i=1,2,\dots,n$ and for any $k\in K$.

Four assumptions are quite clear and simple, and they almost always hold in classical models about immunization. In particular, assumption H3 means that if no shock over the interest rate takes place, then the i-th bond value is proportional to its present price.

If $C>0$ represents the total amount to invest, and the vector

$$q=(q_1,q_2,\dots q_n)$$

gives us the number of units (q_i) of the i-th bond that the investor has bought, then the constraints

$$\sum_{i=1}^{n} q_i P_i = C \quad , \quad q_i \geq 0 \quad i=1,2,\dots,n \tag{1,1}$$

must hold, and the functional

$$V(q,k) = \sum_{i=1}^{n} q_i V_i(k) \tag{1,2}$$

gives us the value at time m of portfolio q if the shock k takes place. Obviously, V is a convex functional in the k variable since H2 guarantees that it is a non negative linear combination of convex functionals.

The following result only means that if there is no shock over the interest rate, then the value (at m) of portfolio q is proportional to the capital C.

Proposition 1.1. $V(q,0)=RC$ for any q subject to (1,1).

Proof. From the assumptions we have

$$V(q,0) = \sum_{i=1}^{n} q_i V_i(0) = \sum_{i=1}^{n} q_i RP_i = RC$$

■

Because of latter proposition we will say that RC is the promised value and it is common for all feasible portfolios q (that is, portfolios q such that (1,1) holds). Let us introduce the guaranteed value by portfolio q which will be

$$\overline{V}(q) = \text{Inf}\ \{V(q,k); k\in K\} \tag{1,3}$$

that is, the infimum of all possible values (at m) of portfolio q depending of the shock $k\in K$.

The following result shows that the promised amount is greater than the guaranteed one.

Proposition 1.2. The following inequalities hold for any feasible portfolio q

$$0 \leq \overline{V}(q) \leq RC$$

Proof. First inequality follows from H4 and from H1 we have

$\overline{V}(q) = \text{Inf } \{V(q,K); k \in K\} \le V(q,0) = RC$ ∎

A portfolio is called maxmin if it guarantees as much as possible and immunized if it guarantees the promised amount RC. To introduce this concepts in a formal way we will consider the optimization program

$$\left.\begin{array}{c} \text{Max } \overline{V}(q) \\ \text{subject to } (1,1) \end{array}\right\} \qquad \text{(P1)}$$

Definition 1.3. A feasible portfolio q is maxmin if it solves program P1, and immunized if $\overline{V}(q) = RC$. ∎

Theorem 1.4. If q is a immunized portfolio, then it is maxmin.

Proof. If q is a immunized portfolio then $\overline{V}(q) = RC$ and applying proposition 1.2. to any feasible portfolio q' we have

$$\overline{V}(q') \le RC = \overline{V}(q)$$

and therefore q solves P1. ∎

Latter result has been proved in a extraordinarily simple way because of the apparent power of the introduced notation. It was established at first time by Bierwag and Khang (1979) for a problem in which the bonds pay a continuous coupon, the shocks are additives and there are two bonds with duration greater and less than m respectively. Later, Prisman (1986) generalized the result of Bierwag and Khang.

We are going to study conditions under which a immunized (or maxmin) portfolio exists. First we need the following lemma.

Lemma 1.5. Let $\mu_0 \ge 0$. Then, there is a feasible portfolio q such that

$$\overline{V}(q) \ge \mu_0 C$$

if and only if for every admissible shock $k \in K$ there is at least a bond i (which depends of k) such that

$$V_i(k) \ge \mu_0 P_i$$

Proof. Let us assume the existence of portfolio q. Then

$$V(q,k) \ge \mu_0 C$$

for any admissible shock k. From (1,1) and (1,2)

$$\sum_{i=1}^{n} q_i V_i(k) \ge \sum_{i=1}^{n} q_i \mu_0 P_i$$

for any k. Since the terms in both sides of last inequality are non negative, this is only possible if at least for one i we have

$$V_i(k) \ge \mu_0 P_i$$

Conversely, let us consider that the given condition holds and

let us prove the existence of q portfolio.
The following set is obviously convex

$$A = \{(\alpha_1, \alpha_2, \dots, \alpha_n); \quad \alpha_j \le \mu_0 P_j \quad j=1,2,\dots,n\}$$

Consider also the set

$$B = \{(\alpha_1, \alpha_2, \dots, \alpha_n); \ \exists k \in K \text{ with } \alpha_j \ge V_j(k) \quad j=1,2,\dots,n\}$$

Let us prove that B is a convex set. In fact, if $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and $(\beta_1, \beta_2 \dots, \beta_n)$ are in B, we can find two shocks k and k' in K such that

$$\alpha_j \ge V_j(k) \quad , \quad \beta_j \ge V_j(k') \quad , \ j=1,2,\dots,n$$

sine K is a convex set, given τ with $0 \le \tau \le 1$, $\tau k+(1-\tau)k'$ is in K and being V_j a convex functional for any j, we have that

$$\tau\alpha_j + (1-\tau)\beta_j \ge \tau V_j(k) + (1-\tau)V_j(k') \ge V_j[\tau k + (1-\tau)k']$$

$$j=1,2,\dots,n$$

and $\tau(\alpha_1, \alpha_2, \dots, \alpha_n) + (1-\tau)(\beta_1, \beta_2, \dots, \beta_n)$ is in B.
We will prove now that there are no points in A^o (interior of A) and B simultaneously. In fact, if $(\alpha_1, \alpha_2, \dots, \alpha_n)$ were in both A^o and B, then $\alpha_j < \mu_0 P_j$ $j=1,2,\dots,n$ and we could find a shock k such that

$$\alpha_j \ge V_j(k) \qquad j=1,2,\dots,n$$

Therefore

$$\mu_0 P_j > \alpha_j \ge V_j(k) \qquad j=1,2,\dots,n$$

and it is a contradiction with the assumptions.
The separation theorems (see Luenberger (1969)) show that we can find n real numbers $q'_1, q'_2, \dots, q'_n$ such that q'_i is not zero for at least one i and

$$\sum_{j=1}^{n} q'_j \alpha_j \le \sum_{j=1}^{n} q'_j \beta_j$$

if $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is in A and $(\beta_1, \beta_2, \dots, \beta_n)$ is in B. In particular, taking $\alpha_j = \mu_0 P_j$ and $\beta_j = V_j(k)+r_j$ $j=1,2,\dots,n$ where k is any admissible shock and r_j is any non negative number,

$$\mu_0 \sum_{j=1}^{n} q'_j P_j \le \sum_{j=1}^{n} q'_j (V_j(k) + r_j) \tag{1,4}$$

if k is admissible and $r_j \ge 0$ $j=1,2,\dots,n$.
We have $q'_1 \ge 0$ because if we had $q_1 < 0$ then the right side in last inequality would tend to minus infinite if r_1 tends to infinite and this is not compatible with the inequality. Analogously $q'_2 \ge 0$, $\dots, q'_n \ge 0$. Since at least one q'_i is not zero,

$$S = \sum_{j=1}^{n} q'_j P_j > 0$$

and then, taking

$$q_j = Cq'_j \ / \ S \qquad j=1,2,\dots,n$$

that $(q_1, q_2, \dots, q_n)$ verifies (1,1) and from (1,2) and (1,4) (with $r_j = 0$ for any j)

$$\mu_0 C \le V(q,k)$$

for any shock k ■

As it has been shown, the lemma is proved with technics of convex analysis, which were applied to immunization theory in Prisman (1986).

The first interesting consequence of lemma 1.5. is that under the hypothesis H1 H2 H3 and H4 one can always find a maxmin portfolio.

Theorem 1.6. Program P1 has solution, that is, there always exists a maxmin portfolio.

Proof. Let us consider the following real valued functional over the admissible shocks

$$U(k) = \text{Max} \{ V_1(k)/P_1 \ , \ V_2(k)/P_2 \ , \ \dots \ , \ V_n(k)/P_n \}$$

for $k \in K$.

Define

$$\mu_0 = \text{Inf} \ \{U(k); k \in K\}$$

Then, for any shock k we have $U(k) \ge \mu_0$ and then there exists a bond (which depends of k) such that

$$\frac{V_i(k)}{P_i} \ge \mu_0$$

The latter lemma shows that we can find a portfolio q such that

$$V(q,k) \ge \mu_0 C$$

for any $k \in K$ and then

$$\overline{V}(q) = \text{Inf} \ \{V(q,k); k \in K\} \ge \mu_0 C$$

We will have proved that q is a solution of P1 if we show that $\overline{V}(q') \le \mu_0 C$ for any portfolio $q' = (q'_1, q'_2, \dots, q'_n)$ subject to (1,1).

Clearly, for any feasible shock k we have

$$\overline{V}(q') \le V(q',k) = \sum_{i=1}^{n} q'_i V_i(k) = \sum_{i=1}^{n} q'_i P_i \frac{V_i(k)}{P_i} \le U(k) \sum_{i=1}^{n} q'_i P_i$$

$= C \ U(k)$

Therefore

$$\overline{V}(q') \le C \ \text{Inf}\{U(k); k \in K\} = C\mu_0$$

■

Let us introduce now the "weak immunization condition".

Definition 1.7. We will say that the set of the admissible shocks K and the n bonds considered verify the weak immunization condition if for any shock $k \in K$ there exists al least one bond i (which depends of k) such that

$$V_i(k) \geq RP_i \qquad \blacksquare$$

The interpretation of the latter concept may be as follows. Let us consider a investor interested in a immunized portfolio, that is, a portfolio which guarantees the promised amount RC. Then, if our investor knew the real future shock k then he (or she) would buy that bond which does not lose value, that is, the bond such that $V_i(k) \geq V_i(0) = RP_i$ (see assumption H3). If the investor can find this bond for any feasible shock, then we have the "weak immunization condition" and this name is because if it holds and we know the future shock then we can immunize.

Now we are going to present a surprising result which shows that immunization is possible under the weak immunization condition (of course, without assuming that we know the future shock k). This is perhaps the most important result in the present paper and will be applied in future sections to explain why immunization is not possible in some classical models. We will also present situations in which immunization is viable and will introduce the concept of "set of worst shocks".

Theorem 1.8. The weak immunization condition is necessary and sufficient to guarantee the existence of a immunized portfolio.

Proof. It is a immediate consequence of lemma 1.5. taking μ_0=R ■

Latter theorem has another interpretation. "Immunization is not possible if and only if there is an admissible shock for which all the bonds lose value at m"

Let us remark that theorems 1.6. and 1.8. show that the converse of theorem 1.4. is false in general. In fact, the maxmin portfolio (i.e. a portfolio that makes maximum the guaranteed amount at time m) always exists but it will be seen that the weak immunization condition is not always satisfied, and then immunized portfolio does not exist. Moreover, it is well known that in classical literature one can find many models in which immunization is not possible. Anyway, it can be easily proved in our general context that if immunized portfolio does exist then immunized and maxmin portfolio are equivalent concepts.

II Looking for immunized bond portfolios

Once we have characterized the existence of immunized portfolios, we will show in this section a way to find them. For it, given a feasible portfolio q we consider the following optimization program

$$\left.\begin{array}{l}\text{Min } V(q,k) \\ \text{subject to } k \in K\end{array}\right\} \qquad (Pq)$$

Many authors like for instance Bierwag (1977), Bierwag and Khang (1979), etc, prove in different models (with continuous or discrete capitalization, with additive, multiplicative, polynomial, differentiable shocks, etc) that the immunized portfolios have the zero shock as the "worst shock". The following result shows that in a general framework, like the one we are working with, the property is also valid.

Proposition 2.1. A feasible portfolio q is immunized if and only the zero shock k=0 solves the program Pq.

Proof. If k=0 solves Pq, then

$$V(q,k) \geq V(q,0)$$

for all $k \in K$ and from proposition 1.1.

$$V(q,k) \geq RC$$

Therefore

$$\overline{V}(q) = \text{Inf}\{V(q,k); k \in K\} \geq RC$$

and q is immunized since the opposite inequality follows from proposition 1.2.

Conversely, if q is immunized $\overline{V}(q) = RC$ and given any $k \in K$ we have

$$V(q,k) \geq \text{Inf}\{V(q,k'); k' \in K\} = \overline{V}(q) = RC = V(q,0)$$

and the zero shock solves program Pq ∎

Latter proposition may be useful to obtain extensions of theorem 1.8. We are going to do it applying the local-global theorem of mathematical programming (see for instance Luenberger (1969)) which shows that for a convex optimization program (like program (Pq)) the concepts of local minimum and global minimum are equivalent.

From now on we need to assume the following additional assumption which almost always holds in classical models.

H5 The set K of admissible shocks is a subset of a normed space X whose elements are real valued functions over the interval [0,T].

The concept of normed space can be found for instance in Luenberger (1969), and examples of X could be $C^p[0,T]$ (functions with p continuous derivatives) or $L^p[0,T]$, that is, the space of measurable functions $f: [0,T] \longrightarrow R$ such that

$$\int_0^T |f(t)|^p \, dt < \infty$$

where p is a fixed natural number such that $p \geq 1$. Another many possibilities for X can be considered.

Proposition 2.2. There exists a immunized portfolio q if and only if there exists V neighborhood of zero in X such that $K \cap V$ and the n considered bonds verify the weak immunization condition.

Proof. The given condition is obviously necessary since we can take V as the whole space X and apply theorem 1.8.
Conversely, if the neighborhood V exists, then we can consider that V is convex, and theorem 1.8. guarantees that there exists a

feasible portfolio q such that

$$V(q,k) \geq RC = V(q.0)$$

holds for $k \in K \cap V$. Then, the zero shock is a local solution of program (Pq) and the local-global theorem shows that it is a global solution. Therefore, the result follows from proposition 2.1. ■

Latter result has two interpretations. First one is as follows "immunization is possible if (and only if) for any arbitrary small shock k we can find a bond (which depends of k) which does not lose value after the shock k". The second one may be written in the following way "immunization is not possible if and only if we can find a shock as small as wanted for which all the bonds lose value at m"

Another interesting consequence of proposition 2.1. is that we can apply the necessary optimality conditions to program (Pq) and characterize the immunized portfolios. Let us remark that these necessary optimality conditions are also sufficient since this program is convex. The concepts and results about optimization which will appear from now till the end of this section can be found in Luenberger (1969).

From now on let us assume the following hypothesis

H6 The functionals V_i $i=1,2,\ldots,n$ are Gateaux differentiable with respect to their variable k in an open set containing the zero shock.

Latter hypotheses may be written more easily if we consider shocks k which depend of p parameters (for instance polynomial shocks with p-1 degree). If this dependence is linear, then H6 means that V_i is differentiable with respect to the parameters.

Theorem 2.3. If the zero shock is interior to the set K of admissible shocks, then q is a immunized portfolio if and only if

$$\left.\frac{\partial V(q,k)}{\partial k}\right|_{k=0} = 0$$

where the left side term represents the Gateaux differential of the functional V with respect to its variable k evaluated in k=0.

Proof. It is a immediate consequence of proposition 2.1. ■

Latter expression may be developed in very general situations. To show it we are going to obtain equivalent conditions in the case of continuous capitalization. The q portfolio pays a continuous coupon $c(t) \geq 0$ for $0 \leq t \leq T$. If $g(s)$ $(0<s<T)$ represents the instantaneous forward interest rate and $k(s)$ is a shock on $g(s)$, then the q portfolio value at time m is given by

$$V(q,k) = \int_0^T c(t)\exp\left(\int_t^m (g(s)+k(s))ds \right) dt \qquad (2,1)$$

It is easily proved in this case that the constant R is given by

$$R = \exp\left(\int_0^m g(s)ds \right) \tag{2,2}$$

and manipulating in (2,1) we have

$$V(q,k) = R\int_0^T c(t)\exp\left(-\int_0^t g(s)ds + \int_t^m k(s)ds\right)dt \tag{2,3}$$

The differential of functional V with respect to its variable k evaluated in the zero shock and applied over the shock k (that is, the derivative of functional V evaluated on the zero shock and in the direction given by the shock k) will be given by

$$V'(q,0,k) = R\int_0^T c(t)\left(\exp\left(-\int_0^t g(s)ds\right)\int_t^m k(s)ds\right)dt \tag{2,4}$$

Theorem 2.4. The q portfolio is immunized if and only if one of the tow following equivalent conditions holds

$$\int_0^T c(t)\exp\left(-\int_0^t g(s)ds\right)\int_t^m k(s)ds\, dt = 0 \tag{2,5}$$

for any admissible shock k.

$$\int_0^m k(s)\left(\int_0^s c(t)\exp\left(-\int_0^t g(s)ds\right) dt\right)ds =$$

$$\int_m^T k(s)\left(\int_s^T c(t)\exp\left(-\int_0^t g(s)ds\right)dt\right)ds \tag{2,6}$$

for any admissible shock k.

Proof. (2,5) is a immediate consequence of (2,4) and theorem 2.3. and (2,6) follows from (2,5) if we change the integration order ■

Expression (2,5) allows us the following lecture. The sum (in this case integral) of the current value of each coupon multiplied by the shocks which affect it must be zero. A coupon at time t is affected by the shocks in the interval [t,m]

Expression (2,6) allows us another lecture. The sum (integral) of the shocks till m multiplied by the current value of the coupon stream that the shock affects must be equal to the sum (integral) of the shocks from m till T multiplied by the current value of the coupon stream that the shock affects. Expression (2,5) and (2,6) give us lectures on the immunization condition in more intuitive terms that theorem 2.3.

Expression (2,5) and (2,6) can be applied in very general situations (that is, all situations in which assumptions H1 - H6 hold) to find the immunized portfolio q if it exists, that is, if weak immunization condition (or local weak immunization condition) holds.

III The set of worst shocks

Once we know when immunization is possible and how to find the immunized portfolios if they exists, following the approach of Prisman and Shores (1986) or Fong and Vasicek (1984) we are now interested in measuring how much money we could lose if immunization were not possible. First step is to introduce a basic concept

Definition 3.1. We will say that a set $k_1, k_2, \ldots, k_h$ of admissible shocks is a set of worst shocks, if given any shock $k \in K$ there exists h real numbers $\lambda_1, \lambda_2, \ldots, \lambda_h$ (which depend of k) such that

$$\sum_{j=1}^{h} \lambda_j k_j \in K$$

and

$$V_i(k) \geq V_i\left(\sum_{j=1}^{h} \lambda_j k_j\right) \qquad i=1,2,\ldots,n$$

■

The latter concept just means that the value at m of the n considered bonds is always minored by their values considering linear combinations of elements in the set of worst shocks. A immediate consequence is the following one. Its proof is very simple and omitted

Proposition 3.2. Under the assumptions of the latter definition, the following inequality holds for any feasible portfolio q (that is, for any q subject to (1,1)).

$$V(q,k) \geq V\left(q, \sum_{j=1}^{h} \lambda_j k_j\right)$$

■

As it is well known, a convex function is always minored by its tangent plane. It is also true for convex functionals in normed spaces (see Luenberger (1989)) and we can apply this fact to obtain some properties of the functional V.

Let ν_j j=1,2,...,h be the value of the Gateaux differential of V with respect to its variable k, evaluated at k=0, and applied over k_j (see Lüenberger (1969). See also (2,4)). Then we have the following result

Theorem 3.3 Under the assumptions of proposition 3.2. the following inequality holds

$$V(q,k) - V(q,0) \geq \sum_{j=1}^{h} \nu_j \lambda_j \qquad \blacksquare$$

Latter inequality gives us the minimum value of V(q,k) if we consider shocks such that $\lambda_1, \lambda_2, \dots \lambda_h$ (which depend of k) are bounded. Since V(q,0) = RC (the promised value) the inequality may also be written as follows

$$\frac{V(q,k) - RC}{RC} \geq \sum_{j=1}^{h} \frac{\nu_j}{RC} \lambda_j \qquad (3,1)$$

and we are measuring the losses per promised dollar. It will be shown in next section that in classical models the coefficients ν_j/RC can be interpreted as duration measures, though in the general context we are working with, many different situations could be considered.

IV Some interesting particular situations

Let us apply the developed theory to some particular interesting cases. First, we will analyze some classical results, and later, we will give new properties.

Let us consider that the admissible shocks are polynomial shocks with degree non greater than p. Their general form becomes

$$K(s) = \lambda_0 + \lambda_1 s + \lambda_2 s^2 + \dots + \lambda_p s^p$$

A base of the space of p degree polynomials is

$$k_0(s) = 1 \ , \ k_1(s) = s \ , \dots , \ k_p(s) = s^p \qquad (4,1)$$

and any shock may be written as a linear combination of $k_0, \dots, k_p$.

It may by easily proved that (2,5) holds for any shock k if it holds for any shock in the base.

Clearly

$$\int_t^m s^i ds = \frac{1}{i+1} (m^{i+1} - t^{i+1})$$

and from (2,5)

$$\int_0^T c(t)\, e^{-\int_0^t g(s)ds} (m^{i+1} - t^{i+1})dt = 0 \qquad (4,2)$$

$i=0,1,2,\dots,p$

Manipulating

$$m^{i+1} = \frac{1}{C} \int_0^T t^{i+1} c(t)\, e^{-\int_0^t g(s)ds}\, dt$$

$i=0,1,\dots,p \qquad (4,3)$

what may be written as

$$m^{i+1} = D_{i+1} \qquad i=0,1,\dots,p \qquad (4,4)$$

where D_{i+1} is i-th duration of q portfolio and is given by the right side member of (4,3). That constitutes the duration measures vector obtained by Chambers *et al.* (1988) and characterizes the immunized portfolios. Actually, expression (2,5) is quite general and therefore, many other situations could be considered, and another results would be obtained.

If we only consider additive shocks, we are in the latter case with p=0 and (4,3) becomes

$$m = \frac{1}{C} \int_0^T tc(t)\, e^{-\int_0^t g(s)ds}\, dt \qquad (4,5)$$

which is the standard result on immunization against additive shocks. See for instance Fisher and Weil (1971) or Bierwag (1977).

Bierwag and Khang (1979) proved that a bond with duration greater than m increases its value (at m) if we have a negative additive shock. Moreover if the bond has duration less than m it increases its value if the shock is additive and positive. Now, if we have the two needed bonds we are under the hypotheses of theorem 1.8. and then we can conclude (like Bierwag and Khang did) that a immunized portfolio does exist because weak immunization condition holds.

Prisman and Shores (1988) proved that (4,3) has no solution if $p \geq 1$, and then immunization against polynomial shock is not possible. Their proof is based in analytic arguments and we are going to give another one with a very simple interpretation.

An example of admissible polynomial shock is given by $k_0(t) = \lambda(t-m)$ where λ is any positive number. Since $K_0(t)<0$ if $t<m$ and $k_0(t)>0$ if $t>m$ we have that the instantaneous forward interest rate is going to fall from t=0 till t=m and it is going to increase for t>m. Then, the coupons we have to capitalize (the coupons paid before m) will lose value at m and so will the ones we have to discount (the ones paid later than m). In this situation only the zero coupon bond would not lose value at m, and if this bond is not in the market, to anticipate the shock k_0 does not permit to save it. Therefore, weak immunization condition fails and immunization is not possible.

Latter case will appear very often (not only for polynomial shocks) because in many situations we can find a shock k(t) such that $k(t)<0$ if $t<m$ and $k(t)>0$ if $t>m$. This fact shows that weak immunization condition may be more appropriate to study the existence of immunized portfolios than analytic technics.

Coming back to the case of polynomial shocks, since immunization is not possible we could be interested in the total amount of money we might lose. In such a case, we can apply the results of section three. The set given in (4,1) is a set of worst shocks since it is a base of admissible shocks. The coefficients ν_i of (3,1) are given in this case by (see also (2,4)) $RC(m^{i+1}- D_{i+1})$ and (3,1) becomes

$$\frac{V(q,k) - RC}{RC} \geq \sum_{i=0}^{p} \lambda_i (m^{i+1} - D_{i+1})/(i+1) \qquad (4,6)$$

where λ_i $i=0,1,\dots,p$ are the coefficients of polynomial k. The expression was obtained by Prisman and Shores (1988).

If we consider shocks which are functions with p ($p\geq 1$) continuous derivatives being p any impair number, then Taylor's formula applied to any admissible shock k gives us $k(t)\geq k_0(t)$ if $t<m$ and $k(t)\leq k_0(t)$ if $t>m$, where

$$k_0(t) = \sum_{i=0}^{p-1} \frac{k^{(i}(m)}{i!}(t-m)^i + \frac{\lambda}{p!}(t-m)^p \qquad (4,7)$$

and

$$\lambda = \mathrm{Max}\ \{k^{(p}(t)\ ;\ 0 \leq t \leq T\} \qquad (4,8)$$

Therefore the value of portfolio q (for any q) if shock k takes place is minored by its value if k_0 is the future shock. It means that the set of (4,1) is a set of worst shocks in this case, and (4,6) may be applied to obtain the maximal losses. In particular, if p=1 then

$$k_0(t) = k(m) + \lambda t - \lambda m \quad ; \quad \lambda = \mathrm{Max}\ \{k'(t);\ 0 \leq t \leq T\}$$

$$\frac{V(q,k) - RC}{RC} \geq \frac{V(q,k_0) - RC}{RC} \geq (m-D_1)(k(m)-m\lambda) + (m^2-D_2)\lambda/2$$

(4,9)

and taking q such that $m = D_1$ (that is, immunized against additive shocks)

$$\frac{V(q,k) - RC}{RC} \geq \frac{\lambda}{2}(m^2 - D_2)$$

which is the Fong and Vasicek formula.

Fong and Vasicek formula has been extended in Montrucchio and Peccati (1991) (see also Shiu (1987)) to the case of non differentiable shocks. The authors show how $(m^2 - D_2)$ gives us the maximal losses if $m - D_1 = 0$ and we change λ by the Dini's derivative of the shock k.

Fong-Vasicek and Montrucchio-Peccati formulas can be applied when the shock derivative (or its Dini's derivative) can be bounded. If these derivatives are bounded, then the shock variations

$$\{|k(t_2)-k(t_1)| \; ; \; 0 \leq t_1 \, , \, t_2 \leq T\}$$

are also bounded but the converse is false in general (small shocks could have "very big" derivative). Now we will apply (3,1) to obtain a new expression for maximal losses. This expression gives us the losses depending of the variations of shock k, and in practical situations it could be easier to determine the shock variations than its derivative.

Let us assume that the set K of admissible shocks is the set of bounded and integrable functions in [0,T]. Let us define

$$k_1(t) = \begin{cases} 1 \text{ if } t \leq m \\ 0 \text{ if } t > m \end{cases} \qquad k_2(t) = \begin{cases} 0 \text{ if } t \leq m \\ 1 \text{ if } t > m \end{cases} \qquad (4,10)$$

Then $\{k_1, k_2\}$ is a set of worst shocks. In fact, given any admissible shock k, consider

$$\lambda_1 = \text{Inf } \{k(t) \; ; \; 0 \leq t \leq m\} \quad , \quad \lambda_2 = \text{Sup } \{k(t); \; m \leq t \leq T\} \qquad (4,11)$$

and clearly $k(t) \geq \lambda_1 k_1(t) + \lambda_2 k_2(t)$ if $t<m$ and the opposite inequality holds for $t>m$. Therefore $V(q,k) \geq V(q,\lambda_1 k_1 + \lambda_2 k_2)$ and definition 3.1. holds.

(2,4) shows that

$$V'(q,0,k_1) = R\int_0^m c(t)e^{-\int_0^t g(s)ds} (m-t)dt \qquad (4,12)$$

and

$$V'(q,0,k_2) = R\int_m^T c(t)e^{-\int_0^t g(s)ds}(m-t)dt \qquad (4,13)$$

Let us consider the present values of the coupons paid before m and after m respectively, which are given by

$$C' = \int_0^m c(t)e^{-\int_0^t g(s)ds}dt \quad , \quad C'' = \int_m^T c(t)e^{-\int_0^t g(s)ds}dt \qquad (4,14)$$

and if these numbers are not zero let us define the duration of coupons paid before m and after m respectively by

$$D' = \frac{1}{C'}\int_0^m tc(t)e^{-\int_0^t g(s)ds}dt \quad , \quad D'' = \frac{1}{C''}\int_m^T tc(t)e^{-\int_0^t g(s)ds}dt \qquad (4,15)$$

Then (4,12) and (4,13) can be written as

$$V'(q,0,k_1) = RC'(m-D') \quad , \quad V'(q,0,k_2) = RC''(m-D'')$$

and therefore, from (3,1) we have

$$\frac{V(q,k) - RC}{RC} \geq \lambda_1 \frac{C'}{C}(m-D') + \lambda_2 \frac{C''}{C}(m-D'') \qquad (4,16)$$

where k is any admissible shock and λ_i (i=1,2) is given by (4,11).

Latter expression can be rewritten since it is clear that $C=C'+C''$ and $CD_1= C'D' + C''D''$, and therefore

$$\frac{C'}{C}(m-D') = (m-D_1) + \frac{C''}{C}(D''- m)- \qquad (4,17)$$

and

$$\frac{V(q,k) - RC}{RC} \geq \lambda_1(m-D_1) + \frac{C''}{C}(D''- m)(\lambda_1 - \lambda_2) \qquad (4,18)$$

what gives us another boundlessness for capital losses which may be applied for very general shocks.

Let us remark that if we immunize against additive shocks (that is, if we take $m = D_1$) , then the losses are measured by

$$\frac{C''}{C}(D'' - m)(\lambda_1 - \lambda_2) \qquad (4,19)$$

and a possible strategy (alternative to the Fong and Vasicek or Prisman and Shores ones) could be to minimize $C''(D''- m)$. This strategy is to minimize a dispersion measure (like the Fong and

Vasicek one) because it is clear that if $m = D_1$ then from (4,17)

$$C''(D''- m) = C'(m - D')$$

Another possibilities could be analyzed, since in all considered situations about the shocks (polynomial, continuously differentiable, bounded and integrable) and in others that we could have studied, weak immunization condition shows that immunization against shocks in many convex sets is possible (that is, immunization against shocks in a convex set which does not contain a shock negative for t<m and positive for t>m can be possible if there are appropriate bonds in the market). Then, the investor can immunize against shocks in these sets if it is considered that they describe the reasonable changes in the interest rate. We always have that (3,1) gives the maximal capital losses if the real shock does not belong to the considered set.

V Conclusions

I. Under a simple set of assumptions a maxmin portfolio exists and the weak immunization condition guarantees the existence of a immunized portfolio.

II. This condition shows quite well why immunization is not possible in many models. The reason is that one can find a shock k such that all the bonds lose value at the investor planning period m. This shock k (if it exists) may be found "as small as wanted".

III. To find this k (in many models) one just have to look for a shock such that the interest rate falls before m and increases after m.

IV. If the latter shock k does exist, the possible capital losses can be measured by mean of the set of worts shocks. This set is given for a finite number of shocks (K_i i = 1,2,...,h) such that their linear combinations can verify

$$\sum_{i=1}^{h} \lambda_i k_i(t) \le k(t) \text{ if } t \le m \quad \text{and} \quad \sum_{i=1}^{h} \lambda_i k_i(t) \ge k(t) \text{ if } t \ge m$$

Then (3,1) gives the maximal losses in a general context.

V. Expression (4.18) is a particular case of (3,1) if we are working with bounded and integrable shocks. It can be applied for any portfolio and therefore, a possible strategy for the investor could be to immunize against additive shocks and to minimize $C''(D''-m)$. This strategy is to minimize a dispersion measure, and it is not incompatible with including a bond with m maturity (empirical result appeared in Bierwag *et al.* (1993)) due to this bond, which pays the biggest amount at the instant m, seems to help in minimizing any dispersion measure.
Actually, to minimize the dispersion measure $C''(D''-m)$ or to minimize the Fong-Vasicek one $D_2 - m^2$ (or both if it is possible) depends of the investor opinion about the changes in the interest rate. First one gives capital losses depending of the shock variations, and second one can be applied if we anticipate the

shock derivative.

VI. The investor can also work with reasonable assumptions about the possible shocks on the interest rates. Then, maxmin portfolio (which always exits) can be determined, and later, (3,1) (or its consequences (4,6), (4,9), (4,18), etc) can be applied to measure the possible capital losses in different contexts about the shocks.

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