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Departamento de Economía de la Empresa
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34-1) 624-9608

RANDOM WALKS AND THE TEMPORAL DIMENSION OF RISK

Javier Estrada *

Abstract

The assumption that stock prices follow a random walk has critical implications for investors and firms. Among those implications is the fact that data frequencies and investment horizons are irrelevant (as defined below) when evaluating the risk of a security. However, if stock prices do not follow a random walk, ignoring either issue may lead investors to make misleading decisions. Using data from the first half of the decade for thirteen European securities markets, I first argue that stock prices in these markets (not surprisingly) do not follow a random walk. Then, I show that investors that assume otherwise are bound to underestimate the total and systematic risk (and overestimate the compound and risk-adjusted returns) of European stocks. The underestimation of risk ranges between .53% and 2.94% a month, and averages 1.25% a month.

Keywords: Time series of stock returns. Random Walks. Risk.
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* Department of Business Economics, Universidad Carlos III de Madrid (Madrid, Spain). Email: <estrada@eco.uc3m.es>. I would like to thank seminar participants at Gothenburg University and Carlos III University for valuable comments. The views expressed below and any errors that may remain are entirely my own.

I- INTRODUCTION

It is not uncommon to find firms that compute their beta by running a regression between the firm's daily returns and the market's daily returns; then they plug the beta into a CAPM equation to estimate their daily cost of equity; they subsequently annualize their cost of equity (or, more generally, their cost of capital); and finally use such rate to discount the cash flows of their investment projects. Nor is it uncommon to find investors that use daily or monthly data to evaluate the expected return and risk of their securities, when in fact their investment horizon may be several years.

If stock prices follow a random walk, both procedures are perfectly acceptable. For, under such condition, the risk of a security measured in any given time interval contains all the relevant information to compute the implied risk of the security in any other time interval. And if that is the case, the widely-used linear scaling of volatility (the $T^{1/2}$ rule) is also perfectly acceptable; that is, weekly volatility can be estimated by multiplying daily volatility by the square root of 5, annual volatility can be estimated by multiplying monthly volatility by the square root of 12, and so forth.[1]

However, if stock prices do not follow a random walk, such method of estimating risk may be badly misleading. For, under such condition, the relationship between the risk of a security in different time intervals breaks down. And if that is the case, the volatility of a security in a given time interval cannot be reliably estimated from the volatility of the security in some other time interval through a linear rescaling. In other words, if stock prices do not follow a random walk, scaling volatility according to the square root of time may lead firms and investors to make wrong investment decisions.

The random walk model of stock prices generates several important implications for practitioners. Two of them are: First, that stock returns should follow a Normal distribution;[2] and, second, that the risk of a security should scale proportionally to the square

[1] The $T^{1/2}$ rule follows from a theorem that states that, if a variable x_i follows a Normal distribution with mean μ and variance σ^2, then the sum of k (*i.i.d.*) variables x_i follows a Normal distribution with mean $k\mu$ and variance $k\sigma^2$. In the case of stock returns, if there are T trading subintervals in each trading interval I_t, and returns in each subinterval are *i.i.d.* normal with mean μ and variance σ^2, then returns in the intervals I_t should be normally distributed with mean $T\mu$ and variance $T\sigma^2$. Because risk is usually measured by the standard deviation (rather than the variance) of returns, the appropriate way to convert volatility in the subintervals into volatility in the intervals is to multiply the former by the square root of T; hence, the name $T^{1/2}$ rule.

[2] This follows from the fact that, if stock prices follow a random walk, then stock returns should be *i.i.d.* And, if enough *i.i.d.* returns are collected, the central limit theorem implies that the limiting distribution of these returns should be Normal.

root of time (the $T^{1/2}$ rule). The first implication is important because investors need to assume a given distribution in order to estimate the risk of their securities. The second implication is important because it says that investors can estimate the risk of a security in any time interval, and subsequently estimate the implied risk in any other time interval through a linear rescaling.[3]

The first implication of the random walk theory, the normality of stock returns, has been widely addressed in the literature; see, for example, Mandelbrot (1963), Fama (1965), Peiró (1994), and Aparicio and Estrada (1997), among many others. The consensus of the literature is that daily stock returns do not follow a Normal distribution but some alternative specification with fatter tails. Hence, investors that use the normality assumption typically underestimate the risk of their securities.[4]

The second implication of the random walk theory, the rate at which risk scales, is addressed in this article. Both Holton (1992) and Peters (1991, 1994) address the issue directly and find that, in short horizons, volatility scales at a faster rate than indicated by a random walk. The evidence presented in this article, which also considers a short investment horizon, does support that finding.

The literature on mean reversion and variance ratios is obviously relevant to this debate. Lo and MacKinlay (1988), using weekly data on NYSE and AMEX stocks, find evidence of positive autocorrelation, particularly in portfolios of small stocks. Fama and French (1988), using monthly data on NYSE stocks, find negative autocorrelations (hence, mean reversion) in long horizons, particularly during the 1926-40 period. Poterba and Summers (1988) find positive autocorrelations in short horizons (less than one year) and negative autocorrelations in long horizons, in both U.S. and non-U.S. data. Richardson and Stock (1989), Richardson (1993), Poon (1996), and Lamoureux and Zhou (1996), on the other hand, argue that the evidence against the random walk hypothesis is weaker than the previously cited (and other) articles suggest.

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[3] Another important implication of the random walk theory is that standard statistical methods can be reliably used to test hypotheses about stock prices and stock returns. However, if returns follow some alternative specification (like, for example, a stable Paretian distribution), standard statistical inference may be badly misleading; see Fama (1965) and Peters (1991).

[4] To illustrate, Aparicio and Estrada (1997), using the same data of this study, report that the probability of obtaining a return between four and five (five and six) standard deviations from the mean is, on average, almost fifty (almost *two thousand*) times higher under a fitted scaled-*t* distribution than under a fitted Normal distribution.

The main difference between the mean reversion literature and this article lies in that the former focus on *testing* whether stock prices follow a random walk, whereas this article focus on the *consequences* of assuming a random walk when stock prices do not follow such process. In other words, the mean reversion literature looks at the scaling of volatility as a way to test the random walk hypothesis, whereas this article looks at the same issue from the point of view of quantifying the errors that stem from mistakenly assuming that such theory holds. Thus, the issue analyzed is the same but the point of view is different.

As is well known, the random walk is a more stringent condition than required by market efficiency. A market is efficient with respect to an information set Ω if an investor cannot use the set Ω to earn an abnormal return. Such condition is generally fulfilled if price changes are uncorrelated, a situation usually referred to as a martingale.[5] A random walk, on the other hand, requires that price changes be *i.i.d.* This stronger condition implies that stock returns be uncorrelated in *all* the moments of their distribution.[6]

Given this relationship between market efficiency and random walks, I should emphasize from the outset that I do not attempt in this article to test the efficiency of European securities markets. I do attempt, however, to provide evidence showing that the thirteen markets considered do not follow a random walk; and I do attempt to quantify the mistakes an investor could make if he assumes otherwise.

To that purpose, I review the nature of the problem in part II. In part III, I describe the data and show that daily stock prices in the thirteen European securities markets analyzed do not follow a random walk. In part IV, I show that investors that ignore this fact are bound to underestimate the total and systematic risk (and overestimate the compound and risk-adjusted returns) of European stocks. In part V, I argue that (as should be expected) monthly stock prices behave much more like a random walk than daily stock prices. Finally, in part VI, I summarize the main conclusions of the analysis. An appendix containing a brief description of a Brownian motion process and several tables concludes the article.

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[5] In fact, from an *economic* point of view, a market may be efficient even if statistically-significant (but "low") correlations in returns are present. This would be the case if any profits an investor can make by exploiting the linear dependence in returns are eliminated by transaction costs. Furthermore, Lucas (1978) has shown that rational expectations prices need not necessarily be a martingale, of which the random walk is a special case. (He also showed, however, that returns *appropriately adjusted by risk*, should be a martingale.)

[6] As is standard in the literature, throughout this article price changes should be understood as meaning changes *in the log* of prices. Hence, under the random walk hypothesis, price changes and stock returns are equivalent concepts.

II- THE PROBLEM

The random walk theory of stock prices, proposed way ahead of his time by Bachelier (1900) and independently rediscovered several years later by Osborne (1959), argues that changes in stock prices can be modeled with the same process that describes the movement of a particle in a fluid. Such process, usually known as Brownian motion and briefly described in part 1 of the appendix, eventually came to be known in the financial literature as a random walk.

For our purposes, it suffices to note that in Osborne's (1959) model, investors able to equate price and value trade at an equilibrium price determined by all the information available. Thus, under this and a few other conditions, at every point in time stock prices reflect all available information, and changes in stock prices only stem from unexpected new information. If successive bits of information arise independently over time, then successive price changes will also be independent.

As stated above, this theory of stock prices yields the critical implication that the temporal dimension of risk is irrelevant; that is, the risk of a security in any time interval can be straightforwardly estimated from the risk of the security in any other time interval through a linear rescaling. However, if stock prices do not follow a random walk, the relationship between the risk of a security in different time intervals breaks down, and investment horizons become a relevant issue.

Upon reflection, this should not be very surprising. Figures 1 and 2 below show the returns of two securities over time (neither of which, of course, follows a random walk). A natural question to ask is: Which security is riskier? The answer is not straightforward. The security in Figure 1 is risky for the long-term investor but not very risky for the short-term investor; the security in Figure 2, on the other hand, is risky for the short-term investor but not very risky for the long-term investor. So the next question is: Which security should offer a higher return? Again, the answer is not straightforward; the risk of each security depends on the investor's investment horizon. However, the answer *would* be straightforward if the two securities were random walks; for in that case the risk of either security in any time interval could be estimated from the risk of the security in any other time interval, which is clearly not feasible with the stocks in Figures 1 and 2. Therefore, the investment horizon of investors *is* a relevant issue when analyzing the risk of securities that do not follow a random walk.

Figure 1

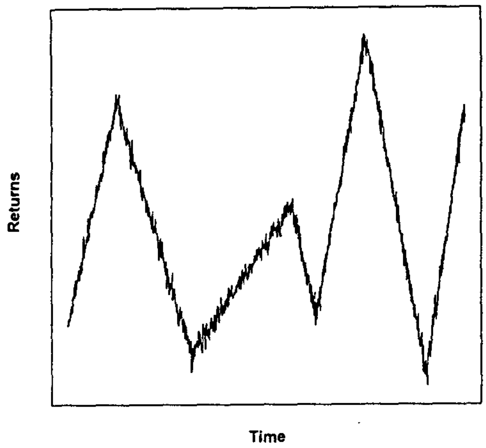
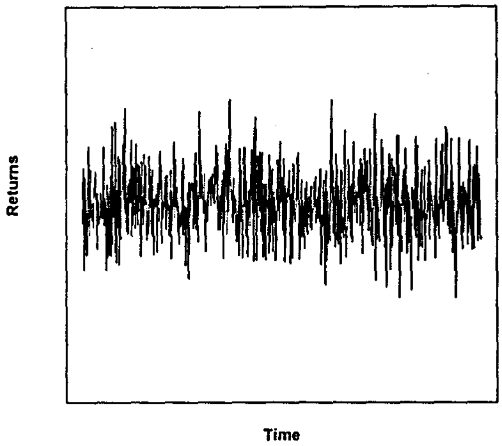


Figure 2



Emerging markets offer an interesting example of this risk puzzle. As is well known, these markets usually exhibit high short-term volatility; hence, in equilibrium, they should (and do) offer high returns. To illustrate, Erb, Harvey, and Viskanta (1996) report that between September 1979 and March 1995, the U.S. market averaged an annual return of 15.4% (with a standard deviation of 14.8%), whereas the Philippines and Poland averaged annual returns of 41.7% and 93.3% (with standard deviations of 36.8% and 90.3%), respectively. Thus, if an investor ignores short-term swings and holds on to his shares in these markets long enough (that is, until the risk-return relationship is in equilibrium), he will be rewarded with high returns. Is then this long-term investor being rewarded for risk borne by short-term investors? Are emerging markets very risky for investors whose investment horizon is one day, one month, or even one year, but perhaps not so risky for long-term investors? If stock prices do not follow a random walk, these questions do not have a straightforward answer.[7]

III- DO STOCK PRICES FOLLOW A RANDOM WALK?

I test in this part whether stock prices behave as a random walk using a sample of thirteen European securities markets, namely, Austria (AUS), Belgium (BEL), Denmark (DEN), England (ENG), Finland (FIN), France (FRA), Germany (GER), Italy (ITA),

[7] Perhaps one way of thinking about long-term risk in emerging markets may be in terms of the uncertainty about *when* the risk-return relationship is going to be in equilibrium. That is, at what point in time in the future an investor will be able to liquidate his positions and realize a return consistent with the risk of these markets.

Netherlands (NET), Norway (NOR), Spain (SPA), Sweden (SWE), and Switzerland (SWI). The behavior of each of these markets is summarized by a Financial Times Actuaries Index (FTAI), measured in local currency and published daily in the *Financial Times*. The sample period extends from January 1, 1990, through December 31, 1994. Sample statistics for the daily and monthly returns of these series are reported in Tables A1 and A2, respectively, in part 2 of the appendix.

The first step of the analysis consists of testing for a unit root in each of the thirteen markets under consideration. To that purpose, logs of the series were first taken and augmented Dickey-Fuller and Phillips-Perron tests were subsequently run. The results of these tests are reported below in Table 1.

TABLE 1: Unit Root Tests (Daily Data)

Augmented Dickey-Fuller Test: $\Delta\ln(I_t) = \alpha + \rho\ln(I_{t-1}) + \beta t + \gamma_1\Delta\ln(I_{t-1}) + \ldots + \gamma_q\Delta\ln(I_{t-q}) + \varepsilon_t$

Phillips-Perron Test: $\ln(I_t) = \alpha + \rho\ln(I_{t-1}) + \beta(t-T/2) + \varepsilon_t$

		Augmented Dickey-Fuller			Phillips-Perron		
Market	ρ	H_0: ρ=0	H_0: α=ρ=0	H_0: α=β=ρ=0	H_0: ρ=0	H_0: α=ρ=0	H_0: α=β=ρ=0
AUS	0.992	-2.3145	3.2161	2.2898	-1.5163	1.1504	0.7765
BEL	0.990	-2.8309	4.6954	3.1362	-2.5388	3.9948	2.6662
DEN	0.987	-1.8359	1.7488	1.1687	-1.4657	1.1457	0.7686
ENG	0.984	-3.2290	5.2279	3.6654	-2.7879	3.9742	2.8482
FIN	0.997	-1.8091	3.3744	2.3529	-1.2655	3.6108	2.7331
FRA	0.988	-2.6460	3.5682	2.3811	-2.4603	3.1140	2.0787
GER	0.988	-2.2590	3.0059	2.0048	-1.9558	2.2258	1.4917
ITA	0.996	-1.7657	2.2715	1.5144	-1.7475	2.2571	1.5053
NET	0.987	-2.4003	3.0788	2.5491	-2.5791	4.0662	3.1803
NOR	0.991	-1.3281	1.2704	0.8497	-1.2990	1.0954	0.7405
SPA	0.991	-2.4670	3.1101	2.0745	-2.1708	2.5990	1.7340
SWE	0.990	-2.3754	3.1449	2.2092	-1.9229	2.3179	1.7316
SWI	0.990	-2.3629	2.9498	2.3989	-2.3915	3.2516	2.6125

Numbers in the table represent test statistics except for ρ, which represents the coefficient from the regression $\ln(I_t)=\mu+\rho\ln(I_{t-1})+\varepsilon_t$, where I_t is the value of an index in day t. Asymptotic critical values for H_0: ρ=0, H_0: α=ρ=0, and H_0: α=β=ρ=0 at the 5% significance level are -3.41, 6.25, and 4.68, respectively. Sample size = 1,305 for all markets.

Table 1 shows that, at the 5% significance level, the unit-root hypothesis cannot be rejected in any of the markets under consideration, which is not surprising given that ρ >.98 in all markets. This finding, however, does not necessarily imply that stock prices in these markets follow a random walk. A unit-root is a necessary but not a sufficient condition for the existence of a random walk; the former (unlike the latter) does not require that stock returns be *i.i.d.* Thus, the next step is to test whether the residuals of the regressions run are in fact *i.i.d.*

Table 2 below reports the results of tests of linear and nonlinear dependence in the residuals. The first two columns show that the first autocorrelation of the residuals is clearly significant in all markets, which is not surprising given the well-known problem of nonsynchronous trading that affects indices; see, for example, Scholes and Williams (1977), Atchison, Butler, and Simonds (1987), and Lo and MacKinlay (1990). The next two columns, which report the results of Ljung-Box tests on the first 6 autocorrelations, further reject the hypothesis of uncorrelated residuals. Higher order Ljung-Box tests (not reported) yield the same results. Hence, stock returns in all thirteen markets are linearly dependent.

TABLE 2: Linear and Nonlinear Dependence in Daily Stock Returns

Market	ϕ_1	*p*-value	$Q(6)$	*p*-value	λ_1	*p*-value	λ_2	*p*-value	$Q^2(6)$	*p*-value
AUS	.2953	0.000	151.82	0.000	.0759	0.000	.8715	0.000	174.71	0.000
BEL	.1809	0.000	65.98	0.000	.1050	0.000	.7705	0.000	26.49	0.000
DEN	.2874	0.000	87.00	0.000	.1355	0.000	.6756	0.000	33.93	0.000
ENG	.1615	0.000	86.07	0.000	.1636	0.000	.7579	0.000	49.04	0.000
FIN	.2450	0.000	118.12	0.000	.0177	0.001	.9774	0.000	29.11	0.000
FRA	.1058	0.000	21.42	0.001	.0820	0.000	.8030	0.000	76.96	0.000
GER	.1040	0.001	14.64	0.012	.0486	0.000	.9054	0.000	44.62	0.000
ITA	.1998	0.000	42.56	0.000	.0890	0.000	.8825	0.000	41.07	0.000
NET	.1212	0.000	67.72	0.000	.0812	0.000	.9048	0.000	53.57	0.000
NOR	.3245	0.000	29.95	0.000	.2619	0.000	.5066	0.000	150.53	0.000
SPA	.1565	0.000	20.57	0.001	.1027	0.000	.7927	0.000	99.88	0.000
SWE	.2214	0.000	84.38	0.000	.1105	0.000	.8340	0.000	139.39	0.000
SWI	.1736	0.000	44.81	0.000	.1884	0.000	.6474	0.000	108.79	0.000

First autocorrelation of residuals estimated from the model $\varepsilon_t=\phi_0+\phi_1\varepsilon_{t-1}$, where ε_t is the residual from the model $\ln(I_t)=\mu+\rho\ln(I_{t-1})+\varepsilon_t$. GARCH coefficients estimated from the model $h_t=\lambda_0+\lambda_1(\varepsilon_{t-1})^2+\lambda_2 h_{t-1}$, where h_t is the conditional variance of ε_t. $Q(6)$ and $Q^2(6)$ are the Ljung-Box statistics for 6 autocorrelations and 6 squared autocorrelations, respectively.

Table 2 also reports two of the three parameters of the GARCH(1,1) models estimated in order to examine nonlinear dependence in the residuals. In all markets, both coefficients are clearly significant. Furthermore, Ljung-Box tests on the first 6 squared autocorrelations clearly reject the null hypothesis of nonlinear independence in the residuals of all markets. In other words, stock returns in all thirteen markets exhibit a significant degree of volatility clustering.

In sum, this evidence shows that although stock prices in the markets analyzed do have a unit root, they do not follow a random walk because they are not *i.i.d.*; they exhibit both linear and nonlinear dependence. In fact, if stock returns in these markets were *i.i.d.*, they should be normally distributed, a hypothesis that is clearly rejected in the thirteen markets analyzed;[8] see Aparicio and Estrada (1997).

[8] Some evidence on the normality of each distribution of daily stock returns can be gathered from the last four columns of Table A1. Under the assumption of normality, the coefficients of skewness and (excess) kurtosis are asymptotically distributed as $N(0,6/T)$ and $N(0,24/T)$, respectively, where T is the sample size; hence, values of these standardized coefficients outside the range [-1.96,1.96] indicate, at the 5% significance level, significant departures from normality. By these standards, Table A1 shows that all but two distributions are significantly skewed (in different directions), and that all thirteen distributions are leptokurtic.

IV- CONSEQUENCES OF NONRANDOM WALKS

The finding that stock prices in European markets do not follow a random walk due to the existence of (linear and) nonlinear dependence is uninteresting if taken by itself. After all, the same finding has been reported in many other studies and for many other markets. Thus, having established that stock prices in the markets analyzed do not follow a random walk, I go one step further and attempt to quantify the mistakes an investor could make if he assumes otherwise. To that purpose, I analyze the volatility, beta, and Sharpe ratio of each market, as well as portfolios of all thirteen markets.

1.- Volatility

As argued above, if stock prices do not follow a random walk, estimating volatility through a linear rescaling (the $T^{1/2}$ rule) may be badly misleading. Table 3 below reports a quantification of the mistakes an investor could make if he estimates monthly risk on the basis of daily data.[9]

TABLE 3: Daily, Monthly, and Implied Monthly Volatility

Market	ODV	OMV	IMV	OSF	RD	AD
AUS	1.1853	8.3512	5.5595	49.64	1.50	2.79
BEL	0.7190	4.6504	3.3724	41.83	1.38	1.28
DEN	0.8232	5.1179	3.8612	38.65	1.33	1.26
ENG	0.8185	4.7308	3.8391	33.41	1.23	0.89
FIN	1.2440	8.7771	5.8349	49.78	1.50	2.94
FRA	0.9996	5.4079	4.6885	29.27	1.15	0.72
GER	1.0478	5.6230	4.9146	28.80	1.14	0.71
ITA	1.3691	7.4437	6.4216	29.56	1.16	1.02
NET	0.7273	3.9566	3.4113	29.59	1.16	0.55
NOR	1.3307	6.8552	6.2415	26.54	1.10	0.61
SPA	1.1249	6.4968	5.2762	33.36	1.23	1.22
SWE	1.2504	7.6259	5.8649	37.19	1.30	1.76
SWI	0.9313	4.8953	4.3682	27.63	1.12	0.53
Averages	1.0439	6.1468	4.8965	35.02	1.25	1.25

Volatility (measured by standard deviation) reported in percentages. ODV = Observed Daily Volatility; OMV = Observed Monthly Volatility; IMV = Implied Monthly Volatility; OSF = Observed Scaling Factor; RD = Relative Difference; AD = Absolute Difference. IMV = $(22)^{1/2}$ODV; OSF = $(OMV/ODV)^2$; RD = OMV/IMV; AD = OMV-IMV.

The first two columns of the table show the observed volatility, measured by the standard deviation of stock returns, computed on the basis of daily data (ODV) and monthly data (OMV) for the thirteen markets considered. In all markets, there is an average of 22 trading days in each month. Thus, under the random walk hypothesis, the OMV column

[9] Throughout the article, the issue of rescaling is analyzed by comparing daily data to monthly data. The number of annual observations for the selected sample period is too small to draw any reliable inference about the behavior of annual data.

could be obtained from the ODV column by multiplying the latter by the square root of 22. However, such products, which I will refer to as *implied* monthly volatility (IMV) are, as shown in the third column, significantly lower than the OMV in all markets.

The fourth column shows the observed scaling factor (OSF); that is, the number whose square root, multiplied by the observed daily volatility, yields the observed monthly volatility.[10] As can be seen from the table, this factor is significantly larger than 22 in all markets, with an average of 35.02.[11] The last two columns report the relative difference (RD) and absolute difference (AD) between the observed monthly volatility and the implied monthly volatility, defined simply as the ratio and the difference between the OMV and the IMV, respectively. The fifth column shows that the observed monthly volatility is larger than the implied monthly volatility in all markets; the former is between 10% and 50% larger than the latter, in relative terms, and 25% on average. The last column, on the other hand, shows that the observed monthly volatility is between .53% to 2.94% larger than the implied monthly volatility in absolute terms, and on average 1.25%.

These findings thus support those of several other empirical studies (cited above) that show that, in short horizons, volatility scales at a *faster* rate than implied by the random walk. They also show that mistakenly assuming that stock prices follow such process will lead investors to seriously underestimate the risk of investing in European stocks. Such underestimation may be as high as almost 3% *a month*, and on average 1.25% *a month.* Does it not make a lot of difference for the average European firm to underestimate its cost of equity by 1.25% a month? Will this not lead that firm to accept *many* projects it should reject?

2.- Betas

This section addresses the standard practice of computing betas on the basis of daily data, and then use those betas to make inferences about the risk of securities in a different time interval. As shown below, and consistent with the results reported in the previous section, this practice may lead to a serious underestimation of risk. In order to estimate

[10] More precisely the OSF is the number that solves the *observed* relationship σ_M^2 = (OSF)σ_D^2, where σ_M^2 and σ_D^2 represent the monthly and daily variance of stock returns, respectively; hence, OSF = $\sigma_M^2/\sigma_D^2 = (\sigma_M/\sigma_D)^2$.

[11] It could be argued that the observed monthly volatility is lower than the implied monthly volatility simply because 22 trading days are being considered (thus omitting the volatility over the weekends). However, note that an OSF of 35 implies that these series do not follow a random walk even in the extreme case in which volatility over the weekend were assumed to be equal to the volatility of two trading days (which empirical evidence shows it is clearly not the case).

betas, the market portfolio used is the FTAI world market portfolio (WMP), which is computed on the basis of 2,249 stocks worldwide.

Table 4 below reports the observed betas for each market analyzed computed on the basis of both daily data (ODB) and monthly data (OMB). Under the random walk hypothesis, the betas should be independent from the frequency of the data used to compute them.[12] However, Table 4 shows that monthly betas are larger than daily betas in all markets; on average, the former are 37% larger than the later in relative terms. Hence, use of daily data to compute monthly betas will lead investors to underestimate the systematic risk of European securities markets.

TABLE 4: Betas

Market	ODB	OMB	RD
AUS	0.765	0.766	1.00
BEL	0.524	0.733	1.40
DEN	0.452	0.708	1.57
ENG	0.720	0.857	1.19
FIN	0.362	1.131	3.12
FRA	0.851	0.918	1.08
GER	0.822	0.835	1.02
ITA	0.748	0.934	1.25
NET	0.598	0.694	1.16
NOR	0.726	0.782	1.08
SPA	0.939	1.229	1.31
SWE	0.880	1.330	1.51
SWI	0.799	0.929	1.16
Averages	0.707	0.911	1.37

ODB = Observed Daily Beta; OMB = Observed Monthly Beta; RD = Relative Difference. RD = OMB/ODB.

The fact that monthly betas are larger than daily betas is explained as follows. Table A3 in part 2 of the appendix shows that the average OSF and average RD for the covariances between each market and the world market portfolio are 53.01 and 2.41, respectively. However, the same table also shows that the OSF and RD for the variance of the market portfolio are 38.64 and 1.76, respectively. Hence, monthly betas are larger than daily betas because the covariance between each market and the market portfolio (the numerator of the betas) scale at a faster rate than the variance of the market portfolio (the denominator of the betas).

[12] This follows from the fact that, under such condition, variances and covariances scale at the same rate, and, therefore, the scaling factor in the numerator of the beta cancels with the scaling factor in its denominator.

3.- Sharpe Ratios

One way of evaluating the risk-return relationship of any stock or market is by means of the so-called Sharpe ratios, which measure the return obtained by an investor per unit of risk borne (or, put simply, risk-adjusted returns).[13] The first two columns of Table 5 below show the observed Sharpe ratio computed on the basis of daily data (ODSR) and monthly data (OMSR). Under the random walk hypothesis, Sharpe ratios should scale (just as volatility) according to the square root of time;[14] thus, the OMSR column could be obtained by multiplying the ODSR column by the square root of 22. Such products, which I will refer to as *implied* monthly Sharpe ratios (IMSR) are reported in the third column.

TABLE 5: Sharpe Ratios

Market	ODSR	OMSR	IMSR	OSF	RD
AUS	-0.0054	-0.0166	-0.0252	9.53	0.66
BEL	-0.0028	-0.0096	-0.0134	11.30	0.72
DEN	-0.0037	-0.0128	-0.0172	12.23	0.75
ENG	0.0220	0.0829	0.1034	14.15	0.80
FIN	0.0303	0.0934	0.1422	9.50	0.66
FRA	-0.0026	-0.0104	-0.0121	16.15	0.86
GER	0.0043	0.0177	0.0204	16.51	0.87
ITA	-0.0012	-0.0049	-0.0057	16.01	0.85
NET	0.0331	0.1322	0.1552	15.97	0.85
NOR	0.0052	0.0218	0.0243	17.81	0.90
SPA	-0.0018	-0.0069	-0.0086	14.18	0.80
SWE	0.0226	0.0804	0.1058	12.71	0.76
SWI	0.0328	0.1358	0.1539	17.11	0.88
Averages	0.0102	0.0387	0.0479	14.09	0.80

ODSR = Observed Daily Sharpe Ratio; OMSR = Observed Monthly Sharpe Ratio; IMSR = Implied Monthly Sharpe Ratio; OSF = Observed Scaling Factor; RD = Relative Difference. IMSR = $(22)^{1/2}$ODSR; OSF = $(\text{OMSR}/\text{ODSR})^2$; RD = OMSR/IMSR.

As can be seen from the table, the observed monthly Sharpe ratios are, in absolute value, *lower* than the implied monthly Sharpe ratios in all markets. This finding is explained as follows. Table A4, in part 2 of the appendix, shows that stock returns do scale almost exactly proportionally to 22;[15] however, as already noted, volatility scales at faster rate than

[13] The Sharpe ratio is computed as $(R_i\text{-}R_f)/\sigma_i$, where R_i and σ_i are the return and risk of security i and R_f is a risk-free rate. The ratios reported in Table 5 ignore the risk-free rate and are computed as R_i/σ_i.

[14] Note that monthly returns and volatility are obtained by multiplying daily returns by T, and daily volatility by $(T)^{1/2}$, respectively. Hence, the ratio between the two implied monthly magnitudes is multiplied by $(T)^{1/2}$.

[15] Table A4 shows that observed monthly returns and implied monthly returns are, on average, 0.2213% and 0.2239%, respectively, obviously close to each other. Further, the average OSF is 21.68 (obviously close to 22), and the average RD is 0.99 (obviously close to 1). Note that in the case of returns, the OSF is the number that solves the *observed* relationship μ_M = (OSF)μ_D, where μ_M and μ_D are the mean monthly and daily stock returns, respectively; hence, OSF = μ_M/μ_D.

the square root of 22. Hence, it follows that the Sharpe ratios must scale at a slower rate than the square root of 22. Note from the last row of the table that the average OSF is 14.09 (clearly lower than 22), and that the average observed monthly Sharpe ratio is only 80% of the average implied monthly Sharpe ratio. Therefore, an investor that mistakenly assumes that stock prices follow a random walk would overestimate risk-adjusted returns by an average of 25% in relative terms.

4.- Portfolios

The results obtained thus far show that firms and investors that mistakenly assume that stock prices follow a random walk, and follow the implications of the theory, are bound to make significant mistakes. These mistakes, which basically consist of underestimating (total and systematic) risk and overestimating risk-adjusted returns, apply (as shown in this section) not only to individual stocks but also to portfolios.

Booth and Fama (1992) show that the continuously compounded rate of return of a portfolio (C_p) can be expressed as

$$C_p = \ln(1+\mu_p) - \frac{\sigma_p^{\,2}}{2(1+\mu_p)^2},$$

where μ_p and σ_p^2 are the expected return and variance of the portfolio, respectively. Using monthly data, an equally-weighted portfolio of the thirteen markets considered yields a monthly compound return of 0.1088%. However, if the return and variance of an equally-weighted portfolio are computed on the basis of daily data, then these two parameters are converted into monthly magnitudes under the assumption of a random walk (that is, through a linear rescaling), and finally plugged in the expression above, the implied compound return of the portfolio would be 0.1706%; that is, almost 57% higher in relative terms.

Recall that Table 3 shows that the average OSF and average RD for volatility are 35.02 and 1.25, respectively. An analysis of the covariance matrix (excluding variances) of the thirteen markets analyzed shows that the average OSF and average RD for these covariances are 53.65 and 2.44, respectively;[16] hence, covariances scale *at an even faster* rate than variances.

[16] To make a consistent comparison between the RD of standard deviations reported in Table 3 (1.25), and the RD of covariances just reported (2.44), the RD in Table 3 should be squared (thus obtaining 1.56). This follows from the fact that, according to the random walk theory, variances and covariances scale at the same rate (T), but standard deviations scale at the square root of such rate ($T^{1/2}$).

If we put together the fact that returns scale at the rate suggested by the random-walk theory, that variances scale at a faster rate than suggested by this theory, and that covariances scale at an even faster rate, simple inspection of the formula above shows where the underestimation of compound returns comes from.

An alternative way of analyzing the risk-return tradeoff in portfolios, and its relationship to the random walk theory, is by comparing efficient frontiers. Table A5 in the appendix shows two efficient frontiers, one computed on the basis of monthly data, and the other computed on the basis of daily data but later converted into implied monthly data under the assumption that stock prices follow a random walk.[17] As can be seen from the table, for any given return, the monthly risk of the portfolio is larger than its implied monthly risk, with the former being larger than the latter by an average of 21% in relative terms. Hence, investors that mistakenly assume that stock prices follow a random walk, and follow the implications of the theory, underestimate the true risk of their portfolios.

V- MONTHLY DATA

All the results reported so far imply that it is dangerous for investors to make investment decisions assuming (implicitly or explicitly) that stock prices follow a random walk. However, it is fair to ask at this point whether the problems discussed are specific to daily data and thus disappear when the data is aggregated. Tables A6-A7 in part 2 of the appendix consider this issue.

Table A6 shows that the unit-root hypothesis for monthly stock returns cannot be rejected, with marginal exceptions in a few markets under some tests. To be sure, in no market all tests simultaneously reject the unit-root hypothesis. Table A7, on the other hand, explores the linear and nonlinear dependencies in monthly stock returns. Ljung-Box tests for six autocorrelations fail to reject (at the 5% significance level) the hypothesis of linearly uncorrelated returns in all markets. Ljung-Box tests for six squared autocorrelations, on the other hand, reject the null hypothesis of nonlinear independence in returns in only two markets (Belgium and Italy), and marginally in one market (Austria). Finally, the last column of Table A7 shows the results of Kolmogorov-Smirnov tests for normality; the null

[17] More precisely, the implied efficient frontier was computed in two steps: First, the efficient frontier was computed on the basis of daily data. Then, each daily risk-return pair was converted into an implied monthly risk-return pair by multiplying the daily return by 22 and the daily risk by the square root of 22.

hypothesis of normally-distributed stock returns (another implication of the random walk theory) cannot be rejected in any market.

Therefore, monthly stock returns do not seem to suffer from the problems of daily stock returns; put differently, the evidence shows that monthly stock returns do appear to be *i.i.d*-Normal. This result should not be too surprising; under the Central Limit Theorem, the longer the time interval for which returns are computed, the more the resulting distribution should conform to the Normal distribution. Whether monthly stock returns can be reliably used to make inferences on annual investment horizons is an interesting issue that cannot be addressed with the sample size selected for this study.

VI- CONCLUSIONS

The results reported in this article question some standard practices in the estimation of risk. More precisely, they raise the issue that if stock prices do not follow a random walk, use of daily data in order to draw inferences about different investment horizons may be badly misleading.

The analysis has shown that none of the series of daily stock prices of the thirteen European markets analyzed follows a random walk. Although a unit-root is present in all markets, daily stock returns, not surprisingly, fail to be *i.i.d.*; they all exhibit linear and nonlinear dependence and display non-normal (fat-tailed) distributions. Under these conditions, both data frequencies and investment horizons become relevant issues

If stock prices follow a random walk, high-frequency data contain all the relevant information to forecast risk in other time frequencies; in other words, short-term risk can be reliably used to forecast long-term risk. If stock prices do not follow a random walk, however, the relationship between volatility in different time intervals breaks down and short-term risk carries no reliable information about long-term risk. This breakdown of the informational content of short-term risk brings about an interesting risk puzzle which is neatly illustrated by emerging markets. On the one hand, these markets are very risky in the short term. On the other hand, investors would not invest in these markets if they were not compensated by high returns. Thus, if an investor waits "long enough" until the emerging market yields a return consistent with its risk, this investor will obtain a high return without suffering from short-term swings. In other words, if markets eventually yield the equilibrium return for their inherent risk, and if an investor waits for the "eventually" to arrive, he would be obtaining a high long-term return without bearing a high long-term risk. I am very hesitant to call this a market inefficiency; rather, I tend to think that it is one of the interesting characteristics of nonrandom walk data that deserves further analysis.

The analysis in this article has shown that ignoring the fact that stock prices do not follow a random walk would lead investors to underestimate total and systematic risk, as well as to overestimate compound and risk-adjusted returns. More precisely, monthly risk implied by daily data underestimates the observed monthly risk by an average of 25% in relative terms, or 1.25% in absolute terms. Observed monthly betas are on average 37% larger, in relative terms, than monthly betas implied by daily data. Monthly risk-adjusted returns implied by daily data are on average 25% higher, in relative terms, than observed monthly risk-adjusted returns. The monthly return of an equally-weighted portfolio of the thirteen markets analyzed implied by daily data is 57% higher, in relative terms, than the observed monthly return. Finally, efficient frontiers implied by daily data consistently underestimate the observed monthly risk at all reasonable levels of monthly return, and on average by over 20% in relative terms.

The fast scaling of volatility in short horizons has further important implications for investors. Among them is the fact that (as suggested above) long-term investors face less risk than that indicated by daily volatility, as well as the fact that investors can benefit from buying a stock right after its price has fallen sharply; see De Bondt and Thaler (1985).

The analysis of monthly data, however, shows that monthly stock prices behave much more in line with the random walk theory. Monthly stock returns in the markets analyzed seem to be both *i.i.d.* and normally distributed. As pointed out above, however, the sample size considered is not large enough to address the issue of whether monthly data can be reliably used to make inferences about annual risk and return. The market data used, on the other hand, does not make it possible to infer how severe may be the underestimation of the cost of capital of firms that compute their betas on the basis of daily data. Both of these interesting issues are not addressed in this article but are left open for further research.

APPENDIX

1.- Brownian Motion

Let $P(t)$ be the price of a security at time t, and h be the minimum length of time between transactions in this security. Further, let a finite time interval between t=0 and t=T be divided into n subintervals, such that nh=T. Then, the change in the price of this security between t=0 and t=T is given by

$$P_T - P_0 = \sum_{k=1}^{n} (P_k - P_{k-1}) , \tag{A1}$$

where P_k and $P_{k\text{-}1}$ are shorthands for P_{kh} and $P_{(k\text{-}1)h}$, respectively. The conditional expected return of the security per unit of time (R_k) is given by

$$R_k = E_{k-1}(P_k - P_{k-1}) / h , \tag{A2}$$

where E_k is the expectation operator conditional on all the information available at time k.

Let ε_k be the unanticipated price change in the security between k and k-1; that is,

$$\varepsilon_k = (P_k - P_{k-1}) - E_{k-1}(P_k - P_{k-1}) = (P_k - P_{k-1}) - R_k h , \tag{A3}$$

such that $E_{k\text{-}1}(\varepsilon_k)$=0 and $\Sigma_k \varepsilon_k$ is a martingale. Define σ_k^2=$E_{k\text{-}1}(\varepsilon_k^2)/h$ as the variance per unit of time of the security's return, and u_k=$\varepsilon_k/(\sigma_k^2 h)^{1/2}$. Then, the price change in the security can be written as

$$P_k - P_{k-1} = R_k h + \varepsilon_k = R_k h + \sigma_k u_k h^{1/2} , \tag{A4}$$

and the differential equation describing the path of P_t as

$$\mathrm{d}P_t = R_t \,\mathrm{d}t + \sigma_t u_t (\mathrm{d}t)^{1/2} . \tag{A5}$$

Let p_t be a random variable whose time path is described by an equation like (A4), and such that R_k=0 and σ_k=1. Thus, the change in p_t between t=0 and t=T is given by

$$p_T - p_0 = \sum_{k=1}^{n} (p_k - p_{k-1}) = h^{1/2} \sum_{k=1}^{n} u_k = T^{1/2} \frac{\sum_{k=1}^{n} u_k}{n^{1/2}} . \tag{A6}$$

Note that the $\{u_k\}$ are *i.i.d.* and then, by the central limit theorem, $\Sigma_k u_k / n^{1/2}$ follows a standard Normal distribution regardless of the distribution of the $\{u_k\}$. Note, further, that the differential equation describing the path of p_t is given by

$$\mathrm{d}p_t = u_t (\mathrm{d}t)^{1/2} . \tag{A7}$$

Thus, a variable p_t whose temporal path is described by an equation like (A7), such that the $\{u_t\}$ are *i.i.d.* and standard normally distributed, is said to follow a Brownian motion process. See Merton (1990).

2.- Complementary Tables

TABLE A1: Sample Moments of the Distributions of Daily Stock Returns

Market	Mean	SD	Min	Max	Skw	SSkw	Krt	SKrt
AUS	-0.0064	1.1853	-7.5998	6.9370	0.1758	2.5918	6.1939	45.6561
BEL	-0.0020	0.7190	-5.5734	6.9116	0.1069	1.5761	14.6793	108.2030
DEN	-0.0030	0.8232	-5.8997	4.9312	-0.0936	-1.3803	5.9536	43.8846
ENG	0.0180	0.8185	-3.9943	5.5348	0.3340	4.9238	3.4455	25.3974
FIN	0.0377	1.2440	-5.4757	5.2919	0.2328	3.4316	2.1259	15.6700
FRA	-0.0026	0.9996	-7.2685	5.4874	-0.2973	-4.3833	3.8782	28.5867
GER	0.0046	1.0478	-10.3649	5.2958	-0.6131	-9.0381	9.7663	71.9886
ITA	-0.0017	1.3691	-8.2403	5.2754	-0.1918	-2.8278	2.2403	16.5131
NET	0.0241	0.7273	-3.6398	3.0222	-0.3147	-4.6391	1.6304	12.0176
NOR	0.0069	1.3307	-8.8584	10.8018	0.3662	5.3988	8.8477	65.2171
SPA	-0.0021	1.1249	-8.6287	6.7887	-0.2627	-3.8733	5.8811	43.3505
SWE	0.0282	1.2504	-6.8453	9.3145	0.5003	7.3755	5.7016	42.0272
SWI	0.0306	0.9313	-7.2125	5.3787	-0.6884	-10.1489	7.3683	54.3125
WOR	-0.0023	0.6535	-4.2796	3.9281	-0.0142	-0.2088	5.5388	40.8272

Mean returns, standard deviations (SD), minimum returns (Min), and maximum returns (Max) are all expressed in percentages. Skw = Skewness = m_3/s^3 and Krt = Kurtosis = m_4/s^4-3, where m_i and s are the ith central sample moment and the sample standard deviation of each distribution, respectively; both coefficients are computed with a finite-sample adjustment. SSkw = Standardized skewness and SKrt = Standardized kurtosis. Sample size = 1,304 for all markets.

TABLE A2: Sample Moments of the Distributions of Monthly Stock Returns

Market	Mean	SD	Min	Max	Skw	SSkw	Krt	SKrt
AUS	-0.1384	8.3512	-25.4979	20.2526	-0.3408	-1.0776	1.5646	2.4739
BEL	-0.0445	4.6504	-12.2494	13.2179	-0.2539	-0.8028	1.0643	1.6828
DEN	-0.0655	5.1179	-11.0931	11.6782	-0.0524	-0.1658	-0.3183	-0.5032
ENG	0.3923	4.7308	-8.6594	10.4902	0.0773	0.2445	-0.3561	-0.5631
FIN	0.8200	8.7771	-20.4114	20.9633	0.0505	0.1596	-0.1234	-0.1951
FRA	-0.0560	5.4079	-14.7757	11.9082	-0.2230	-0.7051	0.0088	0.0139
GER	0.0993	5.6230	-19.7969	9.3121	-1.0676	-3.3760	2.5415	4.0185
ITA	-0.0363	7.4437	-16.1316	20.8624	0.3993	1.2626	0.1139	0.1801
NET	0.5231	3.9566	-9.9611	9.0035	-0.0950	-0.3006	-0.3188	-0.5041
NOR	0.1498	6.8552	-16.4028	12.6052	-0.3229	-1.0210	-0.4515	-0.7139
SPA	-0.0446	6.4968	-20.7705	12.8100	-0.4144	-1.3104	0.4918	0.7775
SWE	0.6132	7.6259	-23.9370	23.0205	-0.2962	-0.9366	1.5848	2.5057
SWI	0.6646	4.8953	-14.5733	12.0524	-0.5547	-1.7542	1.6346	2.5845
WOR	-0.0507	4.0607	-12.9982	9.7972	-0.5708	-1.8051	1.5585	2.4642

Mean returns, standard deviations (SD), minimum returns (Min), and maximum returns (Max) are all expressed in percentages. Skw = Skewness = m_3/s^3 and Krt = Kurtosis = m_4/s^4-3, where m_i and s are the ith central sample moment and the sample standard deviation of each distribution, respectively; both coefficients are computed with a finite-sample adjustment. SSkw = Standardized skewness and SKrt = Standardized kurtosis. Sample size = 60 for all markets.

TABLE A3: Daily, Monthly, and Implied Monthly Covariances to the Market Portfolio

Market	ODC	OMC	IMC	OSF	RD
AUS	0.000033	0.0013	0.00072	38.70	1.76
BEL	0.000022	0.0012	0.00049	53.99	2.45
DEN	0.000019	0.0012	0.00042	60.47	2.75
ENG	0.000031	0.0014	0.00068	45.99	2.09
FIN	0.000015	0.0019	0.00034	120.57	5.48
FRA	0.000036	0.0015	0.00080	41.65	1.89
GER	0.000035	0.0014	0.00077	39.28	1.79
ITA	0.000032	0.0015	0.00070	48.27	2.19
NET	0.000026	0.0011	0.00056	44.81	2.04
NOR	0.000031	0.0013	0.00068	41.60	1.89
SPA	0.000040	0.0020	0.00088	50.54	2.30
SWE	0.000038	0.0022	0.00083	58.36	2.65
SWI	0.000034	0.0015	0.00075	44.93	2.04
Averages	0.000030	0.0015	0.00066	53.01	2.41
WMP	0.000043	0.0016	0.00094	38.64	1.76

ODC = Observed Daily Covariance; OMC = Observed Monthly Covariance; IMC = Implied Monthly Covariance; OSF = Observed Scaling Factor; RD = Relative Difference. IMC = (22)ODC; OSF = OMC/ODC; RD = OMC/IMC.

TABLE A4: Daily, Monthly, and Implied Monthly Expected Returns

Market	ODR	OMR	IMR	OSF	RD
AUS	-0.0064	-0.1384	-0.1408	21.63	0.98
BEL	-0.0020	-0.0445	-0.0440	22.25	1.01
DEN	-0.0030	-0.0655	-0.0660	21.83	0.99
ENG	0.0180	0.3923	0.3960	21.79	0.99
FIN	0.0377	0.8200	0.8294	21.75	0.99
FRA	-0.0026	-0.0560	-0.0572	21.54	0.98
GER	0.0046	0.0993	0.1012	21.59	0.98
ITA	-0.0017	-0.0363	-0.0374	21.35	0.97
NET	0.0241	0.5231	0.5302	21.71	0.99
NOR	0.0069	0.1498	0.1518	21.71	0.99
SPA	-0.0021	-0.0446	-0.0462	21.24	0.97
SWE	0.0282	0.6132	0.6204	21.74	0.99
SWI	0.0306	0.6646	0.6732	21.72	0.99
Averages	0.0102	0.2213	0.2239	21.68	0.99

ODR = Observed Daily Returns; OMR = Observed Monthly Returns; IMR = Implied Monthly Returns; OSF = Observed Scaling Factor; RD = Relative Difference. IMR = (22)ODR; OSF = OMR/ODR; RD = OMR/IMR.

TABLE A5: Efficient Frontiers

	MR	MV	IMV	RD	AD
	0.4284	3.6411	2.6172	1.39	1.02
	0.4504	3.6421	2.6285	1.39	1.01
	0.4724	3.6443	2.6414	1.38	1.00
	0.4944	3.6475	2.6560	1.37	0.99
	0.5164	3.6518	2.6723	1.37	0.98
	0.5384	3.6572	2.6901	1.36	0.97
	0.5604	3.6636	2.7095	1.35	0.95
	0.5824	3.6711	2.7305	1.34	0.94
	0.6044	3.6797	2.7529	1.34	0.93
	0.6264	3.6893	2.7768	1.33	0.91
	0.6484	3.6999	2.8021	1.32	0.90
	0.6704	3.7115	2.8288	1.31	0.88
	0.6924	3.7242	2.8569	1.30	0.87
	0.7144	3.7379	2.8862	1.30	0.85
	0.7364	3.7526	2.9168	1.29	0.84
	0.7584	3.7682	2.9487	1.28	0.82
	0.7804	3.7848	2.9817	1.27	0.80
	0.8024	3.8024	3.0159	1.26	0.79
	0.8244	3.8210	3.0512	1.25	0.77
	0.8464	3.8404	3.0875	1.24	0.75
	0.8684	3.8608	3.1249	1.24	0.74
	0.8904	3.8821	3.1633	1.23	0.72
	0.9124	3.9043	3.2027	1.22	0.70
	0.9344	3.9273	3.2430	1.21	0.68
	0.9564	3.9512	3.2842	1.20	0.67
	0.9784	3.9760	3.3262	1.20	0.65
	1.0004	4.0016	3.3691	1.19	0.63
	1.0224	4.0280	3.4127	1.18	0.62
	1.0444	4.0552	3.4572	1.17	0.60
	1.0664	4.0831	3.5023	1.17	0.58
	1.0884	4.1119	3.5482	1.16	0.56
	1.1104	4.1414	3.5948	1.15	0.55
	1.1324	4.1716	3.6420	1.15	0.53
	1.1544	4.2025	3.6898	1.14	0.51
	1.1764	4.2341	3.7383	1.13	0.50
	1.1984	4.2665	3.7873	1.13	0.48
	1.2204	4.2994	3.8369	1.12	0.46
	1.2424	4.3331	3.8870	1.11	0.45
	1.2644	4.3674	3.9377	1.11	0.43
	1.2864	4.4023	3.9889	1.10	0.41
	1.3084	4.4378	4.0405	1.10	0.40
	1.3304	4.4739	4.0926	1.09	0.38
	1.3524	4.5106	4.1451	1.09	0.37
	1.3744	4.5479	4.1981	1.08	0.35
	1.3964	4.5857	4.2515	1.08	0.33
	1.4184	4.6240	4.3052	1.07	0.32
	1.4404	4.6629	4.3594	1.07	0.30
	1.4624	4.7023	4.4139	1.07	0.29
	1.4844	4.7422	4.4688	1.06	0.27
	1.5064	4.7826	4.5240	1.06	0.26
Avgs:	0.9764	4.0050	3.3957	1.21	0.65

Monthly returns (MR) are the same for both efficient frontiers. MV = Monthly Volatility; IMV = Implied Monthly Volatility; RD = Relative Difference; AD = Absolute Difference. RD = MV/IMV; AD = MV-IMV.

TABLE A6: Unit Root Tests (Monthly Data)

Augmented Dickey-Fuller Test: $\Delta\ln(I_t) = \alpha + \rho\ln(I_{t-1}) + \beta t + \gamma_1\Delta\ln(I_{t-1}) + ... + \gamma_q\Delta\ln(I_{t-q}) + \varepsilon_t$

Phillips-Perron Test: $\ln(I_t) = \alpha + \rho\ln(I_{t-1}) + \beta(t-T/2) + \varepsilon_t$

		Augmented Dickey-Fuller			Phillips-Perron		
Market	ρ	H_0: ρ=0	H_0: α=ρ=0	H_0: α=β=ρ=0	H_0: ρ=0	H_0: α=ρ=0	H_0: α=β=ρ=0
AUS	0.905	-2.0027	2.0060	1.3430	-2.1879	2.4111	1.6121
BEL	0.931	-2.8415	4.5762	3.0529	-2.8767	4.6488	3.1011
DEN	0.911	-1.7277	1.5440	1.0327	-1.8031	1.6791	1.1225
ENG	0.916	-3.2782	5.4420	3.7862	-3.4349	5.9898	4.1277
FIN	0.969	-1.8152	4.0420	2.8908	-1.4850	2.4479	1.7887
FRA	0.895	-2.6614	3.6111	2.4098	-2.8228	4.0668	2.7132
GER	0.901	-2.1386	2.5636	1.7156	-2.2045	2.6938	1.8020
ITA	0.952	-1.7964	2.2342	1.4899	-1.8357	2.2672	1.5119
NET	0.940	-2.7128	4.2932	3.2507	-2.7574	4.3709	3.2673
NOR	0.925	-1.3685	1.1671	0.7877	-1.4117	1.2190	0.8218
SPA	0.905	-2.4055	3.0772	2.0525	-2.4983	3.2975	2.1992
SWE	0.927	-1.9223	2.2674	1.7644	-2.3624	3.1018	2.1867
SWI	0.947	-3.6239	7.2494	5.4200	-2.6014	3.7104	2.8149

Numbers in the table represent test statistics except for ρ, which represents the coefficient from the regression $\ln(I_t)=\mu+\rho\ln(I_{t-1})+\varepsilon_t$. Asymptotic critical values for H_0: ρ=0, H_0: α=ρ=0, and H_0: α=β=ρ=0 at the 5% significance level are -3.41, 6.25, and 4.68, respectively. Sample size = 61 for all markets.

TABLE A7: Linear and Nonlinear Dependence, and Normality in Monthly Stock Returns

Market	$Q(6)$	p-value	$Q^2(6)$	p-value	KS	p-value
AUS	9.17	0.103	12.66	0.049	0.0958	0.6410
BEL	6.58	0.254	20.26	0.002	0.1169	0.3851
DEN	1.95	0.856	6.12	0.410	0.0622	0.9745
ENG	3.45	0.631	6.29	0.392	0.0715	0.9190
FIN	7.77	0.169	4.03	0.673	0.0873	0.7498
FRA	6.17	0.290	6.70	0.349	0.0755	0.8835
GER	9.46	0.092	12.37	0.054	0.0910	0.7035
ITA	3.91	0.562	13.50	0.036	0.0804	0.8324
NET	4.79	0.442	6.02	0.421	0.0648	0.9630
NOR	1.37	0.928	5.90	0.435	0.0977	0.6153
SPA	2.41	0.790	7.51	0.277	0.0652	0.9608
SWE	5.23	0.388	1.17	0.978	0.0630	0.9712
SWI	6.57	0.225	8.59	0.198	0.1025	0.5534

$Q(6)$ and $Q^2(6)$ are the Ljung-Box statistics for 6 autocorrelations and 6 squared autocorrelations, respectively. KS = Kolmogorov-Smirnov test statistic.

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