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OPTIMAL ALLOCATION OF INTEREST RATE RISK

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Abstract

Based on the work of Hellwig (1994), this paper characterizes the optimal allocation of technology-induced interest rate risk in a competitive system of financial intermediation and its interdependence with the provision of liquidity. The analysis is carried out under the assumptions of complete and incomplete information respectively.

The implementation of the second best allocation by a financial intermediary is compared to the one achieved in an equity economy in which individuals hold the assets directly.

Key Words and Phrases

Banking, Deposit Contracts, liquidity, interest rate risk.

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Optimal allocation of interest rate risk

1.- Introduction

In a very general analysis, the problem of interest rate risk is one that concerns the relation between the maturity structure of real assets in the economy and the time pattern of aggregate consumption. If there was perfect maturity matching between the two, the individual's exposure to interest rate risk would be reduced to zero. Concerning this issue, two remarks were pointed out by Stiglitz [15]:

a.- An allocation involving perfect maturity matching, rules out the possibility that the individual's consumption plan may provide for a dependence on observed interest rates, in order to take advantage of changes in relative intertemporal prices. For this reason, an allocation of this sort, with no response of consumption to changes in technologies is unlikely to be efficient.

b.- The time pattern of aggregate consumption should be known from the beginning.

However, individuals may be uncertain about their future time preferences, they may be subject to a preference shock which leads to a demand for liquidity and this gives a rationale for the existence of banks. One of the most important functions of financial intermediaries is to transform highly illiquid assets into more liquid liability payoffs through the demand deposit contract.

By performing this transformation service, banks are exposed to interest rate risk as they take in short term funds to finance long term investments. The control of this interest rate risk is a matter of concern in banking regulation.

The objective of this work is to characterize the optimal allocation of technology-induced interest rate risk in a competitive system of financial intermediation, and its interdependence with the provision of liquidity.

The analysis will be carried out under different information assumptions. In the complete information case, it is assumed the realization of the timing of the consumption needs is publicly observable; in the incomplete information case it is private information of the consumer and therefore an allocation can only be implemented if it is incentive compatible, that is, if it gives no consumer an incentive to lie or deviate about what he actually wants to consume.

This paper of the thesis is based on the work of Hellwig [11], that considers a Diamond Dybvig type economy (individuals have corner preferences), and in which there is stochastic (short-term) investment between dates 1

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and 2.

This work differs from Hellwig's paper, in that it assumes individuals have smooth preferences, that is, they derive utility from consumption in the two periods of their life.

The first motivation for this extension is to test the robustness of Hellwig's model (under the simplifying and somewhat misleading corner preference assumption).

In very general terms, the basic result holds in this extended framework. It is shown that as interest rate increases the optimal rate of return of deposits withdrawn at date 1 decreases and that of deposits that remain until date 2 increases. The intuition is that given the high interest rate, it becomes advantegous to reinvest some of the unused resources at date 1 in this new short term investment. Nevertheless, there are some minor differences with respect to Hellwig's results, that will be commented throughout the work.

As Jacklin [12] has noted, and is confirmed also in the work of Hellwig, the Diamond Dybvig specification with no aggregate uncertainty about preferences, has the feature that the ex-ante optimal allocation is also implementable through trading, where shares of the investment portfolio (of short and long term assets) could be traded at date 1 as with a mutual fund. This rules out any specialness on the side of a financial intermediary.

In a later paper, Jacklin [13] shows that unless there is both aggregate uncertainty about preferences and banks assets are risky, with depositors asymmetrically informed about asset quality, then traded equity contracts can provide the same services as demand deposit contracts, without the possibility of panics. The message of his paper is that liquidity transformation can and should be provided using equity contracts where the underlying assets may or not be risky, but where there is little or no potential for asymmetries of information about asset quality.

The above papers considered models in which individuals have corner preferences, that are not considered a realistic characterisation of individuals' preferences.

With smooth preferences and no aggregate uncertainty about preferences, Jacklin [12] has shown that non traded demand deposit contracts and traded equity are not welfare equivalent. In fact, demand deposits are shown to provide greater risk sharing than equity shares.

Jacklin and Bhattacharya [2] also considered the relative degree of risk sharing provided by traded and non traded contracts, in a framework in which bank assets are risky and individuals (with smooth preferences) are informed concerning bank asset quality. The basic result is that deposit contracts tend to be better for financing low risk assets.

A second motivation for this extension is to compare the traded and non traded solutions in this economy, with smooth preferences and in which there exists a reinvestment opportunity from date 1 to date 2. It will be shown, that in this framework, demand deposits and traded equity are not always welfare equivalent.

The structure of the paper is as follows: The basic framework of the model is presented in section 2. The first-best and second-best allocations under complete and incomplete information respectively, are presented in subsections 2.1 (preferences represented by a multiplicative utility function) and 2.2 (in the case of an additive utility function). Subsection 2.3 shows elasticities of consumption with respect to the short-term random technology. Section 3 compares the second-best allocation (non-traded solution) with an equity economy (traded solution) in which individuals could hold the assets directly. Section 4 concludes the paper.

2.- Description of the model

The hypothesis of the model are summarized as follows:

a.- Three period economy: T = 0, 1, 2

b.- One good per period

c.- There are three investment opportunities:

  i.- A short-term asset at T=0 that yields a sure return b_{o1} at T=1

  ii.- A long-term asset at T=0 that yields a sure return b_{o2} at T=2, premature liquidation of the asset is feasible but the rate of return is only $b_1 < b_{o1}$

  iii.- A short-term asset at T=1 that yields a random return b_{12} at T=2. The random variable is known at T=1 but not at T=0, at date 0, only the distribution function is known[1].

d.- On the household side of the economy, there is a continuum of unit mass of ex ante identical consumers that are uncertain at T=0 about their consumption needs. They are subject at T=1 to a privately observed uninsurable risk of being of type-1 with probability t or of type-2 with probability $1-t$.

  For comparative purposes, preferences will be represented by an additive utility function and by a multiplicative one, which are of the form:

Additive utility function:

$$U^i(c_1, c_2, \rho_i) = \frac{c_1^{1-\gamma}}{1-\gamma} + \rho_i \frac{c_2^{1-\gamma}}{1-\gamma} \qquad [1]$$

where: $0 \le \rho_i \le 1$, $i = 1, 2$ (type), and $\rho_1 > \rho_2$.

Multiplicative utility function:

$$U^i(c_1, c_2, \delta_i) = c_1^{\delta_i} c_2^{\alpha - \delta_i} \qquad [2]$$

where: $0 < \alpha < 1$, $0 < \delta_i < 1$, $i = 1, 2$ (type), and $\delta_1 > \delta_2$.

As commented in the introduction, it is assumed a more general preference structure with respect to Hellwig [11], as individuals derive utility from consumption in both periods.

e.- Consumers are endowed with k_o units of the good at T=0 to be divided between short-term and long-term investments.

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[1] *For simplicity, a triangular distribution for the random return is assumed*

f.- It is assumed no aggregate uncertainty, so that with probability one a fraction t of consumers are of type-1 and a fraction $1-t$ of type-2.

The economy must deal with the following allocation problem:

a.- At T=0 the initial endowment must be divided between short and long term investments $(k_o = k_{o1} + k_{o2})$

b.- At T=1 the fraction $(0 \leq \mu \leq 1)$ of the long-term investment that is liquidated must be determined, this may depend on the observed value of $\tilde{b}_{12}$

c.- At T=1 the returns from short-term assets and possibly liquidated long-term investments must be divided between consumption and new short-term investments, this may also depend on the observed realization of $\tilde{b}_{12}$

The objective of this work is to characterize efficient allocations and to see how the initial uncertainty about the random return affects consumption allocations as well as the initial investment choices. This analysis will be carried out under the assumptions of complete and incomplete information respectively.

2.1.- Multiplicative Utility function

2.1.1.- First-best allocations under complete information

In the complete information case, it is assumed the type of the consumer is publicly observable and in this situation, the efficient allocation will be the solution to the following problem:

$$\max_{\tilde{c}_{ij}, k_{o1}, k_{o2}, \tilde{\mu}} E\left[t U(\tilde{c}_{11}, \tilde{c}_{21}) + (1-t) U(\tilde{c}_{12}, \tilde{c}_{22}) \right]$$

$$\text{s.t} \quad k_{o1} + k_{o2} = k_o$$

$$t\tilde{c}_{11} + (1-t)\tilde{c}_{12} \leq b_{o1} k_{o1} + b_1 \tilde{\mu} k_{o2}$$

$$t\tilde{c}_{21} + (1-t)\tilde{c}_{22} = b_{o2}(1-\tilde{\mu})k_{o2} + \tilde{b}_{12}\left[b_{o1} k_{o1} + b_1 \tilde{\mu} k_{o2} - t\tilde{c}_{11} - (1-t)\tilde{c}_{12} \right] \qquad [3]$$

$$b_1 \tilde{b}_{12} < b_{o2}$$

$$\tilde{\mu} \leq 1$$

$$\tilde{c}_{ij} \geq 0$$

$$\tilde{\mu} \geq 0$$

The utility function is the one described above in Point d of Page 93. c_{11}, c_{21} represents the prior plan indicating the consumption bundle allocated to type-1 consumers and c_{12}, c_{22} the plan allocated to type-2 consumers. The feasibility constraints are the second and third constraints respectively. The second one requires that aggregate consumption at T=1 should be less or equal to aggregate resources per capita available from short-term investments and possibly liquidated long-term ones. Similarly, the third constraint requires that aggregate consumption at T=2 should be covered by non liquidated long-term investments plus short-term reinvestments of unused resources at T=1. The fourth constraint states that at date 1, it is never desirable to liquidate long-term investments in order to make room for new short-term ones.

This maximization problem is solved as a three-step problem:

a.- In a first step, the initial investment choices, (k_{o1}, k_{o2}) are considered as exogenous parameters and the optimal consumption levels and liquidation policy are determined.

b.- In a second step the indirect utility function derived in the first step is maximized on k_{o1} and k_{o2}, and so the optimal levels of the initial investments are obtained.

c.- Finally, the optimal levels of k_{o1} and k_{o2} are substituted back into the first step problem, and the final solution is reached. Although this last step is obvious, it has just been added to clarify how the numerical solutions presented in the figures have been derived.

2.1.1.1.- First step: Optimal consumption levels and liquidation policy.

In this step, b_{o1}, b_{o2}, b_1, b_{12}, k_{o1}, k_{o2} and t are considered as exogenous parameters and so the problem may be rewritten:

$$\max_{c_{ij},\mu} E\left[tU(c_{11},c_{21})+(1-t)U(c_{12},c_{22})\right]$$

$$\text{s.t} \quad k_{o1}+k_{o2}=k_o$$

$$tc_{11}+(1-t)c_{12}\leq b_{o1}k_{o1}+b_1\mu k_{o2}$$

$$tc_{21}+(1-t)c_{22}=b_{o2}(1-\mu)k_{o2}+b_{12}[b_{o1}k_{o1}+b_1\mu k_{o2}-tc_{11}-(1-t)c_{12}] \qquad [4]$$

$$b_1b_{12}<b_{o2}$$

$$\mu\leq 1$$

$$c_{ij}\geq 0$$

$$\mu\geq 0$$

λ_1 and λ_2 are the Lagrange multipliers associated with the first and second resource balance constraints and λ_3 the multiplier associated with μ.

In this first step, the solutions are given in Table I.

Table I

Optimal liquidation solution: $k_{o1}<k_{o1}$ crit and $\mu^*>0$			
CASE C $\lambda_1>0$			
Optimal zero-liquidation solution: $k_{o1}\geq k_{o1}$ crit and $\mu^*=0$			
CASE A $\lambda_1>0$	if $b_{12}<b_{lim}$	CASE B $\lambda_1=0$	if $b_{12}\geq b_{lim}$

The consumption levels of cases A, B and C, the aggregate expected utilites ($U^{*(a)}$, $U^{*(b)}$, $U^{*(c)}$) corresponding

to each of these cases, the value of $k_{oI}crit$ and b_{lim} are given in Appendix A. Figure 1 gives a graphical plot of the frontiers for these three cases in a particular example.

It should be observed that for low exogenously fixed values of k_{oI} (below a critical value $k_{oI}crit$) the optimal solution is characterized by partial liquidation of the long-term technology (this makes sense, if the level of k_{oI} is very low, liquidation of the long-term asset is needed to provide for first period consumption). In this solution, consumption is always constant, independently of the random return, (the multiplier associated with the first resource constraint is positive and therefore, the constraint is binding). For high exogenously fixed values of k_{oI} (above $k_{oI}crit$), the optimal solution involves no liquidation of the long-term technology (given that the level of k_{oI} is high enough to provide for first period consumption). In this case, for low values of the random return, consumption is constant (up to a limit value b_{lim}) and once this limit is attained, first and second period consumption become responsive to the random return (this is explained by the fact that in case B the multiplier $\lambda_1 = 0$ and therefore its corresponding constraint is no longer binding). As will be explained in more detail below, in case B there is reinvestment of resources at date 1 in the new short-term technology.

2.1.1.2.- Second step: Optimal levels of the initial investments

The above optimization problem has been solved considering k_{oI} and k_{o2} as exogenous parameters. The first-best investment levels at date zero are obtained by dynamic programming, maximizing on k_{oI} and k_{o2} the indirect utility function of the problem described above, i.e.:

$$\max_{k_{oI}, k_{o2}} \left[\begin{array}{ll} \int_{b_{min}}^{b_{lim}} U^{*(A)} f(b_{12}) db_{12} + \int_{b_{lim}}^{b_{max}} U^{*(B)} f(b_{12}) db_{12} & \text{if} \quad k_{oI} \geq k_{oI}crit \\ \int_{b_{min}}^{b_{max}} U^{*(C)} f(b_{12}) db_{12} & \text{if} \quad k_{oI} < k_{oI}crit \end{array} \right] \qquad [5]$$

s.t. $k_{oI} + k_{o2} = 1$

Lemma 1. The optimal k_{oI}^* satisfies: $k_{oI}crit \leq k_{oI}^* \leq 1$

where:

$$k_{oI}crit = \frac{b_1 [t\delta_1 + (1-t)\delta_2]}{b_1 [t\delta_1 + (1-t)\delta_2] + b_{oI} [t(\alpha - \delta_1) + (1-t)(\alpha - \delta_2)]} \qquad [6]$$

Proof: See Appendix A

2.1.1.3.- Third step: Final solution .

The optimal levels of k_{oI}, k_{o2} are substituted back into the first-step problem and so the final solution is reached.

This solution to the first best problem gives the main result of the section, expressed by the proposition below:

Proposition 1. Let $(k_{oI}, k_{o2}, c_{11}, c_{12}, c_{21}, c_{22}, \mu)$ be a solution to the first best problem and define:

$$b_{\text{lim}} = \frac{b_{o2}k_{o2}\left[t\delta_1 + (1-t)\delta_2\right]}{b_{ol}k_{ol}\{\alpha - \left[t\delta_1 + (1-t)\delta_2\right]\}}$$ [7]

Then: if $\tilde{b}_{12} < b_{\text{lim}}$ CASE A

$$c_{11}^* = b_{ol}k_{ol}\frac{\delta_1}{t\delta_1 + (1-t)\delta_2} \qquad c_{21}^* = b_{o2}k_{o2}\frac{(\alpha - \delta_1)}{t(\alpha - \delta_1) + (1-t)(\alpha - \delta_2)}$$

[8]

$$c_{12}^* = b_{ol}k_{ol}\frac{\delta_2}{t\delta_1 + (1-t)\delta_2} \qquad c_{22}^* = b_{o2}k_{o2}\frac{(\alpha - \delta_2)}{t(\alpha - \delta_1) + (1-t)(\alpha - \delta_2)} \qquad \mu^* = 0$$

if $\tilde{b}_{12} \geq b_{\text{lim}}$ CASE B

$$\tilde{c}_{11}^* = \delta_1\frac{b_{ol}k_{ol}\tilde{b}_{12} + b_{o2}k_{o2}}{\tilde{b}_{12}\alpha} \qquad \tilde{c}_{21}^* = (\alpha - \delta_1)\frac{b_{ol}k_{ol}\tilde{b}_{12} + b_{o2}k_{o2}}{\alpha}$$

[9]

$$\tilde{c}_{12}^* = \delta_2\frac{b_{ol}k_{ol}\tilde{b}_{12} + b_{o2}k_{o2}}{\tilde{b}_{12}\alpha} \qquad \tilde{c}_{22}^* = (\alpha - \delta_2)\frac{b_{ol}k_{ol}\tilde{b}_{12} + b_{o2}k_{o2}}{\alpha} \qquad \tilde{\mu}^* = 0$$

Proof: See Appendix A

Given that k_{ol} is an endogenous variable, this characterization may seem awkward, but it is understood in terms of dynamic programming considerations. As mentioned before, the maximization problem [3] has been solved as a three step problem: In the first step, k_{ol} and k_{o2} were considered as exogenous parameters and the optimal consumption levels were obtained, in the second step, the optimal levels of the initial investments were derived, maximizing on k_{ol} and k_{o2} the indirect utility function of the first step problem:

- Given that this k_{ol}^* is always above the critical value, $k_{ol}crit$, *the optimal solution involves no liquidation of the long-term asset*. The consumption levels are the ones specified in Proposition 1. It is observed that for *low values of the random return, and up to a limit value, consumption is independent of* $\tilde{b}_{12}$, *but once this limit value is achieved, first period consumption decreases, second period consumption increases with the random return*. The explanation is that given the high value of the random return, it becomes advantageous to reinvest some of the return from the short-term asset available at $T=1$, in the new short-term technology. As mentioned in Hellwig [11], from an ex-ante point of view, the uncertainty about the random return is seen as a source of opportunities rather than a threat. While long-term investments are earmarked for consumption at date 2, short-term investments are not necessarily earmarked for consumption at date 1. The choice between consumption and investment depends on the rate of return $\tilde{b}_{12}$ on the new short-term investments.

2.1.2.- Second best allocations under incomplete information

In this case it is assumed that the realization of the timing of the consumption needs is private information of the consumer.

Given this information asymmetry, an allocation can only be implemented if it is incentive compatible, that is,

if it gives no consumer an incentive to lie or deviate about what he actually wants to consume.

If a type-2 agent claimed to be a type-1 he would get c_{11} units at $T=1$ and c_{21} units at $T=2$. If he reinvested his c_{11} units in the backyard, in the optimal way for him, his optimal consumption levels in periods 1 and 2 would be the solution to the following problem:

$$\max_{c_1, c_2} \left\{ c_1^{\delta_2} c_2^{\alpha - \delta_2} \right\}$$

$$\text{s.t} \quad c_1 \leq c_{11}$$

$$c_2 = (c_{11} - c_1) b_{12} + c_{21}$$

[10]

The optimal solution to this problem yields:

$$c_1^* = \left\{ \frac{\delta_2 (c_{21} + b_{12} c_{11})}{b_{12} \alpha} \right\}^{\delta_2} \leq c_{11} \qquad c_2^* = \left\{ \frac{(\alpha - \delta_2)(c_{21} + c_{11} b_{12})}{\alpha} \right\}^{\alpha - \delta_2}$$

[11]

Incentive compatibility requires that the consumption bundle he receives if he is honest (c_{12}, c_{22}), should be at least as large as what he gets by lying and reinvesting in the backyard; that is:

$$c_{12}^{\delta_2} c_{22}^{\alpha - \delta_2} \geq c_1^{*\delta_2} c_2^{*\alpha - \delta_2}$$

[12]

where c_1^*, c_2^* were derived above.

The incentive constraints for type-1 agents would be obtained in a similar way. However, in solving for the second-best allocation, it can be shown that the incentive compatibility constraints are never binding, and so they need not be taken into account in the maximization problem, but only verify that they are satisfied for the optimal solution, this yields the following:

In the case of a multiplicative utility function, the first-best solution is incentive compatible; that is, the first and second-best solutions coincide.

2.1.2.1.- Numerical simulations

Numerical simulations have been derived for the input data of Table II:

Table II

t	δ_1	δ_2	α	b_{o1}	b_{o2}	b_1	Eb_{12}, α
0.10	0.60	0.30	0.90	1.30	1.60	1.00	1.22 - 0.05

Figure 1 shows the first-step solution, for each exogenously-fixed value of k_{o1}, the solution to the second step problem gives the optimal levels of the initial investments:

Table III

k_{o1}	0.38	k_{o2}	0.62

The final solution to the second-best problem is shown in Figure 2. (Notice that $k_{o1}^* > k_{o1}crit = 0.30$, and so there is no liquidation of the long-term asset).

As it is observed, for low values of b_{12}, these consumption levels are independent of the random return, but once the limit value is attained ($b_{lim} = 1.16$), first period consumption decreases with the random return, second period increases, in this case it becomes advantageous to reinvest some of the return available at $T=1$ in the new short term investments.
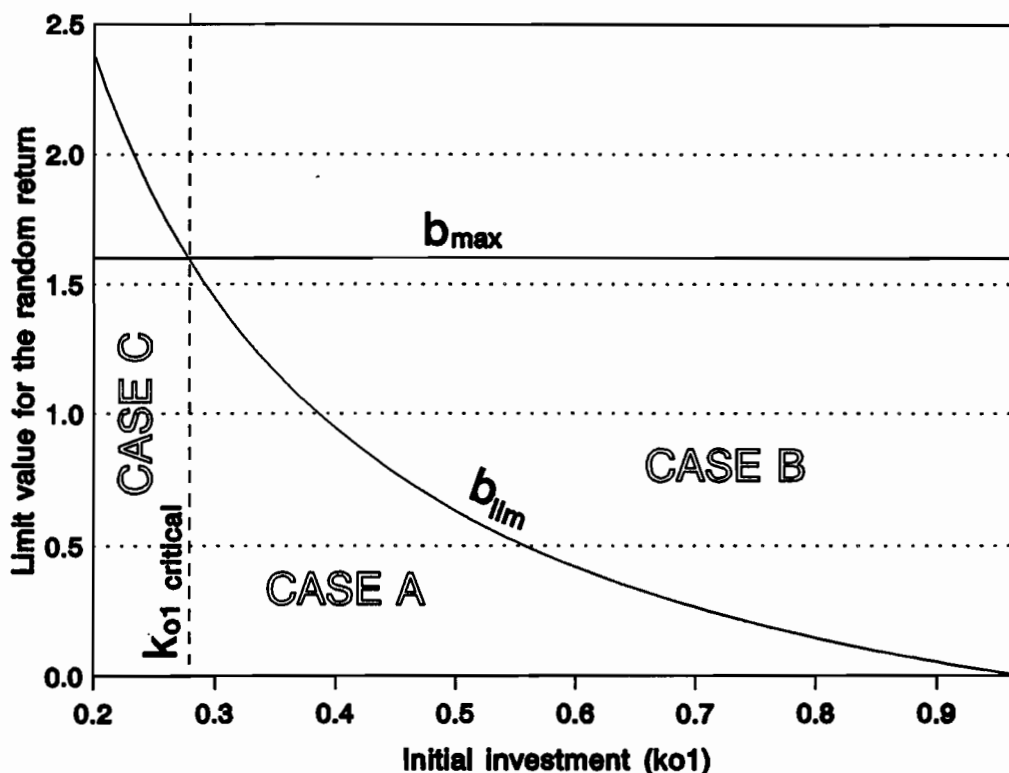
Figure 1.- First-step solution
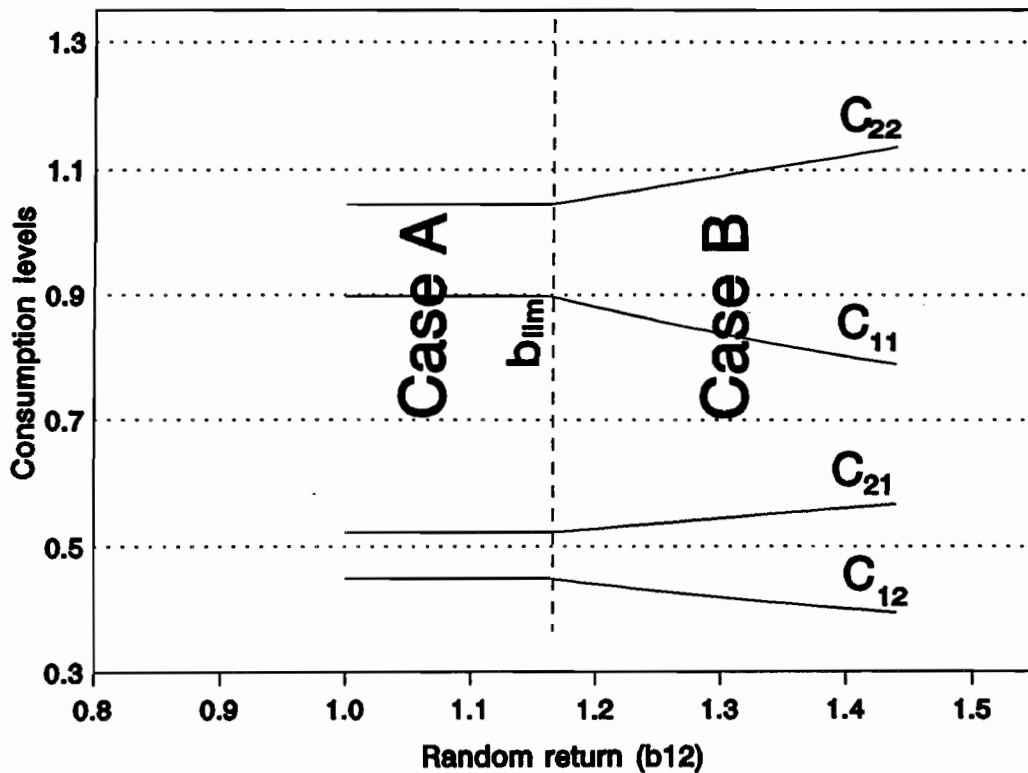
Figure 2.- Optimal consumption levels in the first-best (second-best) allocation

2.1.2.2.- Sensitivity analysis

a.-   Variations in the exogenous parameters

In order to see how the limit value of the random return (that distinguishes cases A and B of the zero-liquidation solution) is affected by variations in other exogenous parameters of the model, a sensitivity analysis has been done with respect to the expected value and the standard deviation of the random return (b_{12}), the proportion of type-1 agents (t) and the relation of long-term versus short-term returns $\left(\dfrac{b_{o2}}{b_{o1}}\right)$, as it is shown in Table IV.

Table IV	Variations in the exogenous parameters			
$E(b_{12})$	1.06	1.14	1.22	1.30
$\sigma_{b_{12}}$	0.03	0.05	0.07	0.08
t	0.00	0.10	0.60	1.00
b_{o2}/b_{o1}	1.10	1.23	1.38	1.5

All these cases are represented in Figure 3 to Figure 18.

It can be observed that the limit value of the random return depends on the characteristics of the investment opportunities that are considered:

i.- When the expected value of b_{12} or the standard deviation of b_{12} increase, b_{lim} moves to the left, up to a limit in which there may only be CASE B (reinvestment solution)

ii.- If the relation of the long-term versus short-term return increases, b_{lim} moves to the right, that is, given the high value of the long-term return, it is not so interesting to take advantage of the reinvestment opportunities, and so this limit value b_{lim} is higher.

iii.- Finally, variations in the proportion of type-1 agents do not have any significant effect on b_{lim}, in general, variations in parameters that correspond to the household side of the economy (risk-aversion coefficient or preference shock) do not influence this limit value (See sensitivity analysis of the additive utility function).

Figure 19 to Figure 22 show the optimal levels of the initial investment for each variation in the exogenous parameters. The optimal level of k_{o1} is increasing in the expected value and the standard deviation of the random return and in the proportion of type-i agents, it is decreasing in the relation of long-term versus short-term returns. It should be noticed that whenever $Eb_{12} > \dfrac{b_{o2}}{b_{o1}}$ the optimal level of $k_{o1}^* = 1$.
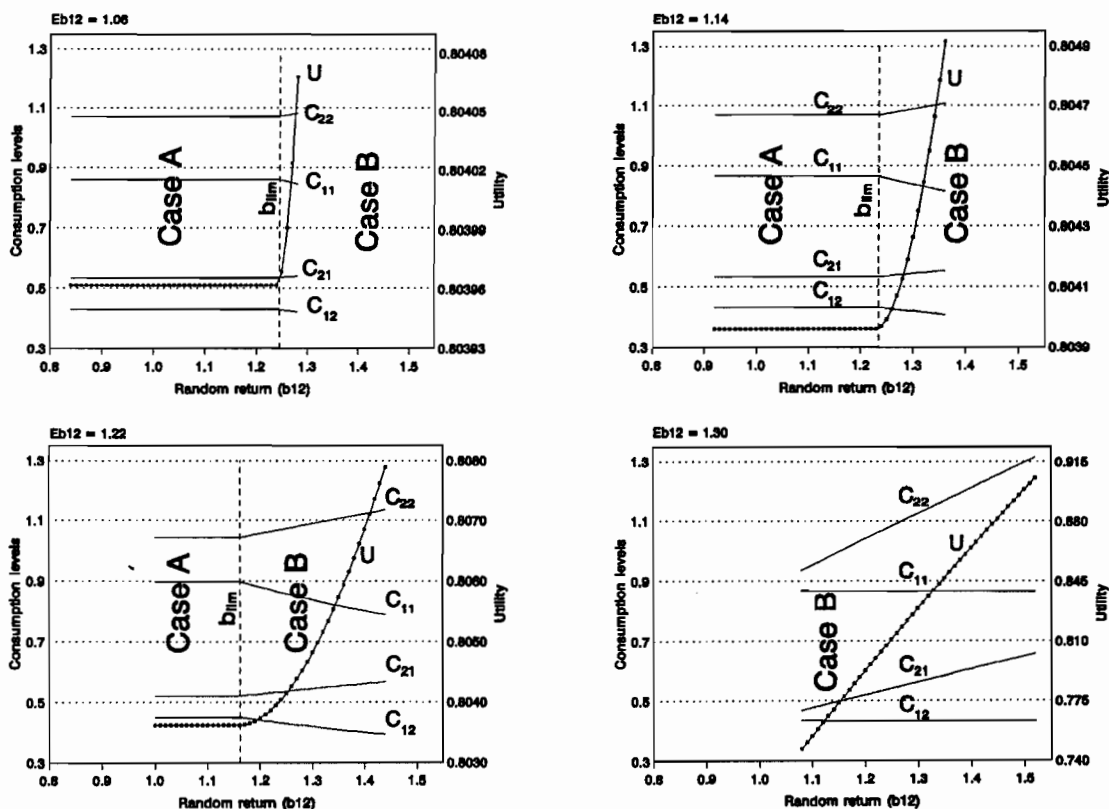

Figure 3 to **Figure 6**.- Variation in the expected value of b_{12}
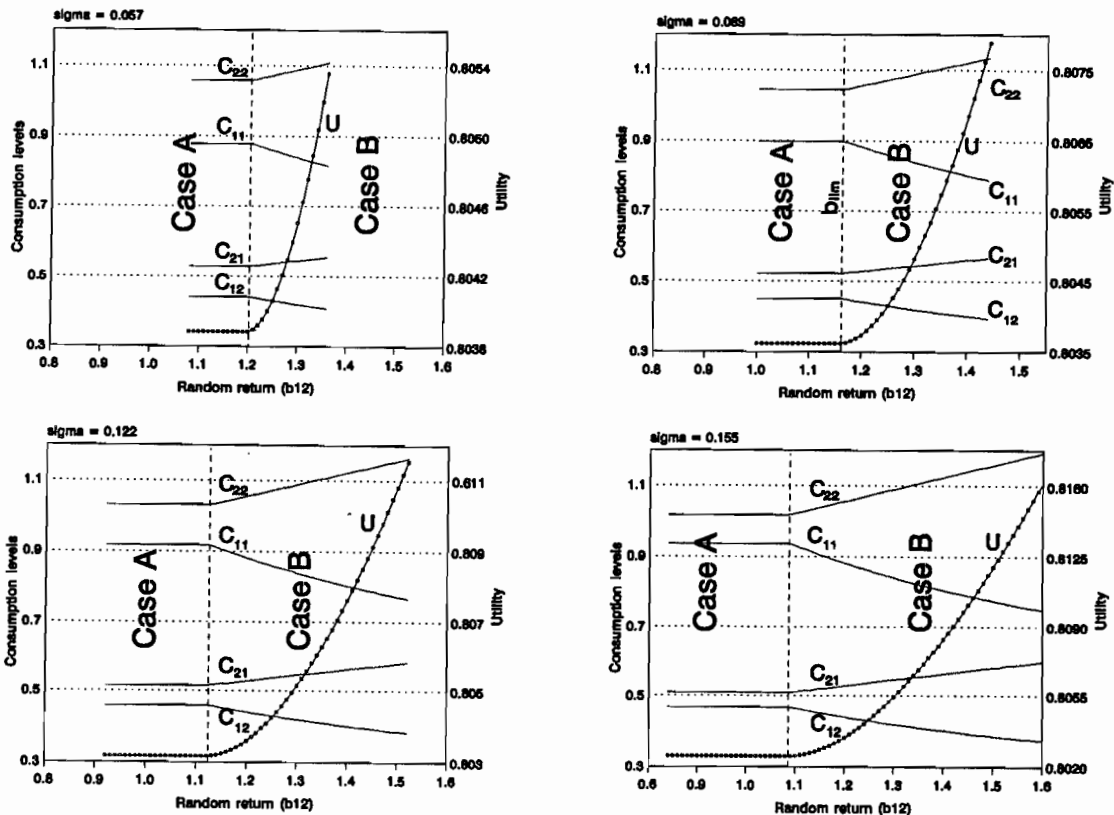
Figures 7 to **Figure 10**.- Variation in σ
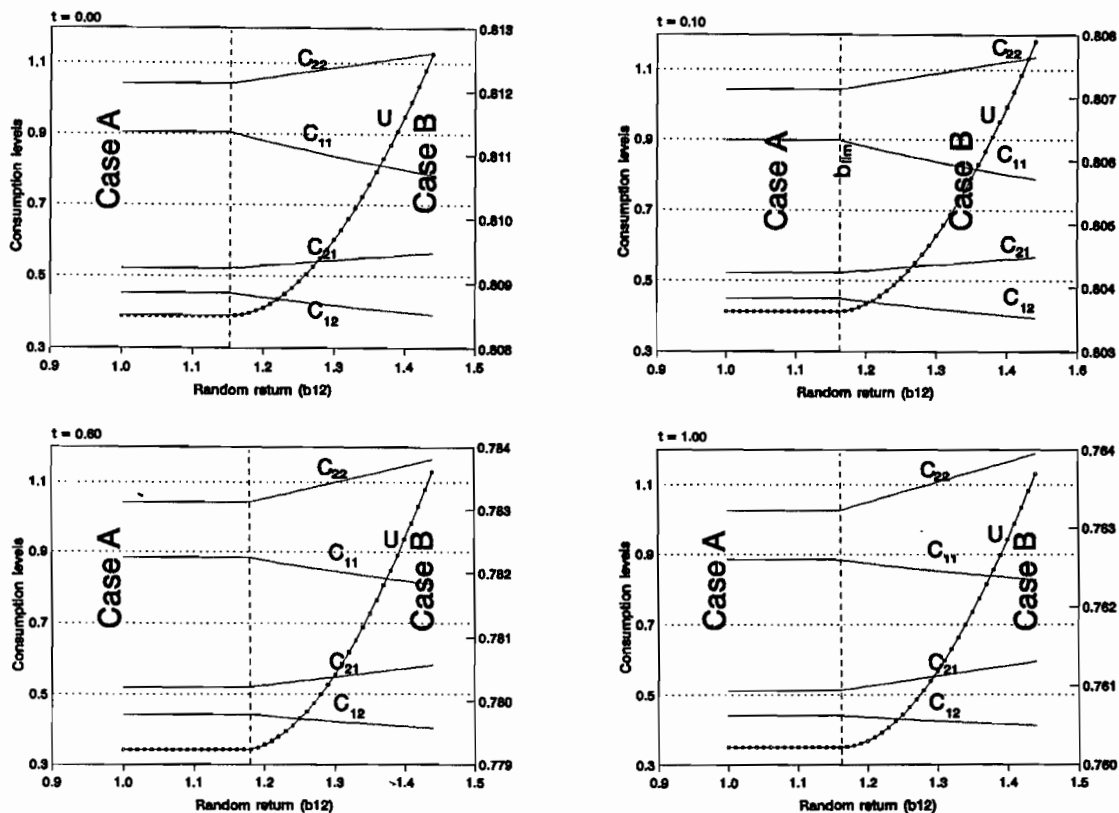
Figure 11 to **Figure 14**.- Variation in the proportion of type-1 agents (*t*)
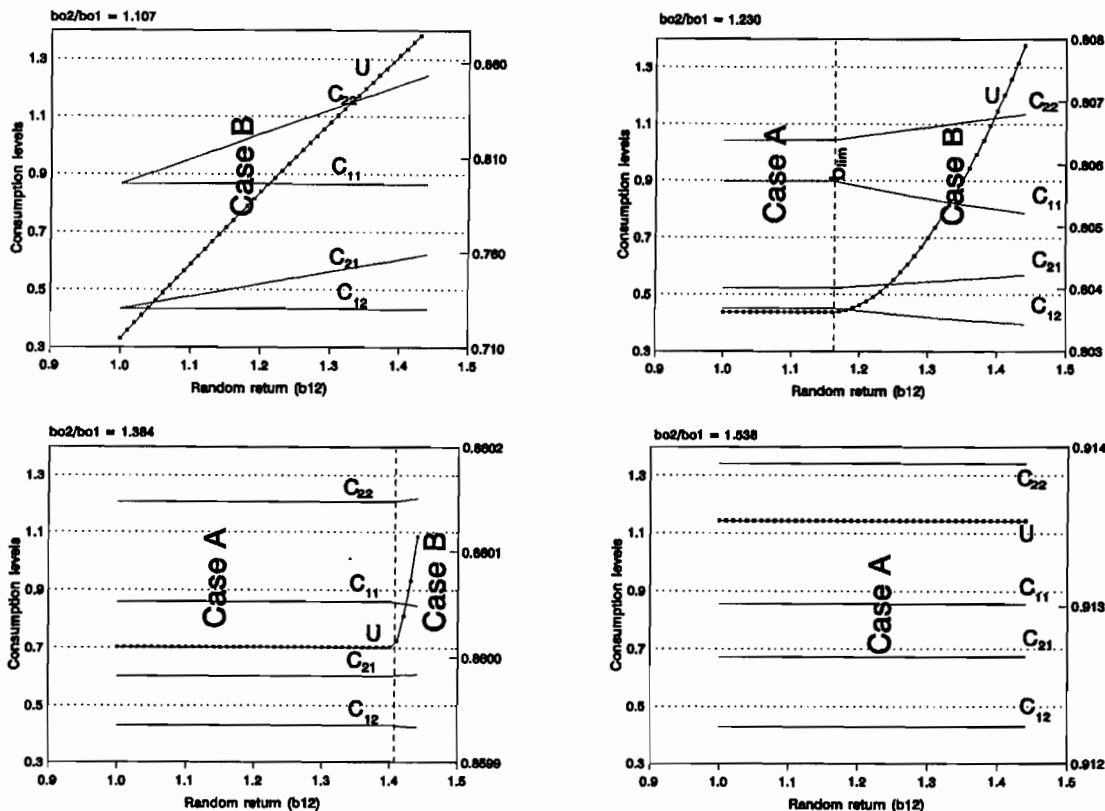
Figure 15 to Figure 18.- Variation in $\dfrac{b_{o2}}{b_{o1}}$
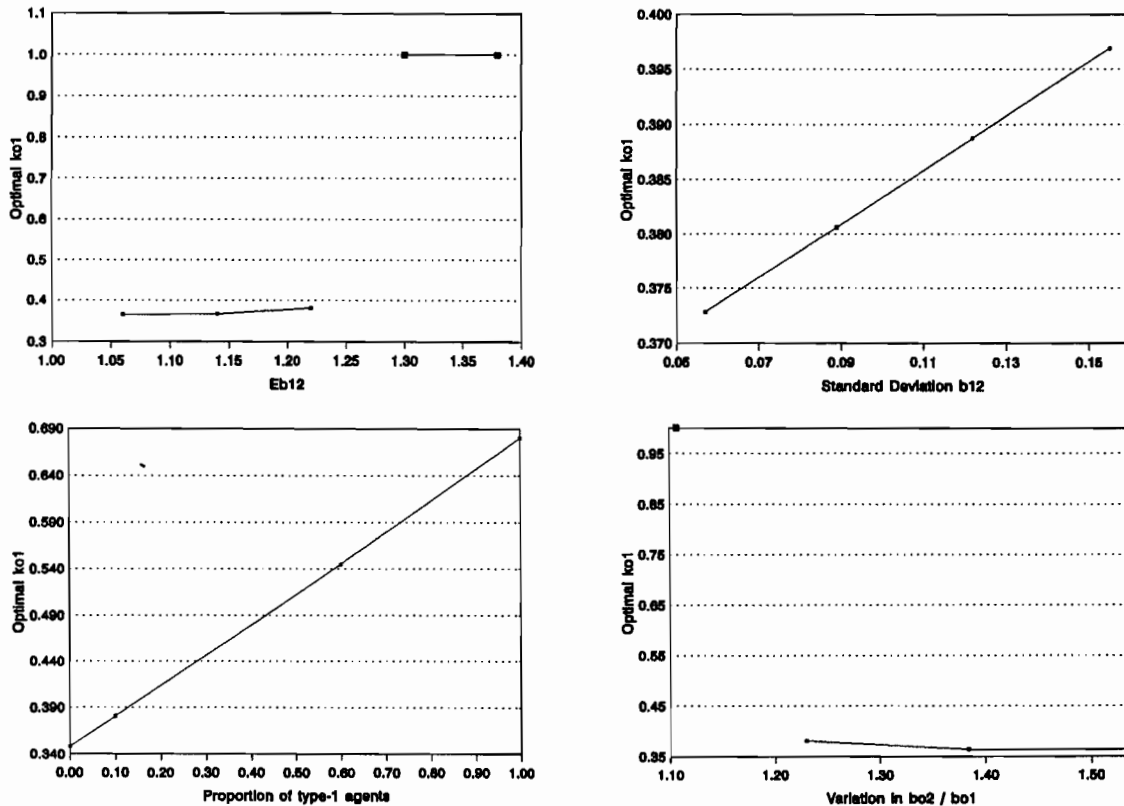
Figure 19 to Figure 22.- Variation in the optimal levels of the initial investment

2.2.- Additive Utility function

2.2.1.- First-best allocations under complete information

Similarly to the multiplicative case, the efficient allocation will be the solution to the following problem:

$$\max_{\tilde{c}_{ij}, k_{o1}, k_{o2}, \tilde{\mu}} E\left[tU(\tilde{c}_{11}, \tilde{c}_{21}) + (1-t)U(\tilde{c}_{12}, \tilde{c}_{22})\right]$$

s.t $\quad k_{o1} + k_{o2} = k_o$

$t\tilde{c}_{11} + (1-t)\tilde{c}_{12} \le b_{o1}k_{o1} + b_1\tilde{\mu}k_{o2}$

$t\tilde{c}_{21} + (1-t)\tilde{c}_{22} = b_{o2}(1-\tilde{\mu})k_{o2} + \tilde{b}_{12}\left[b_{o1}k_{o1} + b_1\tilde{\mu}k_{o2} - t\tilde{c}_{11} - (1-t)\tilde{c}_{12}\right]$ [13]

$b_1\tilde{b}_{12} < b_{o2}.$

$\tilde{\mu} \le 1$

$\tilde{c}_{ij} \ge 0$

$\tilde{\mu} \ge 0$

and where the utility function is the one described above in Point d of Page 5.
As before, this maximization problem is solved as a three-step problem:

2.2.1.1.- First step: Optimal consumption levels and liquidation policy.

In this step, b_{o1}, b_{o2}, b_1, b_{12}, k_{o1}, k_{o2} are considered as exogenous parameters and so the problem may be rewritten:

$$\max_{c_{ij}, \mu} E\left[tU(c_{11}, c_{21}) + (1-t)U(c_{12}, c_{22})\right]$$

s.t $\quad k_{o1} + k_{o2} = k_o$

$tc_{11} + (1-t)c_{12} \le b_{o1}k_{o1} + b_1\mu k_{o2}$

$tc_{21} + (1-t)c_{22} = b_{o2}(1-\mu)k_{o2} + b_{12}\left[b_{o1}k_{o1} + b_1\mu k_{o2} - tc_{11} - (1-t)c_{12}\right]$ [14]

$b_1b_{12} < b_{o2}$

$\mu \le 1$

$c_{ij} \ge 0$

$\mu \ge 0$

λ_1 and λ_2 are the Lagrange multipliers associated with the first and second resource balance constraints and λ_3 the multiplier associated with μ.

In this first step, the solutions are given in Table V.

Table V

Optimal liquidation solution: $k_{ol} < k_{ol}crit$ and $\mu > 0$			
CASE C $\lambda_1 > 0$			
Optimal zero-liquidation solution: $k_{ol} \geq k_{ol}crit$ and $\mu = 0$			
CASE A $\lambda_1 > 0$	if $b_{12} < b_{\lim}$	CASE B $\lambda_1 = 0$	if $b_{12} \geq b_{\lim}$

The consumption levels of cases A, B and C, the aggregate expected utilities ($U^{*(a)}$, $U^{*(b)}$, $U^{*(c)}$) corresponding to each of these cases, the value of $k_{ol}crit$ and $b_{\lim}$ are given in Appendix B. As in the multiplicative case, if the exogenously fixed value of k_{ol} is below the critical one, the optimal solution involves partial liquidation of the long-term asset, and consuption is independent of the short term random return. For values of k_{ol} above the critical one, there is never liquidation of long-term tecnology. In this case, for values of $b_{12} < b_{\lim}$, consumption is constant and once this limit is attained it becomes dependent on the random return.

2.2.1.2.- Second step: Optimal levels of the initial investments

The first-best investment levels at date zero are obtained by dynamic programming, maximizing on k_{ol} and k_{o2} the indirect utility function of the problem described above, i.e.:

$$\max_{k_{ol}, k_{o2}} \left[\begin{array}{ll} \int_{b_{min}}^{b_{\lim}} U^{*(A)} f(b_{12}) db_{12} + \int_{b_{\lim}}^{b_{max}} U^{*(B)} f(b_{12}) db_{12} & \text{if} \quad k_{ol} \geq k_{ol}crit \\ \int_{b_{min}}^{b_{max}} U^{*(C)} f(b_{12}) db_{12} & \text{if} \quad k_{ol} < k_{ol}crit \end{array} \right] \quad [15]$$

s.t. $k_{ol} + k_{o2} = 1$

Lemma 2. The optimal k_{ol}^* satisfies: $k_{ol}crit \leq k_{ol}^* \leq 1$

where:

$$k_{ol}crit = \frac{b_{o2}^{\frac{\gamma-1}{\gamma}} \rho_1^{-1/\gamma}}{b_{ol} b_1^{-1/\gamma} \left[t + (1-t) \left(\frac{\rho_1}{\rho_2} \right)^{-1/\gamma} \right] + b_{o2}^{\frac{\gamma-1}{\gamma}} \rho_1^{-1/\gamma}} \quad [16]$$

Proof: See Appendix B.

2.2.1.3.- Third step: Final solution

The optimal levels of k_{ol}, k_{o2} are substituted back into the first-step problem and so the final solution is reached.

This solution to the first best problem gives the main result of the section, expressed by the proposition below:

Proposition 2. Let $(k_{o1}, k_{o2}, c_{11}, c_{12}, c_{21}, c_{22}, \mu)$ be a solution to the first best problem and define:

$$b_{lim} = \left[\frac{\left\{ b_{o1} k_{o1} \left[t + (1-t)\left(\frac{\rho_1}{\rho_2}\right)^{-1/\gamma} \right] \right\}^{-\gamma}}{b_{o2} k_{o2}} \right] \frac{1}{\rho_1} \qquad [17]$$

Then: if $\tilde{b}_{12} < b_{lim}$ CASE A $(\lambda_1 > 0)$

$$c_{12}^* = b_{o1} k_{o1} \qquad c_{21}^* = \frac{b_{o2} k_{o2}}{\left[t + (1-t)\frac{\rho_1}{\rho_2} \right]^{-1/\gamma}}$$

$$c_{11}^* = b_{o1} k_{o1} \qquad c_{22}^* = \left[\frac{\rho_1}{\rho_2} \right]^{-1/\gamma} c_{21}^* \qquad \mu^* = 0 \qquad [18]$$

if $\tilde{b}_{12} \geq b_{lim}$ CASE B $(\lambda_1 = 0)$

$$\tilde{c}_{12}^* = \tilde{c}_{11}^* \qquad\qquad \tilde{c}_{11}^* = \left[\rho_1 \tilde{b}_{12} \right]^{-\frac{1}{\gamma}} \tilde{c}_{21}^*$$

$$\tilde{c}_{21}^* = \frac{b_{o1} k_{o1} \tilde{b}_{12} + b_{o2} k_{o2}}{\left[t + (1-t)\frac{\rho_1}{\rho_2} \right]^{-1/\gamma} + \tilde{b}_{12}^{\frac{\gamma-1}{\gamma}} \rho_1^{-\frac{1}{\gamma}}} \qquad \tilde{c}_{22}^* = \left[\frac{\rho_1}{\rho_2} \right]^{-\frac{1}{\gamma}} \tilde{c}_{21}^* \qquad \tilde{\mu}^* = 0 \qquad [19]$$

Proof: See Appendix A.

Proposition 2 characterizes the first best allocation under complete information. As in the multiplicative case, for low values of the short term return $(\tilde{b}_{12})$, consumption is independent of it (this is explained by the fact that in Case A the multiplier associated with the first constraint is positive and therefore the constraint is binding), once the limit value is attained, consumption becomes responsive to the random return, first period consumption decreases with the random return and second period consumption increases with it (in Case B $\lambda_1 = 0$ and therefore its associated constraint is not binding). As already mentioned, the intuition is that given the high value of the random return, it becomes advantegous to reinvest some of the return available at date 1 in this new short term asset.

It should be observed that *the first-best allocation involves no liquidation of the long-term technology,* $(\mu^* = 0)$, as the optimal level of the initial investment, (k_{o1}) is always above the critical level $k_{o1} crit$.

2.2.1.4.- Numerical simulations

In order to provide a graphical plot of the optimal solution, some numerical simulations have been developed for the input data given in Table VI:

Table VI

t	ρ_1	ρ_2	γ	$\bar{b}_{o1}$	$\bar{b}_{o2}$	b_1	$Eb_{12} - \sigma$
0.10	0.30	0.60	1.50	1.30	1.60	1.00	1.22 - 0.05

Figure 23 illustrates the solution to the first step problem; there are three possible cases depending on the fixed exogenous value of k_{o1} that is considered. The second step problem gives the optimal levels of the initial investments by integration of the indirect utility function (See Figure 24); the solution to the second step problem is given in Table VII:

Table VII

k_{o1}^*	0.62	k_{o2}^*	0.38

The optimal levels of the initial investments, for each value of the relative-risk aversion coefficient are shown in Figure 25.

These optimal levels of k_{o1} and k_{o2} are substituted back into the first step solution to reach the final solution shown in Figure 26. It should be observed that in this example the value of $k_{o1}crit = 0.5677 \le k_{o1}^*$ and so there is no liquidation of the long-term asset.

Some remarks, concerning the solution to the first-best problem should be pointed out:

a.- For values of $\bar{b}_{12}$ smaller than the limit value b_{lim}, the optimal consumption levels are the ones given by CASE A, and are independent of the random return, otherwise the consumption levels would be given by CASE B and depend on the value of $\bar{b}_{12}$.

This result shows that first period optimal levels of consumption decrease (second period consumption increase) as the random return $\bar{b}_{12}$ increases. (This is explained by the fact that in CASE A the Lagrange multiplier λ_1 is > 0 and therefore the corresponding resource balance constraint is satisfied with equality. On the contrary, in CASE B $\lambda_1 = 0$ and the constraint is satisfied with strict inequality). More exactly, when the random return from new short-term investments exceeds the critical value b_{lim} it is desirable to reinvest some of the return $k_{o1}b_{o1}$ available at T=1 in the new short-term investments, in order to take advantage of this favourable opportunity.

b.- Premature liquidations of the long-term asset are always zero under the first-best solution.

c.- In general, the results that are obtained do not differ from the ones in Hellwig [11].
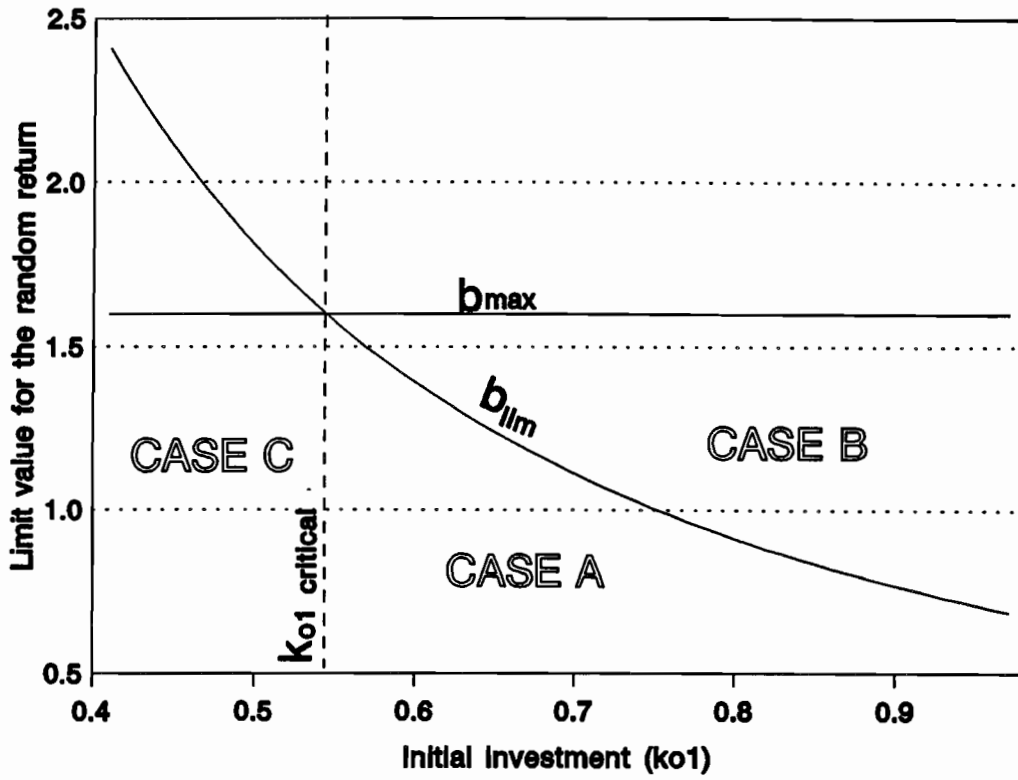
Figure 23.- First-step solution
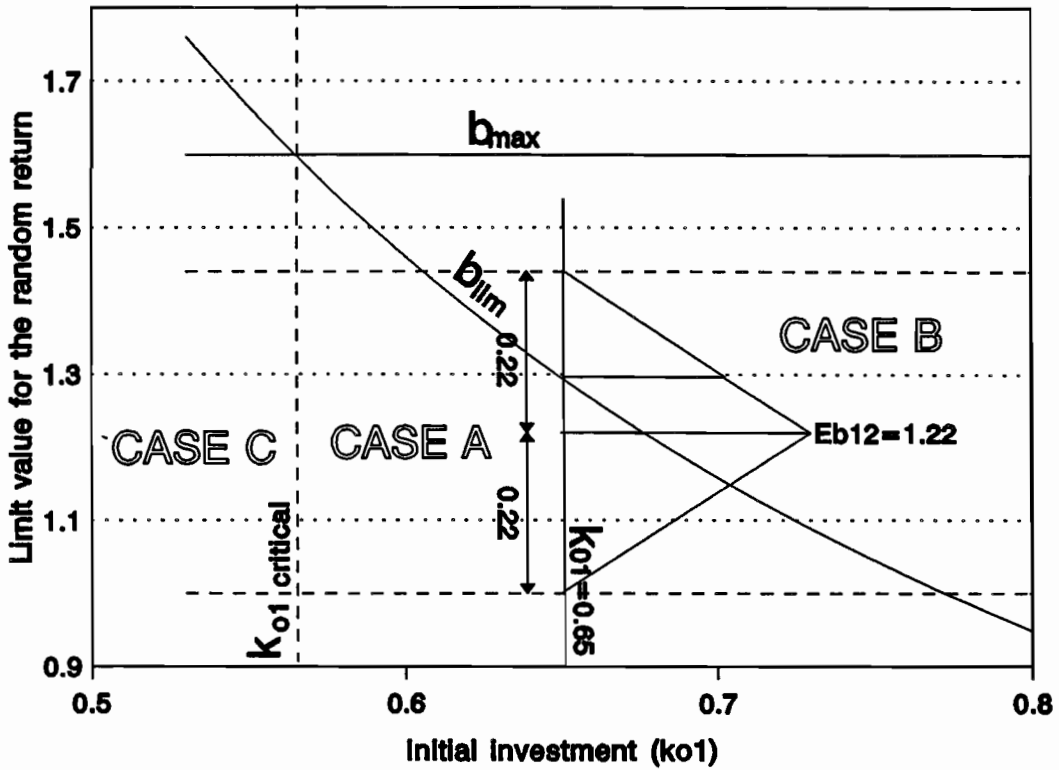
Figure 24.- Integration procedure of the second step solution
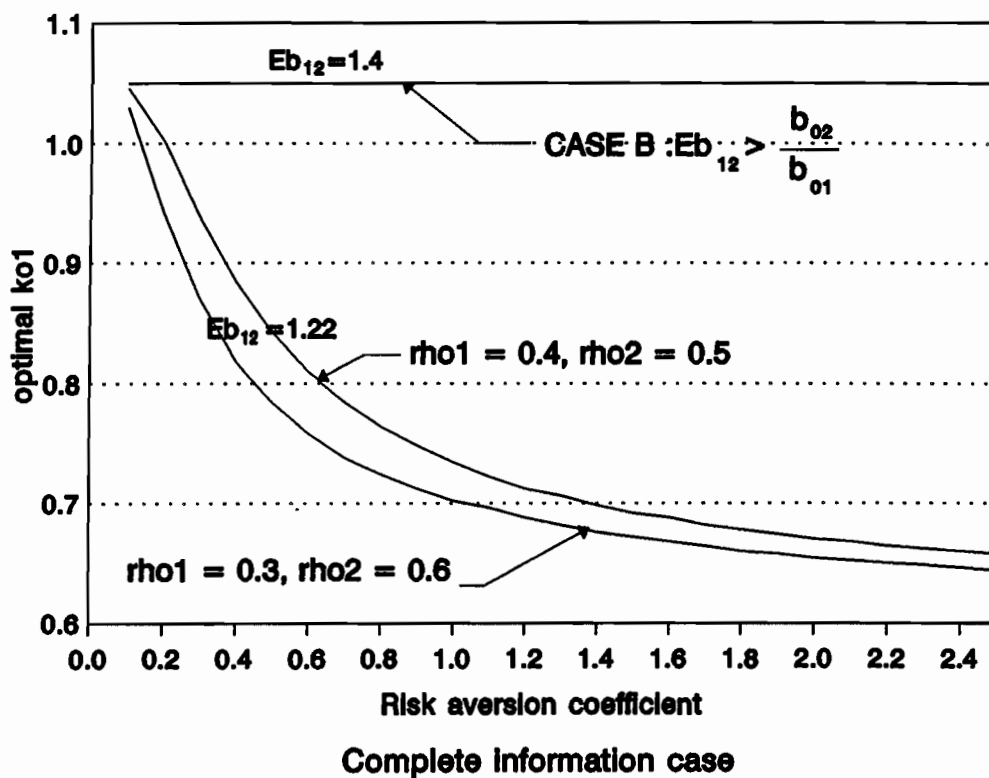
Figure 25.- Second step solution: Optimal levels of the initial investments.
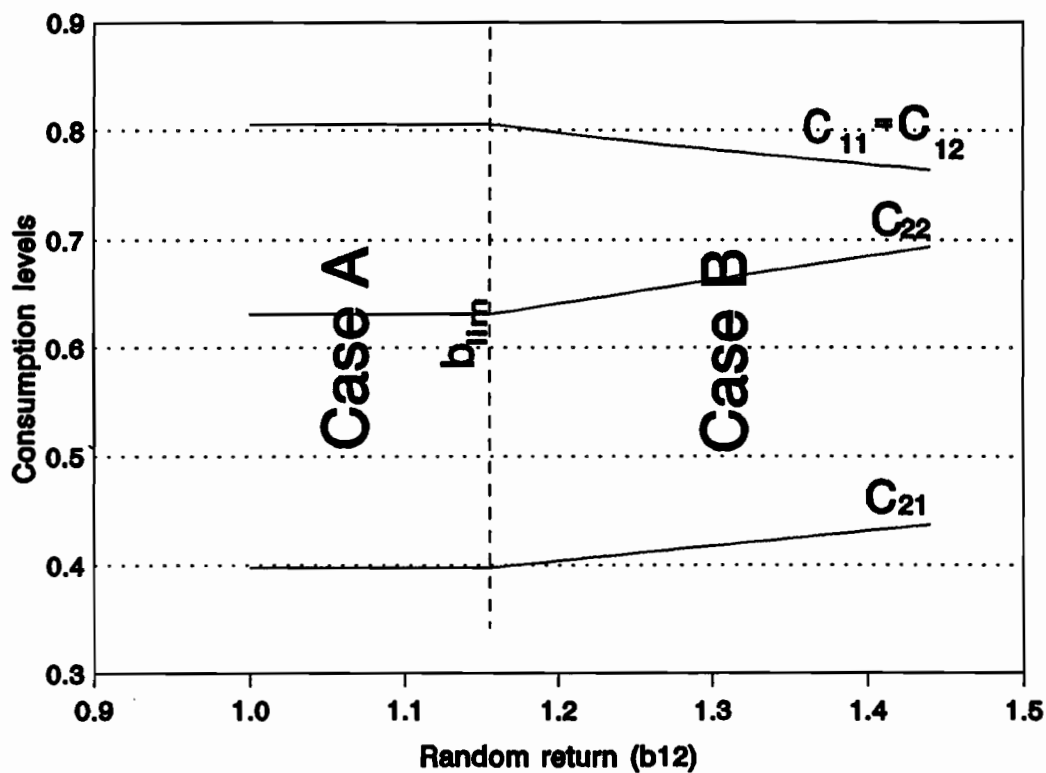
Figure 26.- Optimal consumption levels in the first-best allocation.

2.2.1.5.- Sensitivity analysis

a.- Sensitivity with respect to γ in the case of a large and a small preference shock.

Table VIII	p_1	p_2	Relative Risk-aversion coefficient			
Small shock	0.4	0.5	0.5	1.5	2.0	2.5
Large shock	0.3	0.6	0.5	1.5	2.0	2.5

All these cases are represented in Figure 27 to Figure 34

There are some remarks to be pointed out:

i.- *Comparison between a large and a small shock.* It can be seen that in all figures (independently of risk aversion), in the case of a small preference shock, first period consumption is always higher and the difference between second period consumption, $c_{22} - c_{21}$ is lower.

ii.- *Comparison with respect to γ.* As the relative risk-aversion coefficient increases, first period consumption diminishes, and second period consumption increases. This occurs both in the case of a small and a large shock[2]. Also as γ increases, the difference between $c_{22} - c_{21}$ diminishes.

iii.- It should be noticed that the limit value of the random return b_{12} that distinguishes CASE A ($\lambda_1 > 0$) from CASE B ($\lambda_1 = 0$) varies very slightly from one figure to the other.

---

[2] *This result confirms that of Breeden (1984), he demonstrates that for the class of utility functions with constant relative risk-aversion, if agents are sufficiently risk-averse, they may choose to reverse hedge, that is they choose to consume less now in order to invest more in the future.*
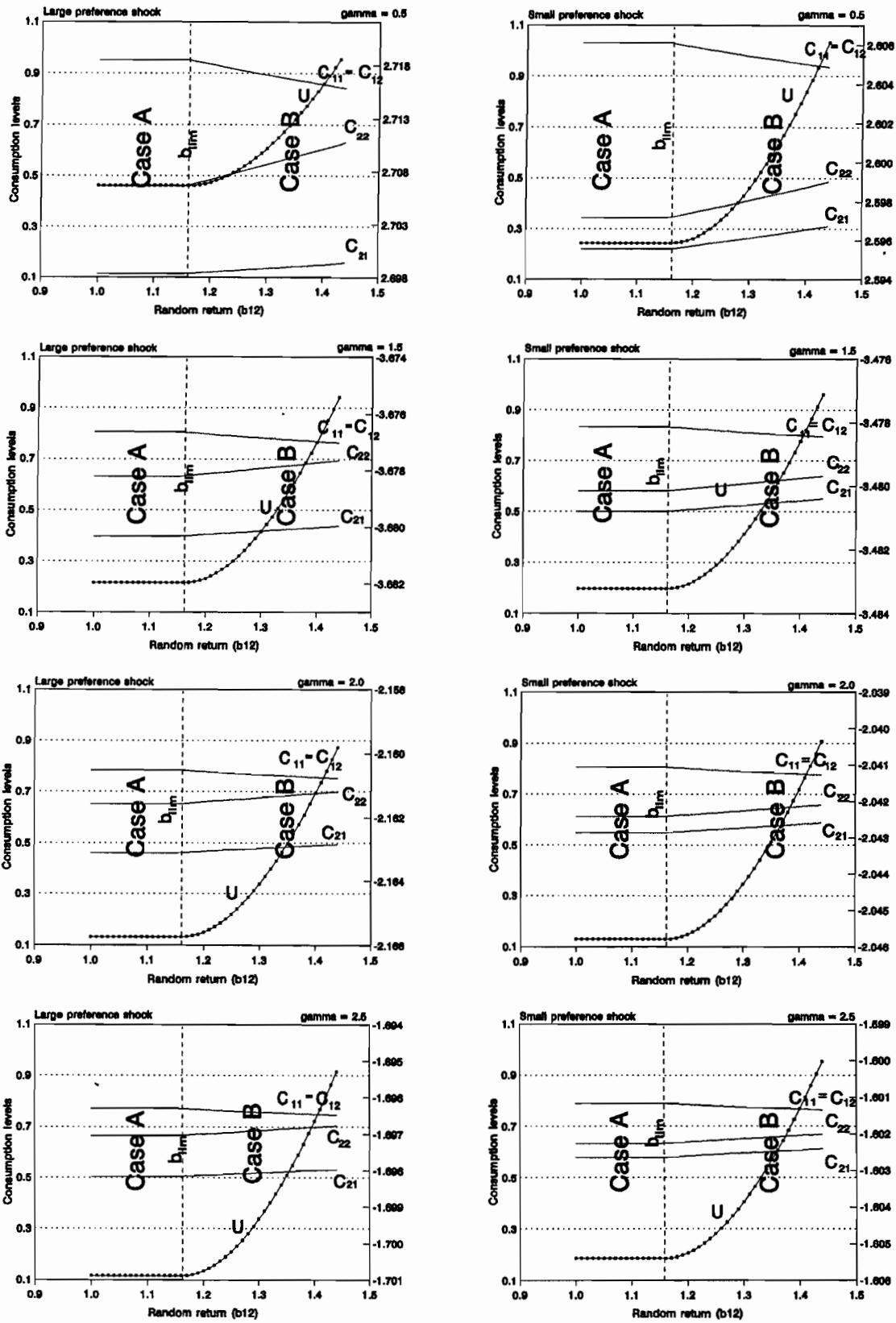
Figure 27 to **Figure 34**.- Variations in ρ and γ

2.2.2.- Second best allocations under incomplete information

In this case it is assumed that the realization of the timing of the consumption needs is private information of the consumer.

Given this information asymmetry, an allocation can only be implemented if it is incentive compatible, that is, if it gives no consumer an incentive to lie or deviate about what he actually wants to consume.

If a type-2 agent claimed to be a type-1 he would get c_{11} units at $T=1$ and c_{21} units at $T=2$.

If he reinvested his c_{11} units in the backyard, in the optimal way for him, his optimal consumption levels in periods 1 and 2 would be the solution to the following problem:

$$
\max_{c_1,c_2} \left\{ \frac{c_1^{1-\gamma}}{1-\gamma} + \rho_2 \frac{c_2^{1-\gamma}}{1-\gamma} \right\}
$$

$$
\text{s.t} \quad c_1 \le c_{11}
$$

$$
c_2 = (c_{11} - c_1) b_{12} + c_{21}
$$

[20]

The optimal solution to this problem yields:

$$
c_1^* = (\rho_2 b_{12})^{-\frac{1}{\gamma}} \frac{c_{21} + b_{12} c_{11}}{1 + b_{12}^{(\gamma-1)/\gamma} \rho_2^{-1/\gamma}} \le c_{11} \qquad c_2^* = \frac{c_{21} + c_{11} b_{12}}{1 + b_{12}^{(\gamma-1)/\gamma} \rho_2^{-1/\gamma}}
$$

[21]

Incentive compatibility requires that the consumption bundle he receives if he is honest (c_{12}, c_{22}), should be at least as large as what he gets by lying and reinvesting in the backyard; that is:

$$
\frac{c_{12}^{1-\gamma}}{1-\gamma} + \rho_2 \frac{c_{22}^{1-\gamma}}{1-\gamma} \ge \frac{c_1^{*1-\gamma}}{1-\gamma} + \rho_2 \frac{c_2^{*1-\gamma}}{1-\gamma}
$$

[22]

where c_1^*, c_2^* were derived above.

The incentive constraints for type-1 agents would be obtained in a similar way.

In the absence of any other backyard technology (for converting date 2 consumption into date 1) there is no other incentive constraint to be considered.

Taking the incentive constraints into account, the second-best problem is a solution to the following one:

$$\max_{\tilde{c}_{ij}, k_{o1}, k_{o2}, \tilde{\mu}} E\left[tU(c_{11}, \tilde{c}_{21}) + (1-t)U(c_{12}, \tilde{c}_{22})\right]$$

$$\text{s.t} \quad k_{o1} + k_{o2} = k_o$$

$$t\tilde{c}_{11} + (1-t)\tilde{c}_{12} \le b_{o1}k_{o1} + b_1\tilde{\mu}k_{o2}$$

$$t\tilde{c}_{21} + (1-t)\tilde{c}_{22} = b_{o2}(1-\tilde{\mu})k_{o2} + \tilde{b}_{12}\left[b_{o1}k_{o1} + b_1\tilde{\mu}k_{o2} - t\tilde{c}_{11}(1-t)\tilde{c}_{12}\right] \qquad [23]$$

$$b_1\tilde{b}_{12} < b_{o2}$$

$$\tilde{\mu} \le 1$$

$$\tilde{c}_{ij} \ge 0$$

$$\tilde{\mu} \ge 0$$

$$[24]$$

IC constraints

2.2.2.1.- Numerical simulations

The analytical treatment of the second-best solution is quite a tedious one, therefore numerical solutions have been computed[3]. The working procedure is the same as for the first-best case, i.e., the problem is solved in three steps.

There are some remarks to be pointed out:

a.- In the second-best allocation the incentive constraint for type-1 agents is binding, whereas that of type-2 agents is never binding.

b.- The second-best allocation does not involve liquidation of the long-term asset (see sensitivity analysis with respect to b_1). This result differs from Hellwig as in his case the second best allocation may involve liquidation of the long-term technology. Although this result is based on numerical analysis, it seems that similarly to the first best allocation, the utility function is always a continous and increasing function in k_{o1} in Case C, and therefore, the optimal level of the initial investment will be at least $k_{o1}crit$. On the contrary, in Hellwig's case, the utility function (in the liquidation solution) is increasing in k_{o1} but it is not continous in the limit case $k_{o1}crit$, that distinguishes the liquidation and non-liquidation solutions, and therefore, the optimal k_{o1}^* may occur in the liquidation case, for values of k_{o1} sufficiently close but below $k_{o1}crit$.

c.- The optimal solution has been derived for the input data of Table VI.

A graphical plot of the optimal solution is given by Figure 36. The optimal initial investment levels are shown

---

[3] *The system of non-linear equations was solved by the Newton Raphson technique, with the use of a computer program, that is explained in Appendix A.*

in Table IX.

Table IX

k_{o1}	0.62	k_{o2}	0.38

The optimal levels of the initial investments for each value of gamma are shown in Figure 35.

2.2.2.2.- Sensitivity analysis

a.- Sensitivity with respect to γ in the case of a large and a small preference shock as was shown in Table VIII, in the first-best case.

All these cases are represented in Figure 37 to Figure 44.

Some remarks should be pointed out:

    i.- *Comparison between a large and a small shock.* It can be seen that in all figures (independently of risk aversion), in the case of a small preference shock, the difference between first period consumption $c_{11}-c_{12}$ and second period consumption, $c_{22}-c_{21}$ is always lower with respect to the case of a large shock.

    ii.- *Comparison with respect to γ.* As the relative risk-aversion coefficient increases, first period consumption diminishes, and second period consumption increases. This occurs both in the case of a small and a large shock. Also, as γ increases the difference $c_{11}-c_{12}$ and $c_{22}-c_{21}$ also diminishes.

    iii.- It should be noticed that the limit value of the random return b_{12} that distinguishes CASE A ($\lambda_1 > 0$) from CASE B ($\lambda_1 = 0$) varies very slightly from one figure to the other.

b.- Sensitivity with respect to b_1

A sensitivity analysis with respect to the liquidation value of the long-term asset, has been done. The following values, given in Table X, for b_1 ($b_1 < b_{o1}$) have been considered.

Table X

Liquidation value of the long-term asset (b_1)			
0.00	0.50	1.00	1.10

There is a unique solution shown in Figure 45 due to the fact that these variations do not affect the optimal solution.
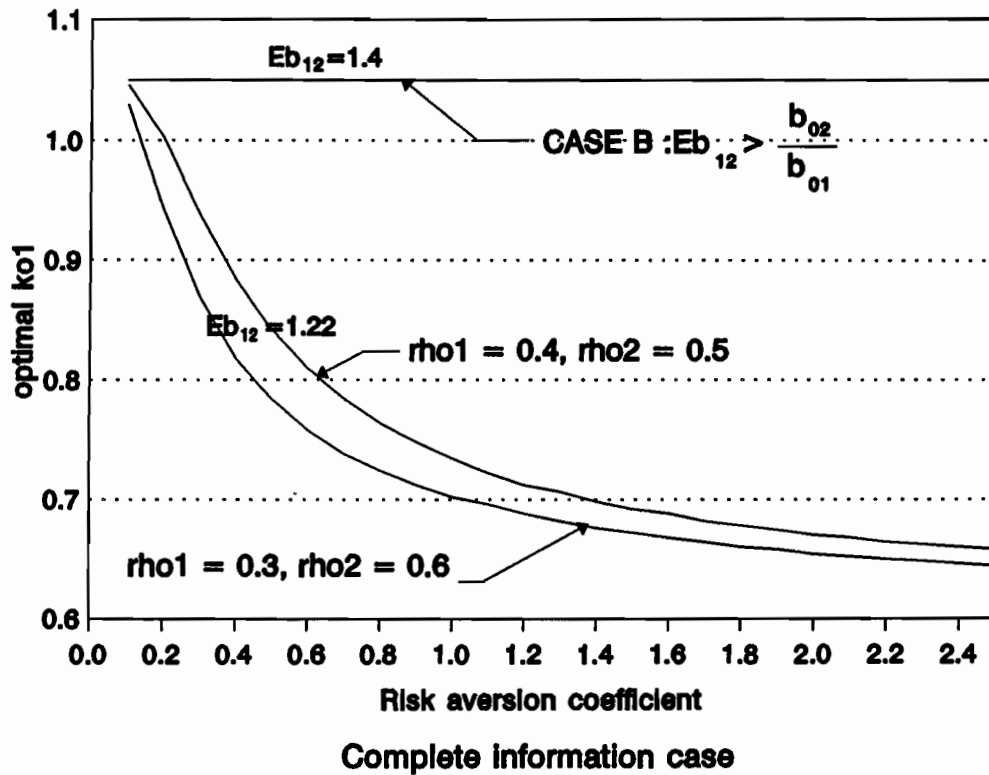
Figure 35.- Optimal levels of the initial investments.
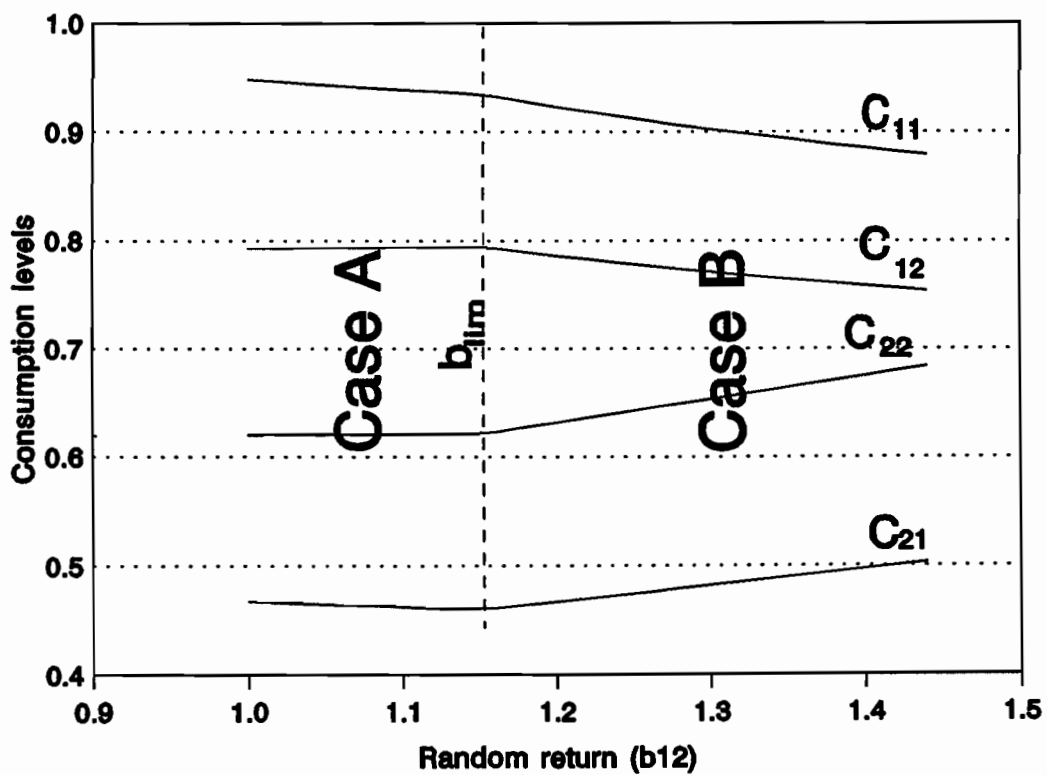
Figure 36.- Optimal consumption levels in the second-best allocation
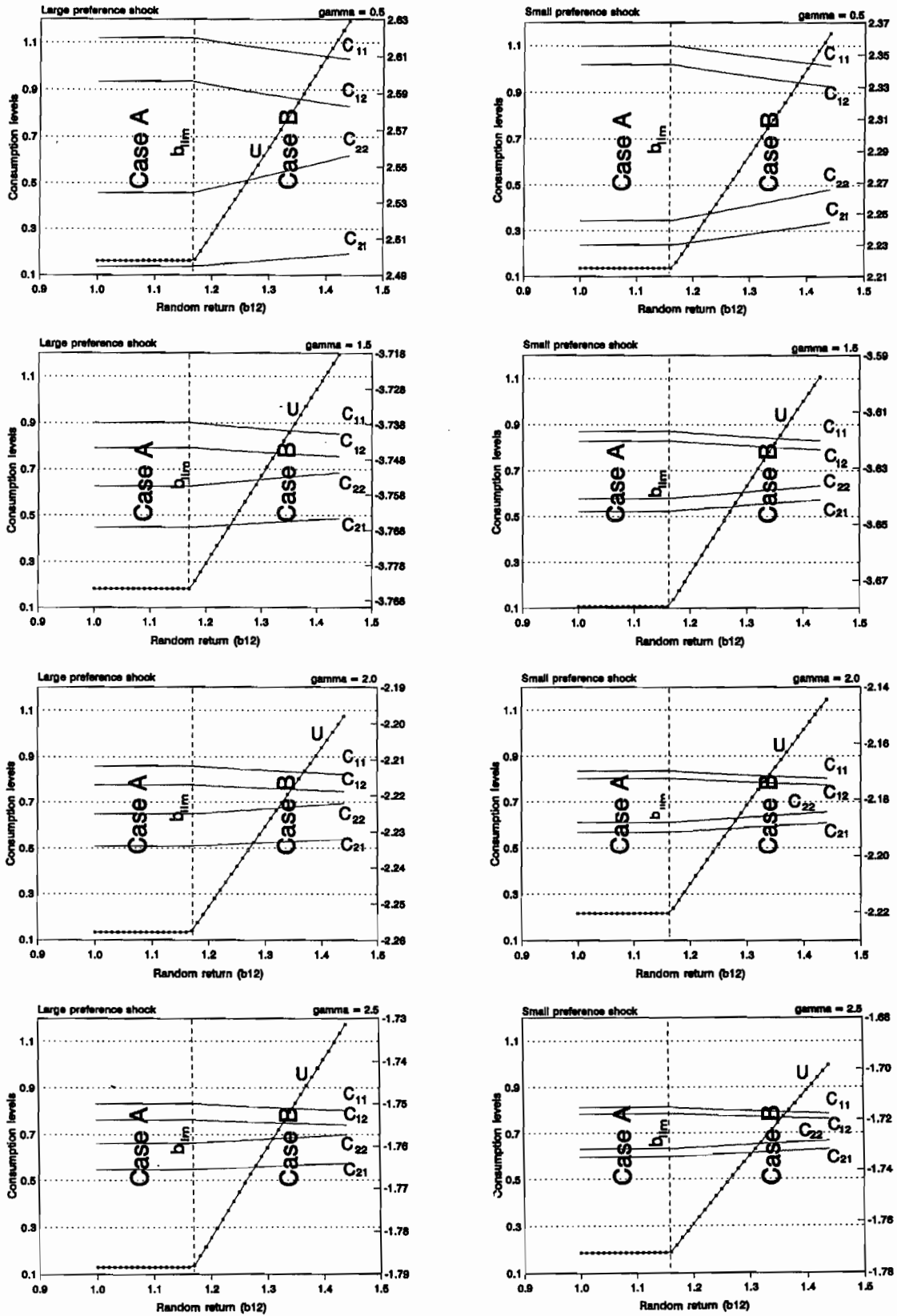
Figures 37 to 44.- Variations in ρ and γ.
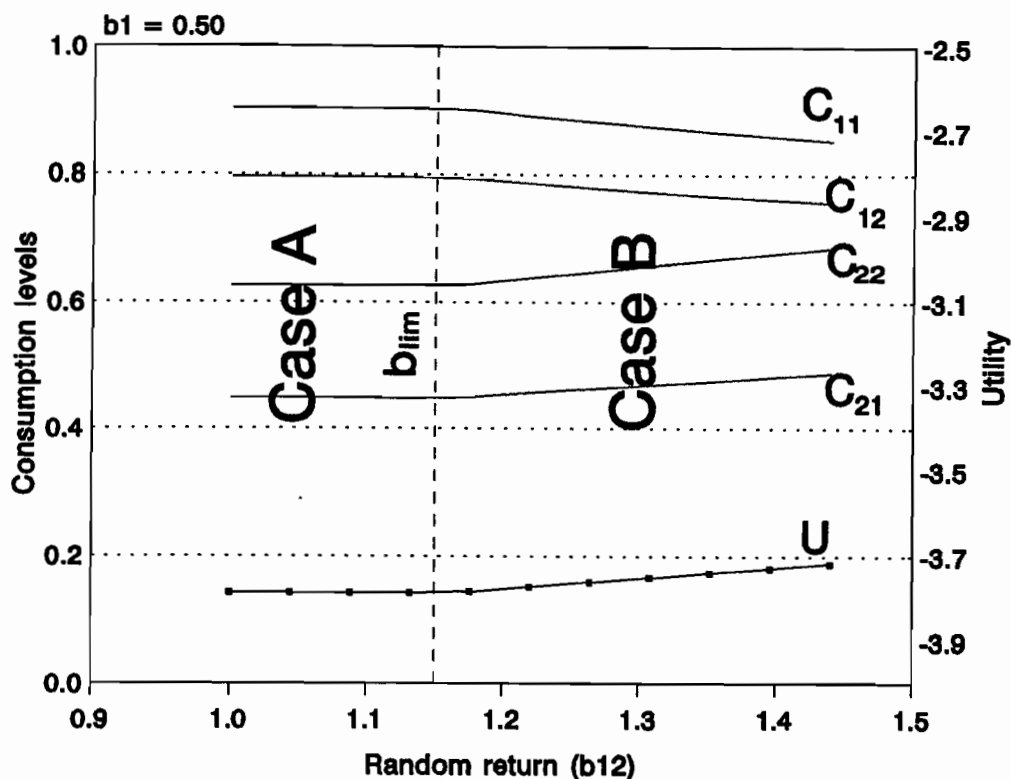
Figure 45.- Variation in the liquidation value of the long-term asset.

It should be noticed that there is never liquidation of the long-term asset, as the optimal solutions are always in cases A and B.

2.2.3.- Comparison between the first-best and the second-best allocations.

The above sections have characterized the first and second-best consumption allocations with respect to the random return from the new short-term investment b_{12}.

A main feature of both solutions is that *for low values of b_{12} up to a limit value (that coincides for the first and second best) the first constraint is binding and so, there is no reinvestment, and once this limit is attained, the first constraint is no longer binding and there is reinvestment in the new short-term asset.*

However, the difference between them is, that in the second-best allocation the optimal consumption levels are always dependent on b_{12}, individuals are always bearing the risk of the random return, whereas in the first-best, consumption is constant for low values of b_{12}.

This difference is due to incentive compatibility reasons.

The optimal levels of the initial investments, are the same in the first and in the second-best.

2.3.- Elasticities with respect to the short-term interim random return

As already mentioned in the introduction, this work is concerned with the efficient allocation of technology-induced interest rate risk in a competitive system of financial intermediation.

This section will study the dependence of the optimal consumption levels on the short-term interim random return, in terms of elasticities, and for the additive and multiplicative utilities. The analytical expression for these elasticities is given in Appendix B.

2.3.1.- Multiplicative utility function

Figure 46 gives the elasticity of first and second period consumption in the first best (=second best) allocation, where $e_i = \dfrac{\partial c_{ij}}{\partial b_{12}} \dfrac{b_{12}}{c_{ij}}$ $i, j = 1, 2.$

	1.0	1.1	1.2	1.3	1.4	1.5
e1 ■	0.00	0.00	0.00	-0.63	-0.62	-0.61	-0.59	-0.58
e2 ✳	0.00	0.00	0.00	0.37	0.38	0.39	0.41	0.42
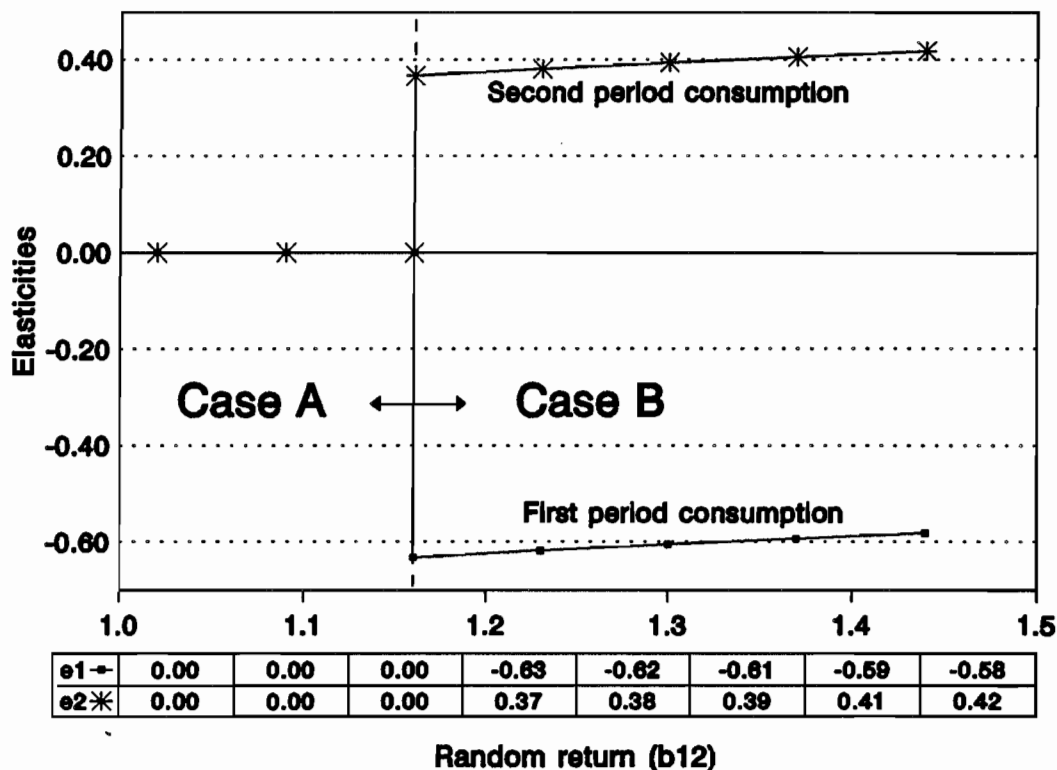
Random return (b12)

Figure 46.- First-best (Second-best) allocation

It can be seen, that in Case A, consumption is inelastic to the random return whereas in Case B there is a negative elasticity of first period consumption and a positive elasticity of the second period one with respect to the random return.

2.3.2.- Additive utility function

Figure 47 and Figure 48 show the elasticities of first and second period consumption in the first and second best allocations respectively, and where $e_{ij} = \dfrac{\partial c_{ij}}{\partial b_{12}} \dfrac{b_{12}}{c_{ij}}$ $i, j = 1, 2$.

	1.0	1.1	1.2	1.3	1.4			1.5
e1 ←	0.00	0.00	0.00	-0.26	-0.25	-0.24	-0.23	-0.22
e2 ✳	0.00	0.00	0.00	0.40	0.42	0.43	0.44	0.45

Random return (b12)
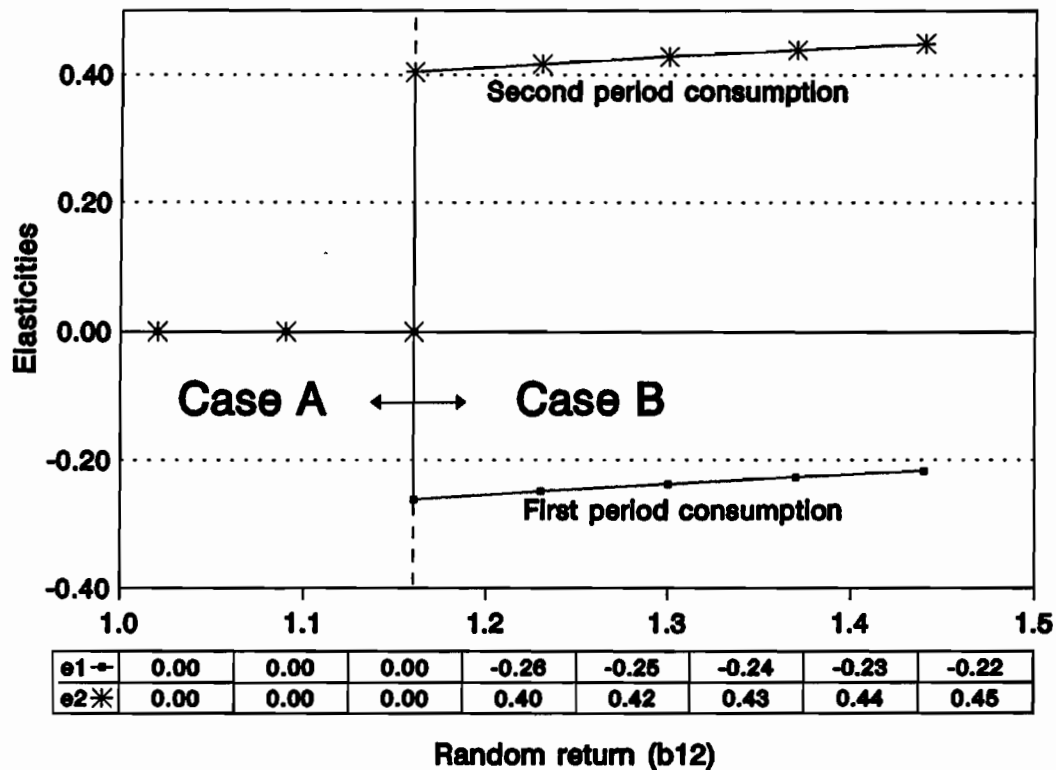
Figure 47.- First-best allocation

It can be observed in both figures that there is a negative elasticity of first period consumption and a positive elasticity of the second period one, with respect to b_{12}.

In the first best case, and up to a limit value of the random return, consumption is inelastic to the random return (Case A), but once the limit is attained, there is reinvestment in the new short-term asset and so first period consumption (second period) diminishes (increases).

In the second best allocation, consumption is always responsive to the random return, although in Case A, elasticities are very small. The elasticities of the consumption of type-2 do not differ from the first best case, however those of type-1 agents do differ, in the first period there is negative elasticity with respect to b_{12} which is higher with respect to the first-best and on the contrary, the elasticity of second period consumption is smaller.

		1.0	1.1		1.2	1.3	1.4	1.5
e11		0.00	0.00	0.00	-0.29	-0.28	-0.27	-0.26	-0.25
e21	✳	-0.01	-0.01	-0.01	0.38	0.39	0.40	0.41	0.42
e12	◆	-0.00	-0.00	-0.00	-0.26	-0.25	-0.24	-0.23	-0.22
e22	■	0.00	0.00	0.00	0.41	0.42	0.43	0.44	0.45

Random return (b12)
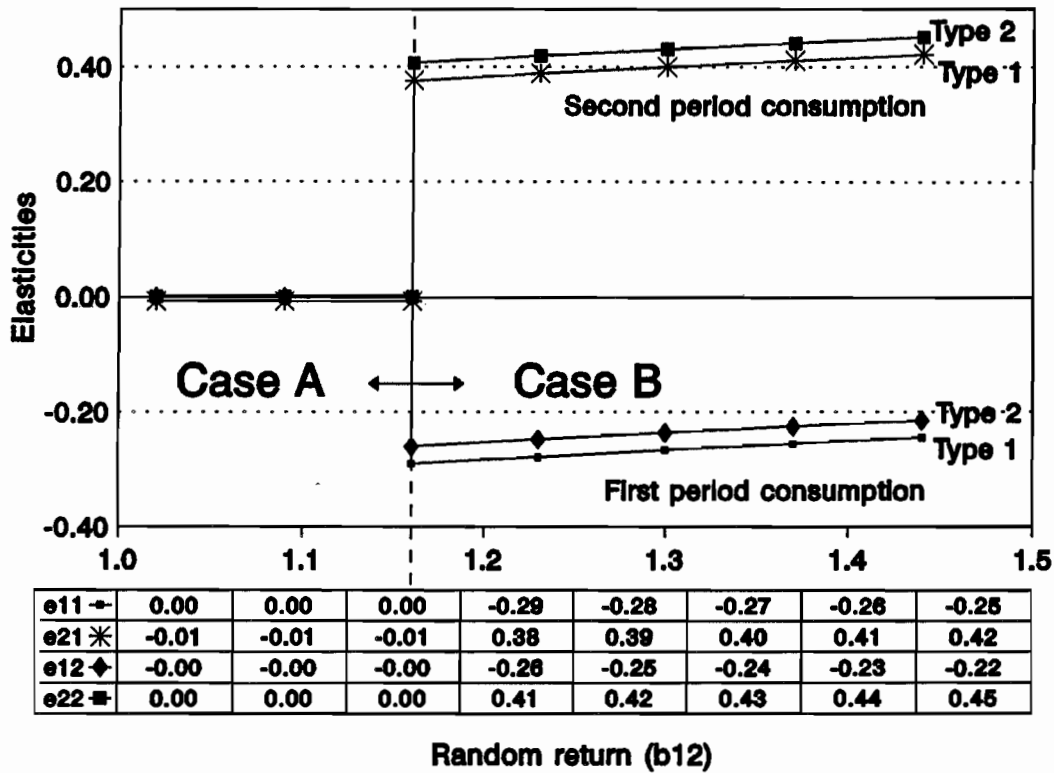
Figure 48.- Second-best allocation

2.3.3.- Comparison among them

A first remark to be made is that in the multiplicative case, *the elasticities of consumption are always the same for type-1 and type-2 agents, compared to the additive function (second best allocation).*

It is also observed that there is a higher negative elasticity of first period consumption in the multiplicative case with respect to the additive case and on the contrary, a smaller positive elasticity with respect to second period consumption.

3.- Comparison with an Equity Economy

This section will compare the second best allocation (non-traded solution) to the competitive equilibrium in an equity economy (traded solution). Suppose that at $T=1$, there was a walrasian market for date 1 and date 2 consumption goods, in which consumers participate with endowments consisting of $b_{o1}k_{o1}$ units of the date 1 good and $b_{o2}k_{o2}$ units of the date 2 good. Let $R_2 = 1 + r$ be some equilibrium interest rate at which individuals are willing to trade good 1 in exchange for good 2, and so that for any agent j:

$$c_{1j} = b_{o1}k_{o1} + B_j$$

$$c_{2j} = b_{o2}k_{o2} - R_2 B_j \qquad j = 1,2$$

[1]

where B_j is the quantity demanded (or supplied) of good 1 in exchange for good 2 and with $\sum_j B_j = 0$ across agents determining R_2, subject to the caveat $R_2 \geq b_{12}$, the short term realized (storage) rate from $T=1$ to $T=2$. If storage (with $R_2 = b_{12}$) is done then $0 \geq \sum_j B_j \geq -b_{o1}k_{o1}$ is the constraint overall.

The individuals' maximization problems are shown below:

3.1.- Multiplicative utility function

Type-1 problem at $T=1$

$$\max_{B_1} \left\{ c_{11}^{\delta_1} c_{21}^{\alpha-\delta_1} \right\}$$

$$\text{s.t} \quad c_{11} = b_{o1}k_{o1} + B_1$$

$$c_{21} = b_{o2}k_{o2} - R_2 B_1$$

[26]

with solution:

$$B_1 = \frac{\delta_1 b_{o2}k_{o2} + b_{o1}k_{o1}R_2(\delta_1 - \alpha)}{\alpha R_2}$$

[27]

Type-2 problem at $T=1$

$$\max_{B_2} \left\{ c_{12}^{\delta_2} c_{22}^{\alpha-\delta_2} \right\}$$

$$\text{s.t} \quad c_{12} = b_{o1}k_{o1} + B_2$$

$$c_{22} = b_{o2}k_{o2} - R_2 B_2$$

[28]

with solution:

$$B_2 = \frac{\delta_2 b_{o2}k_{o2} + b_{o1}k_{o1}R_2(\delta_2 - \alpha)}{\alpha R_2}$$

[29]

A).- If $R_2 \geq b_{12}$

From the equilibrium condition $\sum_j B_j = 0$ the equilibrium interest rate is obtained, that is:

$$t \frac{\delta_1 b_{o2}k_{o2} + b_{o1}k_{o1}R_2(\delta_1 - \alpha)}{\alpha R_2} + (1-t) \frac{\delta_2 b_{o2}k_{o2} + b_{o1}k_{o1}R_2(\delta_2 - \alpha)}{\alpha R_2} = 0$$

[30]

and the value of R_2^* is:

$$R_2^* = \frac{[t\delta_1 + (1-t)\delta_2]b_{o2}k_{o2}}{b_{o1}k_{o1}[t(\alpha - \delta_1) + (1-t)(\alpha - \delta_2)]}$$

[31]

which coincides with the limit value of the random return that distinguishes Cases A and B in the non traded

solution.

Substituting R_2^* in the expressions for B_1 and B_2 yields:

$$B_1^* = \frac{b_{o1}k_{o1}(1-t)(\delta_1 - \delta_2)}{t\delta_1 + (1-t)\delta_2} \qquad B_2^* = \frac{b_{o1}k_{o1}t(\delta_2 - \delta_1)}{t\delta_1 + (1-t)\delta_2} \qquad [32]$$

It should be observed that $B_1^* > 0$ and $B_2^* < 0$ as $\delta_1 > \delta_2$. Type-1 agents demand in the aggregate tB_1^* units of good 1, which are supplied by the type 2 agents in exchange for $(1-t)R_2^*B_2^*$ units of good 2.

The optimal consumption levels for type-1 and type-2 agents would be:

$$c_{11}^* = b_{o1}k_{o1}\frac{\delta_1}{t\delta_1 + (1-t)\delta_2} \qquad c_{21}^* = b_{o2}k_{o2}\frac{(\alpha - \delta_1)}{t(\alpha - \delta_1) + (1-t)(\alpha - \delta_2)}$$

$$c_{12}^* = b_{o1}k_{o1}\frac{\delta_2}{t\delta_1 + (1-t)\delta_2} \qquad c_{22}^* = b_{o2}k_{o2}\frac{(\alpha - \delta_2)}{t(\alpha - \delta_1) + (1-t)(\alpha - \delta_2)} \qquad [33]$$

These allocations coincide with the ones obtained in the non traded solution (Case A).

B).- If $b_{12} > R_2$

In this case the equilibrium interest rate must be $R_2^* = b_{12}$, that is, the realized short term return.

The optimal levels of B_1^* and B_2^* would be:

$$B_1^* = \frac{\delta_1 b_{o2}k_{o2} + b_{o1}k_{o1}b_{12}(\delta_1 - \alpha)}{\alpha b_{12}} \qquad B_2^* = \frac{\delta_2 b_{o2}k_{o2} + b_{o1}k_{o1}b_{12}(\delta_2 - \alpha)}{\alpha b_{12}} \qquad [34]$$

and therefore, the optimal consumption pattern for type-1 and type-2 agents is:

$$\tilde{c}_{11}^* = \delta_1\frac{b_{o1}k_{o1}\tilde{b}_{12} + b_{o2}k_{o2}}{\tilde{b}_{12}\alpha} \qquad \tilde{c}_{21}^* = (\alpha - \delta_1)\frac{b_{o1}k_{o1}\tilde{b}_{12} + b_{o2}k_{o2}}{\alpha}$$

$$\tilde{c}_{12}^* = \delta_2\frac{b_{o1}k_{o1}\tilde{b}_{12} + b_{o2}k_{o2}}{\tilde{b}_{12}\alpha} \qquad \tilde{c}_{22}^* = (\alpha - \delta_2)\frac{b_{o1}k_{o1}\tilde{b}_{12} + b_{o2}k_{o2}}{\alpha} \qquad [35]$$

As before, the allocations obtained in the equity economy coincide with those achieved in the non traded solution.

Finally, it can be shown that in this case, the constraint $0 \geq \sum_j B_j \geq -b_{o1}k_{o1}$ is always satisfied. The result of this section is summarized by the following proposition.

Proposition 3. In the case in which preferences are represented by a multiplicative utility function, the allocations obtained in the traded solution coincide with the ones achieved in the non traded one, for the investment cum storage pattern described.

The following table shows numerical computations of the traded solution for the input data given by Table II.

Table XI

b_{12}	R_2^*	B_1^*	B_2^*	$\sum B_j^*$
1.00	1.17	0.404	-0.045	0.0
1.10	1.17	0.404	-0.045	0.0
1.17	1.17	0.404	-0.045	0.0
1.20	1.20	0.386	-0.054	-0.0098
1.30	1.30	0.344	-0.075	-0.0331
1.40	1.40	0.307	-0.093	-0.0531

3.2.- Additive utility function

Similarly to the multiplicative case, the individuals' maximization problems are defined in the following ones:

Type-1 problem at $T=1$

$$\max_{B_1} \left\{ \frac{c_{11}^{1-\gamma}}{1-\gamma} + \rho_1 \frac{c_{21}^{1-\gamma}}{1-\gamma} \right\}$$

$$\text{s.t} \quad c_{11} = b_{o1} k_{o1} + B_1$$

$$c_{21} = b_{o2} k_{o2} - R_2 B_1$$

[36]

with solution:

$$B_1 = \frac{(\rho_1 R_2)^{-1/\gamma} b_{o2} k_{o2} - b_{o1} k_{o1}}{1 + (\rho_1 R_2)^{-1/\gamma} R_2}$$

[37]

Type-2 problem at $T=1$

$$\max_{B_2} \left\{ \frac{c_{12}^{1-\gamma}}{1-\gamma} + \rho_2 \frac{c_{22}^{1-\gamma}}{1-\gamma} \right\}$$

$$\text{s.t} \quad c_{12} = b_{o1} k_{o1} + B_2$$

$$c_{22} = b_{o2} k_{o2} - R_2 B_2$$

[38]

with solution:

$$B_2 = \frac{(\rho_2 R_2)^{-1/\gamma} b_{o2} k_{o2} - b_{o1} k_{o1}}{1 + (\rho_2 R_2)^{-1/\gamma} R_2} \qquad [39]$$

A).- If $R_2 \geq b_{12}$

From the equilibrium condition $\sum_j B_j = 0$, the following non-linear equation in R_2 is obtained, that is:

$$t \frac{(\rho_1 R_2)^{-1/\gamma} b_{o2} k_{o2} - b_{o1} k_{o1}}{1 + (\rho_1 R_2)^{-1/\gamma} R_2} + (1-t) \frac{(\rho_2 R_2)^{-1/\gamma} b_{o2} k_{o2} - b_{o1} k_{o1}}{1 + (\rho_2 R_2)^{-1/\gamma} R_2} = 0 \qquad [40]$$

The value of R_2^* is obtained as a solution to the above equation, and from it the values of B_1^* and B_2^* are derived. These values are substituted in the expressions for c_{11}, c_{21} and c_{12}, c_{22}, to calculate ex-ante expected utility in this economy.

B).- If $b_{12} > R_2$

In this case the equilibrium interest rate is $R_2^* = b_{12}$, the realized short term return.

The optimal levels of B_1^* and B_2^* are:

$$B_1^* = \frac{(\rho_1 b_{12})^{-1/\gamma} b_{o2} k_{o2} - b_{o1} k_{o1}}{1 + (\rho_1 b_{12})^{-1/\gamma} b_{12}} \qquad B_2^* = \frac{(\rho_2 b_{12})^{-1/\gamma} b_{o2} k_{o2} - b_{o1} k_{o1}}{1 + (\rho_2 b_{12})^{-1/\gamma} b_{12}}$$

and from them, the optimal consumption levels and the value of the expected utility are obtained.

The following table shows the numerical computations of the traded solution for the input data of Table VI.

Table XII

b_{12}	R_2^*	B_1^*	B_2^*	$\sum_j B_j$
1.00	1.17	0.125	-0.014	0.0
1.10	1.17	0.125	-0.014	0.0
1.17	1.17	0.125	-0.014	0.0
1.22	1.22	0.113	-0.023	-0.0092
1.23	1.23	0.102	-0.032	-0.0179
1.44	1.44	0.073	-0.052	-0.0397

In order to compare the expected utility obtained in the non traded solution with respect to the traded one, some numerical examples have been computed. The input data are those corresponding to Table VI, with the variations in the exogenous parameters shown from Figure 49 to Figure 51.

It is observed that in all the examples the non-traded solution is always welfare superior, also, as the value of

the random return increases, the difference in utility diminishes (an exception is the case of $\gamma = 0.5$). This result differs from Hellwig [11], as in his case, the traded and non-traded solutions coincide.

It may be concluded *that if preferences are represented by an additive utility function, the allocations obtained in the non-traded solution are welfare superior with respect to the ones achieved in the traded one.*
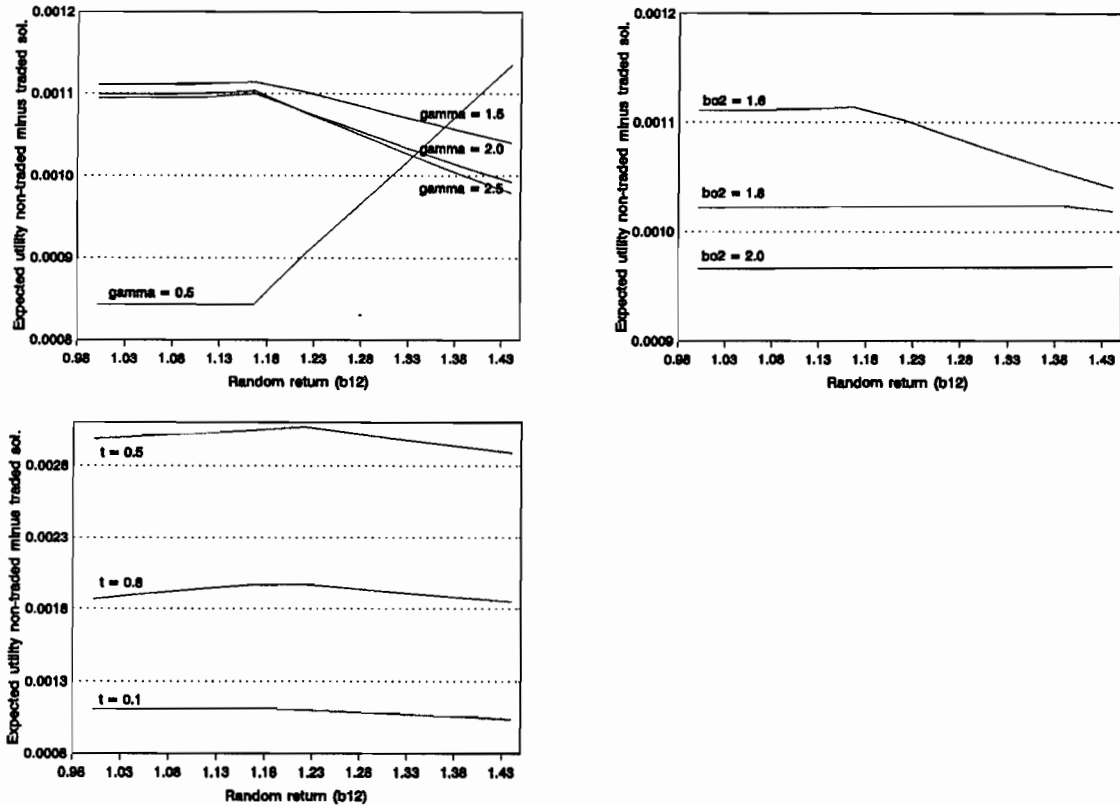
Figure 49 to **Figure 51.-** Expected Utility of non-traded minus traded solution.

4.- Conclusions

This paper has studied the optimal allocation of technology-induced interest rate risk in a competitive system of financial intermediation.

The study has been carried out under different information assumptions and in the case in which preferences are represented by a multiplicative utility function and by an additive one, respectively.

The objective was to analyze how the initial uncertainty about the short-term interim random return affected consumption as well as initial investment choices; it was shown that there is always a limit value of this random return above which, it becomes advantageous to reinvest some of the return available at $T=1$ (and therefore not consume all of it) in the new short-term investment.

The optimal consumption levels in the first-best and second-best allocations depend on the random return, b_{12}.

In the case in which preferences are represented by a *multiplicative utility function* the first and second-best allocations coincide. It is observed that for low values of b_{12} consumption levels are independent of the random return, (CASE A), but once a limit value is attained, consumption levels depend on it (CASE B), and there is a negative elasticity of first period consumption with respect to b_{12} and a positive elasticity of second period consumption with respect to it. These elasticities are the same for each type of agent.

The first-best (=second-best) allocation involves no liquidation of the long-term asset.
In the case in which preferences are represented by an *additive utility function* the first-best allocation is never incentive compatible, it has the characteristic that both types consume the same in the first period but type-2 receive a strictly higher amount in the second period. In the complete information case, as in the multiplicative case, for low values of the random return, consumption is constant and once the limit value is attained first period consumption decreases, second period consumption increases. In the second-best allocation, individuals are always bearing the risk of the short-term asset. For low values of b_{12} elasticities are very small, and once the limit value is attained, there is a negative elasticity of first period consumption with respect to b_{12}, (which is higher for type-1 than for type-2 agents) and a positive elasticity of the second period consumption (which is higher for type-2 than for type-1 agents).

In both the complete and incomplete information cases, there is never liquidation of the long-term technology. This result differs from the one in Hellwig [11], as in his case the second best allocation may involve liquidation of the long-term asset.

The limit value of the random return is not very sensitive to those parameters that characterize the household side of the economy (risk-aversion, preference shock or the proportion of each type of agents), although it depends on the investment opportunities (the distribution function of the random return $(Eb_{12}, \sigma_{b_{12}})$, the certain return on the long versus short-term investments (b_{o1}, b_{o2}) or the liquidation value of the long-term asset (b_l)).

Given this sensitivity analysis that was done with respect to the exogenous parameters of the model (multiplicative utility function), some conclusions may be drawn concerning the optimal contracts as well as investment policies chosen by a financial intermediary[4]:

a.- If the expected value of the random return exceeds the relation of the long-term versus short-term riskless return ($Eb_{12} > \dfrac{b_{o2}}{b_{o1}}$) this would result in implementing *only* the reinvestment solution (CASE B) as the optimal contract. In this situation the optimal initial investment choices would be $k_{o1}^* = 1$ and $k_{o2}^* = 0$, (notice that $b_{lim} = 0$ in equation [7]) that is, the intermediary would invest only in liquid assets and take advantage of the favourable reinvestment opportunities at $T=1$, in order to provide for second period consumption.

Banks holding only liquid securities would correspond to the proposal of 100% Reserve Banking or narrow banking that has been discussed currently in the literature. This proposal has been suggested in the US as a way to avoid the moral hazard problem inherent in deposit insurance and lender-of-last resort interventions although it would alter the character of financial markets by forbidding the financing of illiquid loans by short-term deposits (Diamond and Dybvig [9]).

b.- In the situation in which the expected value of the random return is less than the relation of the long versus short-term return ($Eb_{12} < \dfrac{b_{o2}}{b_{o1}}$), then $k_{o2}^* > 0$, however there is never liquidation of the long-term asset as the

---

[4] *As shown in Jacklin [12] and is commented also in the work of Hellwig [11], the demand deposit contract can be used to achieve the constrained social optimum.*

level of the short-term investment is always high enough to provide for first period consumption ($k_{ol}crit \leq k_{ol}^{*} < 1$). If the rate of return on long-term investments is very high, then the optimal contract would result in giving individuals constant consumption in both periods, independently of b_{12}. In this case, given the high rate of return on the long-term asset, b_{o2}, it is not interesting to take advantage of the reinvestment opportunities at $T = 1$.

The second-best allocation (demand deposit contract or non-traded solution) has been compared to the one achieved through an equity economy in which individuals could trade the assets directly (traded solution). It has been shown that if preferences are represented by *a multiplicative utility function*, the allocations in the traded and non-trade solutions coincide. However, if preferences are represented by an *additive utility function*, demand deposit contracts are shown to provide greater risk sharing that equity contracts.

This result (and contrary to Hellwig's model) shows that financial intermediaries do provide a positive role in the economy.

Appendix A: Utility Functions

A.- Multiplicative Utility Function: First Best (=Second Best) allocation

The first best allocation is obtained as a solution to the following problem:

$$\max_{c_{ij}, \mu} \left\{ E\left[t\, U^1(c_{11},\ c_{21},\ \rho_1) + (1-t)\, U^2(c_{12},\ c_{22},\ \rho_2) \right] \right\} \tag{1}$$

s.t.

$$
\begin{aligned}
& tc_{11} + (1-t)c_{12} \leq b_{o1}k_{o1} + \mu k_{o2}b_1 \\
& tc_{21} + (1-t)c_{22} = (1-\mu)b_{o2}k_{o2} + \left[\mu k_{o2}b_1 + b_{o1}k_{o1} - tc_{11} - (1-t)c_{12} \right] b_{12} \\
& c_{ij} \geq 0 \\
& \mu \geq 0
\end{aligned}
\tag{2}
$$

Where:

$$U(c_1,\ c_2,\ \rho_i) = c_1^{\delta_i} c_2^{\alpha - \delta_i} \tag{3}$$

and:

$$0 < \alpha < 1 \qquad 0 < \delta_i < \alpha \qquad i = 1,\ 2 \qquad \delta_2 < \delta_1 \tag{4}$$

Given that any monotonic transformation of the utility function, represents the same preferences, the work will be done with:

$$U(c_1,\ c_2,\ \rho_i) = \delta_i \ln c_1 + (\alpha - \delta_i) \ln c_2 \tag{5}$$

For simplicity the following notation is used: $\alpha_i = \delta_i$, $\beta_i = \alpha - \delta_i$

The Kuhn-Tucker conditions are:

$$t\frac{\alpha_1}{c_{11}} - \lambda_1 t + t b_{12}\lambda_2 = 0 \qquad \text{if } c_{11} > 0 \quad [a]$$

$$t\frac{\beta_1}{c_{21}} + t\lambda_2 = 0 \qquad \text{if } c_{21} > 0 \quad [b]$$

$$(1-t)\frac{\alpha_2}{c_{12}} - \lambda_1(1-t) + (1-t)b_{12}\lambda_2 = 0 \qquad \text{if } c_{12} > 0 \quad [c]$$

$$(1-t)\frac{\beta_2}{c_{22}} + (1-t)\lambda_2 = 0 \qquad \text{if } c_{22} > 0 \quad [d] \qquad [6]$$

$$k_{o2}b_1\lambda_1 - k_{o2}(b_1 b_{12} - b_{o2})\lambda_2 = 0 \qquad \text{if } \mu > 0 \quad [e]$$

$$b_{o1}k_{o1} + \mu k_{o2}b_1 - t c_{11} - (1-t)c_{12} = 0 \qquad \text{if } \lambda_1 > 0 \quad [f]$$

$$t c_{21} + (1-t)c_{22} - (1-\mu)b_{o2}k_{o2} - [\mu k_{o2}b_1 + b_{o1}k_{o1} - t c_{11} - (1-t)c_{12}]b_{12} = 0 \qquad \forall \lambda_2 \quad [g]$$

A.1.- First-step solution

The following cases may be considered:

Case A	$\lambda_1 > 0$	$\mu^* = 0$
Case B	$\lambda_1 = 0$	$\mu^* = 0$
Case C	$\lambda_1 > 0$	$\mu^* > 0$

In Case A the first constraint is binding (no reinvestment of resources), whereas in Case B it is no longer binding. Conditions on the random return for each case to hold will be derived. Case C involves liquidation of the long-term asset (in the final solution this case never holds).

A.1.1.- CASE A: $(\lambda_1 > 0, \ \mu^* = 0)$

The equations to be solved are:

$$\frac{\alpha_1}{c_{11}} - \lambda_1 + b_{12}\lambda_2 = 0 \qquad [7] \qquad\qquad \frac{\beta_1}{c_{21}} + \lambda_2 = 0 \qquad [8]$$

$$\frac{\alpha_2}{c_{12}} - \lambda_1 + b_{12}\lambda_2 = 0 \qquad [9] \qquad\qquad \frac{\beta_2}{c_{22}} + \lambda_2 = 0 \qquad [10]$$

$$t c_{11} + (1-t)c_{12} - b_{o1}k_{o1} = 0 \qquad [11] \qquad t c_{21} + (1-t)c_{22} - b_{o2}k_{o2} = 0 \qquad [12]$$

From [7] and [9]: \qquad From [8] and [10]:

$$c_{12} = c_{11}\frac{\alpha_2}{\alpha_1} \qquad [13] \qquad\qquad c_{22} = c_{21}\frac{\beta_2}{\beta_1} \qquad [14]$$

From [11] and [13]: \qquad From [12] and [14]:

$$c_{11} = \frac{b_{o1}k_{o1}\alpha_1}{t\alpha_1 + (1-t)\alpha_2} \qquad [15] \qquad\qquad c_{21} = \frac{b_{o2}k_{o2}\beta_1}{t\beta_1 + (1-t)\beta_2} \qquad [16]$$

or substituting back in terms of δ_i, $\alpha - \delta_i$:

The aggregate expected utility is given by:

$$U^{*(a)} = t c_{11}^{*\,\delta_1} c_{21}^{*\,\alpha-\delta_1} + (1-t) c_{12}^{*\,\delta_2} c_{22}^{*\,\alpha-\delta_2} \qquad [18]$$

In Case A it is assumed $\lambda_1 > 0$, from [7] and [8]:

$$\lambda_1 = \frac{\alpha_1}{c_{11}} - b_{12} \frac{\beta_1}{c_{21}} > 0 \qquad [19]$$

Substituting c_{11}^* and c_{21}^* in the expression for λ_1, the following condition on b_{12} for this case to hold is obtained:

$$b_{12} < \frac{b_{o2} k_{o2}\left[t\delta_1 + (1-t)\delta_2\right]}{b_{o1} k_{o1}\left\{\alpha - \left[t\delta_1 + (1-t)\delta_2\right]\right\}} = b_{lim} \qquad [20]$$

Similarly it is assumed $\mu^* = 0$, that means $\dfrac{\partial L}{\partial \mu} \leq 0$

$$\frac{\partial L}{\partial \mu} \leq k_{o2} b_1 \lambda_1 - k_{o2}(b_1 b_{12} - b_{o2})\lambda_2 \leq 0 \qquad [21]$$

Substituting $\lambda_1 = \dfrac{\alpha_1}{c_{11}} - b_{12}\dfrac{\beta_1}{c_{21}}$ and $\lambda_2 = -\dfrac{\beta_1}{c_{21}}$ in the above expression, the condition on k_{o1}, for this case to hold is obtained:

$$k_{o1} \geq \frac{b_1\left[t\delta_1 + (1-t)\delta_2\right]}{b_1\left[t\delta_1 + (1-t)\delta_2\right] + b_{o1}\left[t(\alpha-\delta_1) + (1-t)(\alpha-\delta_2)\right]} = k_{o1}crit \qquad [22]$$

If the optimal level of the initial investment is above this limit value ($k_{o1}crit$) there is no liquidation in the optimal solution.

A.1.2.- CASE B: $(\lambda_1 = 0,\ \mu^* = 0)$

The F.O.C. in this case are:

$$\frac{\alpha_1}{c_{11}} + b_{12}\lambda_2 = 0 \qquad [23]$$

$$\frac{\beta_1}{c_{21}} + \lambda_2 = 0 \qquad [24]$$

$$\frac{\alpha_2}{c_{12}} + b_{12}\lambda_2 = 0 \qquad [25]$$

$$\frac{\beta_2}{c_{22}} + \lambda_2 = 0 \qquad [26]$$

$$tc_{21} + (1-t)c_{22} - b_{o2}k_{o2} - \left[b_{o1}k_{o1} - tc_{11} - (1-t)c_{12}\right]b_{12} = 0 \qquad [27]$$

From [23] and [25]:   From [24] and [26]:

$$c_{12} = c_{11}\frac{\alpha_2}{\alpha_1} \qquad [28]$$

$$c_{22} = c_{21}\frac{\beta_2}{\beta_1} \qquad [29]$$

From [23] and [24]:

$$\alpha_1 c_{21} = \beta_1 b_{12} c_{11} \qquad [30]$$

Substituting [28] and [29] in [27]:

$$b_{o2}k_{o2} + b_{12}b_{o1}k_{o1} = \left[t + (1-t)\frac{\beta_2}{\beta_1}\right]c_{21} + b_{12}\left[t + (1-t)\frac{\alpha_2}{\alpha_1}\right]c_{11} \qquad [31]$$

Equations [30] and [31] yield:

$$c_{11} = \frac{b_{o2}k_{o2} + b_{12}b_{o1}k_{o1}}{\left[t\beta_1 + (1-t)\beta_2\right]\frac{b_{12}}{\alpha_1} + \frac{b_{12}}{\alpha_1}\left[t\alpha_1 + (1-t)\alpha_2\right]}$$

$$c_{21} = \beta_1 b_{12}\frac{b_{o2}k_{o2} + b_{12}b_{o1}k_{o1}}{\left[t\beta_1 + (1-t)\beta_2\right]b_{12} + b_{12}\left[t\alpha_1 + (1-t)\alpha_2\right]} \qquad [32]$$

which can again be expressed in terms of δ_i, $\alpha - \delta_i$:

The aggregate expected utility is given by:

$$U^{*(b)} = tc_{11}^{*\,\delta_1}c_{21}^{*\,\alpha-\delta_1} + (1-t)c_{12}^{*\,\delta_2}c_{22}^{*\,\alpha-\delta_2} \qquad [34]$$

In this case, the condition for the random return obtained from $tc_{11} + (1-t)c_{12} \le b_{o1}k_{o1}$ yields:

$$b_{12} \ge \frac{b_{o2}k_{o2}\left[t\delta_1 + (1-t)\delta_2\right]}{b_{o1}k_{o1}\left\{\alpha - \left[t\delta_1 + (1-t)\delta_2\right]\right\}} = b_{\lim} \qquad [35]$$

A.1.3.- CASE C: $(\lambda_1 > 0, \mu^* > 0)$

The equations to be solved are the [6].

From [6][a] and [6][c]:

$$c_{12} = c_{11} \frac{\alpha_2}{\alpha_1} \qquad [36]$$

From [6][b] and [6][d]:

$$c_{22} = c_{21} \frac{\beta_2}{\beta_1} \qquad [37]$$

From [6][a] and [6][b] considering [6][e]:

$$\beta_1 b_{o2} c_{11} = \alpha_1 b_1 c_{21} \qquad [38]$$

Eliminating μ in [6][f] and [6][g]:

$$b_{o2}(k_{o2} b_1 + b_{o1} k_{o1}) = b_{o2} \left[t c_{11} + (1-t) c_{12} \right] + b_1 \left[t c_{21} + (1-t) c_{22} \right] \qquad [39]$$

Equations [36], [37] and [39]:

$$b_{o2}(k_{o2} b_1 + b_{o1} k_{o1}) = \left[t + (1-t) \frac{\beta_2}{\beta_1} \right] b_1 c_{21} + b_{o2} \left[t + (1-t) \frac{\alpha_2}{\alpha_1} \right] c_{11} \qquad [40]$$

[38] and [40] yield:

$$c_{11} = \alpha_1 \frac{k_{o2} b_1 + b_{o1} k_{o1}}{\left[t\beta_1 + (1-t)\beta_2 \right] + \left[t\alpha_1 + (1-t)\alpha_2 \right]} \qquad c_{21} = \frac{\beta_1}{b_1} \frac{b_{o2}(k_{o2} b_1 + b_{o1} k_{o1})}{\left[t\beta_1 + (1-t)\beta_2 \right] + \left[t\alpha_1 + (1-t)\alpha_2 \right]} \qquad [41]$$

which can again be expressed in terms of $\alpha_i = \delta_i$ and $\beta_i = \alpha - \delta_i$:
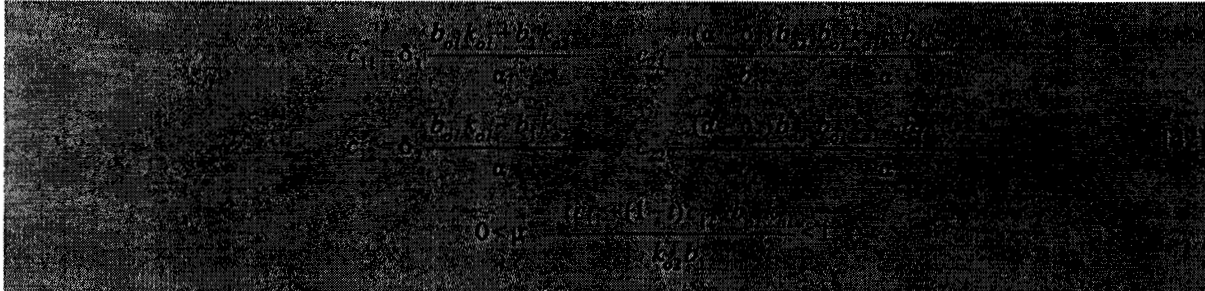
The aggregate expected utility is given by:

$$U^{*(c)} = t c_{11}^{* \, \delta_1} c_{21}^{* \, \alpha - \delta_1} + (1-t) c_{12}^{* \, \delta_2} c_{22}^{* \, \alpha - \delta_2} \qquad [43]$$

In this case, it is assumed $\mu^* > 0$, that is substituting c_{11}^* and c_{12}^* in the expression for μ^*, the condition on k_{o1} for this case to be satisfied is obtained:

$$k_{o1} < \frac{b_1 b_{o2} \left[t\delta_1 + (1-t)\delta_2 \right]}{b_1 b_{o2} \left[t\delta_1 + (1-t)\delta_2 \right] + b_{o2} b_{o1} \left[t(\alpha - \delta_1) + (1-t)(\alpha - \delta_2) \right]} = k_{o1} crit \qquad [44]$$

A.2.- Second-step solution:

The second step is the solution to the expression:

$$
\max_{k_{o1}, k_{o2}} \left[\begin{array}{ll} \int_{b_{min}}^{b_{lim}} U^{*(A)} f(b_{12}) db_{12} + \int_{b_{lim}}^{b_{max}} U^{*(B)} f(b_{12}) db_{12} & \text{if} \quad k_{o1} \geq k_{o1} crit \\ \int_{b_{min}}^{b_{max}} U^{*(C)} f(b_{12}) db_{12} & \text{if} \quad k_{o1} < k_{o1} crit \end{array} \right] \tag{45}
$$

Then, if $0 \leq k_{o1} < k_{o1} crit$:

$$
\max_{k_{o1}} \int_{b_{min}}^{b_{max}} U^{*(c)} f(b_{12}) db_{12} \quad \rightarrow \quad \int_{b_{min}}^{b_{max}} \frac{\partial U^{*(c)}}{\partial k_{o1}} f(b_{12}) db_{12} = 0 \tag{46}
$$

That is, if $\dfrac{\partial U^{*(c)}}{\partial k_{o1}} > 0$ in the interval $[b_{min}, b_{max}]$, the maximum is reached in $k_{o1} crit$. The proof is given by:

$$
\frac{\partial U^{*(c)}}{\partial k_{o1}} = t \left[\delta_1 c_{11}^{*\,\delta_1 - 1} c_{21}^{*\,\alpha - \delta_1} \frac{\partial c_{11}^*}{\partial k_{o1}} + (\alpha - \delta_1) c_{11}^{*\,\delta_1} c_{21}^{*\,\alpha - \delta_1 - 1} \frac{\partial c_{21}^*}{\partial k_{o1}} \right] +
$$
$$
+ (1-t) \left[\delta_2 c_{12}^{*\,\delta_2 - 1} c_{22}^{*\,\alpha - \delta_2} \frac{\partial c_{12}^*}{\partial k_{o1}} + (\alpha - \delta_2) c_{12}^{*\,\delta_2} c_{22}^{*\,\alpha - \delta_2 - 1} \frac{\partial c_{22}^*}{\partial k_{o1}} \right] \tag{47}
$$

where:

$$
\frac{\partial c_{11}^*}{\partial k_{o1}} = \delta_1 \frac{b_{o1} - b_1}{\alpha} \qquad \frac{\partial c_{21}^*}{\partial k_{o1}} = (\alpha - \delta_1) \frac{b_{o2}}{b_1} \frac{b_{o1} - b_1}{\alpha}
$$
$$
\frac{\partial c_{12}^*}{\partial k_{o1}} = \delta_2 \frac{b_{o1} - b_1}{\alpha} \qquad \frac{\partial c_{22}^*}{\partial k_{o1}} = (\alpha - \delta_2) \frac{b_{o2}}{b_1} \frac{b_{o1} - b_1}{\alpha} \tag{48}
$$

By assumption $b_1 < b_{o1}$ and therefore, $\dfrac{\partial U^{*(c)}}{\partial k_{o1}} > 0$ which implies $k_{o1}^* \geq k_{o1} crit$, *this means the optimal solution falls always in Cases A and B, with no liquidation of the long-term asset.*

In the case of a multiplicative utility function, the first best and second best allocations coincide.

The Incentive constraints are not binding, and therefore they are just checked for the optimal solution.

The Incentive constraints should impose that the consumption bundle an agent obtains by saying the truth should be at least as large as what he obtains by lying and reinvesting in the backyard technology in the optimal way for him.

The type-1 agent that claims to be a type-2 would receive c_{12} and c_{22} units, if he decides to reinvest his c_{12} in the backyard he solves the following problem:

$$
\max_{c_1, c_2} \left\{ c_1^{\delta_1} c_2^{\alpha - \delta_1} \right\} \tag{49}
$$

s.t.

$$
c_1 \leq c_{12}
$$
$$
c_2 = c_{22} + (c_{12} - c_1) b_{12} \tag{50}
$$

with solution:

$$c_1^* = \delta_1 \frac{c_{22} + b_{12} c_{12}}{b_{12} \alpha} \le c_{12} \qquad c_2^* = \alpha - \delta_1 \frac{c_{22} + b_{12} c_{12}}{\alpha} \qquad [51]$$

Therefore, I.C. should impose:

$$c_{11}^{\delta_1} c_{21}^{\alpha - \delta_1} \ge c_1^{*\,\delta_1} c_2^{*\,\alpha - \delta_1} \qquad [52]$$

and similarly for type-2 agents.

B.- Additive Utility Function: First Best allocation

The first best allocation is obtained as a solution to the following problem:

$$\max_{c_{ij}, \mu} \left\{ t \left[\frac{c_{11}^{1-\gamma}}{1-\gamma} + \rho_1 \frac{c_{21}^{1-\gamma}}{1-\gamma} \right] + (1-t) \left[\frac{c_{12}^{1-\gamma}}{1-\gamma} + \rho_2 \frac{c_{22}^{1-\gamma}}{1-\gamma} \right] \right\} \qquad [53]$$

s.t.

$$tc_{11} + (1-t)c_{12} \le b_{o1} k_{o1} + \mu k_{o2} b_1$$
$$tc_{21} + (1-t)c_{22} = (1-\mu) b_{o2} k_{o2} + \left[\mu k_{o2} b_1 + b_{o1} k_{o1} - tc_{11} - (1-t)c_{12} \right] b_{12} \qquad [54]$$
$$c_{ij} \ge 0$$
$$\mu \ge 0$$

The Kuhn-Tucker conditions are:

$$tc_{11}^{-\gamma} - \lambda_1 t + t b_{12} \lambda_2 = 0 \qquad \text{if } c_{11} > 0 \quad [a]$$

$$t \rho_1 c_{21}^{-\gamma} + t \lambda_2 = 0 \qquad \text{if } c_{21} > 0 \quad [b]$$

$$(1-t) c_{12}^{-\gamma} - \lambda_1 (1-t) + (1-t) b_{12} \lambda_2 = 0 \qquad \text{if } c_{12} > 0 \quad [c]$$

$$(1-t) \rho_2 c_{22}^{-\gamma} + (1-t) \lambda_2 = 0 \qquad \text{if } c_{22} > 0 \quad [d] \qquad [55]$$

$$k_{o2} b_1 \lambda_1 - k_{o2} (b_1 b_{12} - b_{o2}) \lambda_2 = 0 \qquad \text{if } \mu > 0 \quad [e]$$

$$tc_{11} + (1-t)c_{12} - b_{o1} k_{o1} - \mu k_{o2} b_1 = 0 \qquad \text{if } \lambda_1 > 0 \quad [f]$$

$$tc_{21} + (1-t)c_{22} - (1-\mu) b_{o2} k_{o2} - \left[\mu k_{o2} b_1 + b_{o1} k_{o1} - tc_{11} - (1-t)c_{12} \right] b_{12} = 0 \qquad \forall \lambda_2 \quad [g]$$

B.1.- First-step solution:

The following cases may be considered:

Case A	$\lambda_1 > 0$	$\mu^* = 0$
Case B	$\lambda_1 = 0$	$\mu^* = 0$
Case C	$\lambda_1 > 0$	$\mu^* > 0$

In Case A the first constraint is binding whereas, in Case B, it is no longer binding, that means there is reinvestment in the new short term asset (the return available at date 1 is not consumed completely in this

period). Conditions on the random return for each case to hold will be derived. Case C implies liquidation of the long-term asset (however, in the optimal solution this case never holds).

B.1.1.- CASE A: $(\lambda_1 > 0)$

The equations to be solved are:

$$c_{11}^{-\gamma} - \lambda_1 + b_{12}\lambda_2 = 0 \qquad [56]$$

$$\rho_1 c_{21}^{-\gamma} + \lambda_2 = 0 \qquad [57]$$

$$c_{12}^{-\gamma} - \lambda_1 + b_{12}\lambda_2 = 0 \qquad [58]$$

$$\rho_2 c_{22}^{-\gamma} + \lambda_2 = 0 \qquad [59]$$

$$tc_{11} + (1-t)c_{12} - b_{ol}k_{ol} = 0 \qquad [60]$$

$$tc_{21} + (1-t)c_{22} - b_{o2}k_{o2} = 0 \qquad [61]$$

From [56] and [58]:

$$c_{11} = c_{12} \qquad [62]$$

From [57] and [59]:   From [60] and [62]:

$$c_{22} = \left[\frac{\rho_1}{\rho_2}\right]^{\frac{-1}{\gamma}} c_{21} \qquad [63]$$

$$c_{11}^* = c_{12}^* = b_{ol}k_{ol} \qquad [64]$$

From [61] and [63]:

$$c_{21}^* = \frac{b_{o2}k_{o2}}{t + (1-t)\left[\dfrac{\rho_1}{\rho_2}\right]^{\frac{-1}{\gamma}}} \qquad [65]$$

That is:
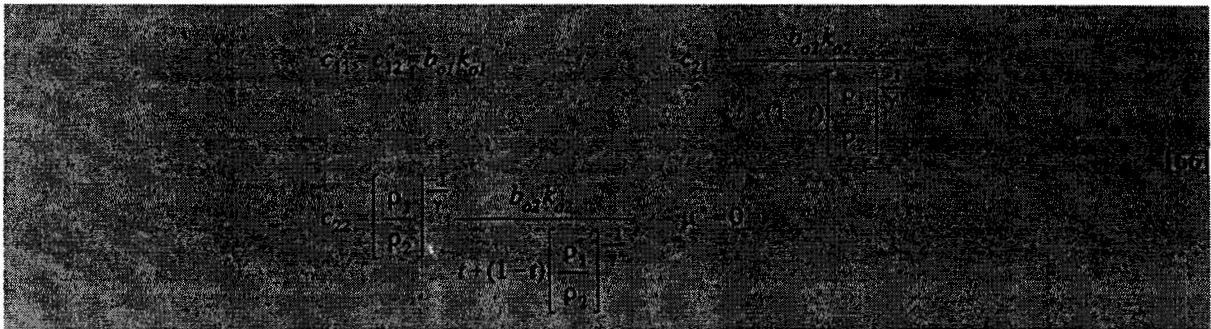
In Case A it is assumed $\lambda_1 > 0$, from [56] and [57]:

$$\lambda_1 = c_{11}^{-\gamma} - b_{12}\rho_1 c_{21}^{-\gamma} > 0 \qquad [67]$$

Substituting c_{11}^* and c_{21}^* in the expression for λ_1, the following condition on b_{12} for this case to hold is obtained:

$$b_{12} < \frac{1}{\rho_1} \left\{ \frac{b_{ol}k_{ol}\left[t + (1-t)\left(\dfrac{\rho_1}{\rho_2}\right)^{\frac{-1}{\gamma}}\right]}{b_{o2}k_{o2}} \right\}^{-\gamma} = b_{lim} \qquad [68]$$

Similarly it is assumed $\mu^* = 0$, that means $\dfrac{\partial L}{\partial \mu} \leq 0$

$$\frac{\partial L}{\partial \mu} \le k_{o2} b_1 \lambda_1 - k_{o2}(b_1 b_{12} - b_{o2}) \lambda_2 \le 0 \qquad [69]$$

Substituting $\lambda_1 = c_{11}^{-\gamma} - b_{12} \rho_1 c_{21}^{-\gamma}$ and $\lambda_2 = -\rho_1 c_{21}^{-\gamma}$ in the above expression, the condition on k_{o1}, for this case to hold is obtained:

$$k_{o1} \ge \frac{b_{o2}^{\frac{\gamma-1}{\gamma}} \rho_1^{-1/\gamma}}{b_{o1} b_1^{-1/\gamma} \left[t + (1-t)\left(\frac{\rho_1}{\rho_2}\right)^{-1/\gamma} \right] + b_{o2}^{\frac{\gamma-1}{\gamma}} \rho_1^{-1/\gamma}} = k_{o1} crit \qquad [70]$$

If the optimal level of the initial investment is above this limit value ($k_{o1}crit$) there is no liquidation in the optimal solution.

B.1.2.- CASE B: ($\lambda_1 = 0$)

The F.O.C. in this case are:

$$[71] \qquad\qquad\qquad\qquad [72]$$

$$[73] \qquad\qquad\qquad\qquad [74]$$

$$t c_{21} + (1-t) c_{22} - b_{o2} k_{o2} - \left[b_{o1} k_{o1} - t c_{11} - (1-t) c_{12} \right] b_{12} = 0 \qquad [75]$$

From [71] and [73]:   From [72] and [74]:

$$c_{11} = c_{12} \qquad [76] \qquad\qquad c_{22} = \left[\frac{\rho_1}{\rho_2}\right]^{\frac{-1}{\gamma}} c_{21} \qquad [77]$$

From [71] and [72]:   From [77] and [78] in [75]:

$$c_{11} = \left[b_{12} \rho_1 \right]^{\frac{-1}{\gamma}} c_{21} \qquad [78] \qquad c_{21}^{*} = \frac{b_{o1} k_{o1} b_{12} + b_{o2} k_{o2}}{t + (1-t)\left[\frac{\rho_1}{\rho_2}\right]^{\frac{-1}{\gamma}} + b_{12}^{\frac{\gamma-1}{\gamma}} \rho_1^{\frac{-1}{\gamma}}} \qquad [79]$$

That is:
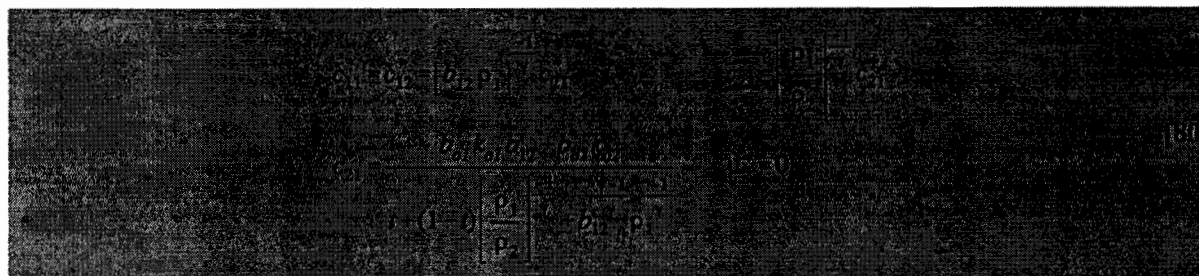
In Case B it is assumed $\lambda_1 = 0$, or equivalently:

$$tc_{11}^* + (1-t)c_{12}^* \le b_{ol}k_{ol} \tag{81}$$

Substituting the optimal consumption levels, the following expression for the random return is obtained:

$$b_{12} \ge \frac{1}{\rho_1}\left\{\frac{b_{ol}k_{ol}\left[t+(1-t)\left[\dfrac{\rho_1}{\rho_2}\right]^{\frac{-1}{\gamma}}\right]}{b_{o2}k_{o2}}\right\}^{-\gamma} = b_{\lim} \tag{82}$$

B.1.3.- CASE C: $(\lambda_1 > 0, \mu^* > 0)$

The equations to be solved are the [55].

From [55][a] and [55][c]:   From [55][b] and [55][d]:

$$c_{11} = c_{12} \tag{83}$$

$$c_{22} = \left[\frac{\rho_1}{\rho_2}\right]^{\frac{-1}{\gamma}} c_{21} \tag{84}$$

From [55][a] and [55][b], considering [55][e]:

$$c_{21} = \left[\frac{b_1}{\rho_1 b_{o2}}\right]^{\frac{-1}{\gamma}} c_{11} \tag{85}$$

Eliminating μ in [55][f] and [55][g]:

$$b_{o2}\left[tc_{11}+(1-t)c_{12}\right]+b_1\left[tc_{21}+(1-t)c_{22}\right]=b_{o2}\left[b_{ol}k_{ol}+b_1k_{o2}\right] \tag{86}$$

Substituting [83], [84] in [86]:

$$b_{o2}c_{11}+b_1\left[t+(1-t)\left(\frac{\rho_1}{\rho_2}\right)^{\frac{-1}{\gamma}}\right]c_{21}=b_{o2}\left[b_{ol}k_{ol}+b_1k_{o2}\right] \tag{87}$$

[85] and [87] yield:

In this case, it is assumed $\mu^* > 0$, that is substituting c_{11}^* and c_{12}^* in the expression for μ^*, the condition on k_{ol} for this case to be satisfied is obtained:

$$k_{ol} < \frac{b_{o2}^{\frac{\gamma-1}{\gamma}} \rho_1^{-1/\gamma}}{b_{ol} b_1^{-1/\gamma} \left[t + (1-t) \left(\frac{\rho_1}{\rho_2} \right)^{-1/\gamma} \right] + b_{o2}^{\frac{\gamma-1}{\gamma}} \rho_1^{-1/\gamma}} = k_{ol} crit$$ [89]

B.2.- Second-step solution:

The second step is the solution to the expression:

$$\max_{k_{ol}, k_{o2}} \begin{bmatrix} \int_{b_{min}}^{b_{lim}} U^{*(A)} f(b_{12}) db_{12} + \int_{b_{lim}}^{b_{max}} U^{*(B)} f(b_{12}) db_{12} & \text{if} & k_{ol} \geq k_{ol} crit \\ \int_{b_{min}}^{b_{max}} U^{*(C)} f(b_{12}) db_{12} & \text{if} & k_{ol} < k_{ol} crit \end{bmatrix}$$ [90]

Then, if $0 \leq k_{ol} < k_{ol} crit$:

$$\max_{k_{ol}} \int_{b_{min}}^{b_{max}} U^{*(c)} f(b_{12}) db_{12} \quad \rightarrow \quad \int_{b_{min}}^{b_{max}} \frac{\partial U^{*(c)}}{\partial k_{ol}} f(b_{12}) db_{12} = 0$$ [91]

That is, if $\frac{\partial U^{*(c)}}{\partial k_{ol}} > 0$ in the interval $[b_{min}, b_{max}]$, the maximum is reached in $k_{ol} crit$. The proof is given by:

$$\frac{\partial U^{*(c)}}{\partial k_{ol}} = t \left[c_{11}^{*-\gamma} \frac{\partial c_{11}^*}{\partial k_{ol}} + \rho_1 c_{21}^{*-\gamma} \frac{\partial c_{21}^*}{\partial k_{ol}} \right] + (1-t) \left[c_{12}^{*-\gamma} \frac{\partial c_{12}^*}{\partial k_{ol}} + \rho_2 c_{22}^{*-\gamma} \frac{\partial c_{22}^*}{\partial k_{ol}} \right]$$ [92]

where:

$$\frac{\partial c_{21}^*}{\partial k_{ol}} = \frac{b_1^{\frac{-1}{\gamma}} b_{o2} (b_{ol} - b_1)}{\left\{ t + (1-t) \left[\frac{\rho_1}{\rho_2} \right]^{\frac{-1}{\gamma}} \right\} b_1^{\frac{\gamma-1}{\gamma}} + b_{o2}^{\frac{\gamma-1}{\gamma}} \rho_1^{\frac{-1}{\gamma}}}$$ [93]

$$\frac{\partial c_{11}^*}{\partial k_{ol}} = \frac{\partial c_{12}^*}{\partial k_{ol}} = \left[\frac{b_{o2} \rho_1}{b_1} \right]^{\frac{-1}{\gamma}} \frac{\partial c_{21}^*}{\partial k_{ol}} \qquad \frac{\partial c_{22}^*}{\partial k_{ol}} = \left[\frac{\rho_1}{\rho_2} \right]^{\frac{-1}{\gamma}} \frac{\partial c_{21}^*}{\partial k_{ol}}$$

By assumption $b_1 < b_{ol}$ and therefore, $\frac{\partial U^{*(c)}}{\partial k_{ol}} > 0$ which implies $k_{ol}^* \geq k_{ol} crit$, *this means the optimal solution falls always in Cases A and B, with no liquidation of the long-term asset.*

C.- Additive Utility Function: Second Best allocation

The Newton-Raphson technique is an iterative method, that at every step, it takes the Taylor's series as the solution of the equation root. This is mathematically described as follows:

$$f(x^{(n)} + h) = f(x^{(n)}) + h \frac{df(x^{(n)})}{dx} = 0 \qquad\qquad [94]$$

Therefore, the new value of the iterative solution is given by:

$$x^{(n+1)} = x^{(n)} + h = x^{(n)} - \frac{f(x^{(n)})}{\dfrac{df(x^{(n)})}{dx}} \qquad\qquad [95]$$

A criterion to stop the iteration with a suitable solution would be:

$$|x^{(n+1)} - x^{(n)}| < \varepsilon \qquad\qquad [96]$$

In a more general way, the same method can be applied to a system of non-linear equations using the expression:

$$[x_i^{(n+1)}] = [x_i^{(n)}] - [J(x^{(n)})]^{-1} [f(x^{(n)})] \qquad\qquad [97]$$

This method has been applied to the system of non-linear equations (the F.O.C corresponding to the second-best allocation), that are given by Equation [99].

where:

$$g = \frac{\gamma - 1}{\gamma} \qquad a = 1 + b_{12}^g \rho_2^{\frac{-1}{\gamma}} \qquad b = 1 + b_{12}^g \rho_1^{\frac{-1}{\gamma}} \qquad\qquad [98]$$

In order to derive the numerical solutions, some computer programs have been written in Ms-Dos Qbasic. In the next pages some flow-charts of the programs are presented. The first two charts correspond to the additive utility function and the next three to the multiplicative one, the last flow-chart shows the distribution function that has been used.

$$tc_{11}^{-\gamma} - \lambda_1 t + tb_{12}\lambda_2 + c_{11}^{-\gamma}\lambda_4 - \left\{ \left[\rho_2^{\frac{-1}{\gamma}} \frac{b_{12}^{\frac{-1}{\gamma}} c_{21} + b_{12}^g c_{11}}{a} \right]^{-\gamma} \rho_2^{\frac{-1}{\gamma}} \frac{b_{12}^g}{a} + \left[\frac{c_{21} + b_{12}c_{11}}{a} \right]^{-\gamma} \rho_2 \frac{b_{12}}{a} \right\} \lambda_5 = 0 \qquad \text{if } c_{11} > 0$$

$$tp_1 c_{21}^{-\gamma} + t\lambda_2 + \rho_1 c_{21}^{-\gamma}\lambda_4 - \left\{ \left[\rho_2^{\frac{-1}{\gamma}} \frac{b_{12}^{\frac{-1}{\gamma}} c_{21} + b_{12}^g c_{11}}{a} \right]^{-\gamma} \rho_2^{\frac{-1}{\gamma}} \frac{b_{12}^{\frac{-1}{\gamma}}}{a} + \left[\frac{c_{21} + b_{12}c_{11}}{a} \right]^{-\gamma} \frac{\rho_2}{a} \right\} \lambda_5 = 0 \qquad \text{if } c_{21} > 0$$

$$(1-t)c_{12}^{-\gamma} - \lambda_1(1-t) + (1-t)b_{12}\lambda_2 + c_{12}^{-\gamma}\lambda_5 - \left\{ \left[\rho_1^{\frac{-1}{\gamma}} \frac{b_{12}^{\frac{-1}{\gamma}} c_{22} + b_{12}^g c_{12}}{b} \right]^{-\gamma} \rho_1^{\frac{-1}{\gamma}} \frac{b_{12}^g}{b} + \left[\frac{c_{22} + b_{12}c_{12}}{b} \right]^{-\gamma} \rho_1 \frac{b_{12}}{b} \right\} \lambda_4 = 0 \quad \text{if } c_{12} > 0$$

$$(1-t)\rho_2 c_{22}^{-\gamma} + (1-t)\lambda_2 + \rho_2 c_{22}^{-\gamma}\lambda_5 - \left\{ \left[\rho_1^{\frac{-1}{\gamma}} \frac{b_{12}^{\frac{-1}{\gamma}} c_{22} + b_{12}^g c_{12}}{b} \right]^{-\gamma} \rho_1^{\frac{-1}{\gamma}} \frac{b_{12}^{\frac{-1}{\gamma}}}{b} + \left[\frac{c_{22} + b_{12}c_{12}}{b} \right]^{-\gamma} \frac{\rho_1}{b} \right\} \lambda_4 = 0 \qquad \text{if } c_{22} > 0 \qquad [99]$$

$$k_{o2}b_1\lambda_1 - k_{o2}(b_1 b_{12} - b_{o2})\lambda_2 - \lambda_3 = 0 \qquad \text{if } \mu > 0$$

$$tc_{11} + (1-t)c_{12} - b_{o1}k_{o1} - \mu k_{o2}b_1 = 0 \qquad \text{if } \lambda_1 > 0$$

$$tc_{21} + (1-t)c_{22} - (1-\mu)b_{o2}k_{o2} - \left[\mu k_{o2}b_1 + b_{o1}k_{o1} - tc_{11} - (1-t)c_{12} \right]b_{12} = 0 \qquad \forall \lambda_2$$

$$1 - \mu = 0 \qquad \text{if } \lambda_3 > 0$$

$$c_{11}^{1-\gamma} + \rho_1 c_{21}^{1-\gamma} - \left[\rho_1^{\frac{-1}{\gamma}} \frac{b_{12}^{\frac{-1}{\gamma}} c_{22} + b_{12}^g c_{12}}{b} \right]^{1-\gamma} - \rho_1 \left[\frac{c_{22} + b_{12}c_{12}}{b} \right]^{1-\gamma} = 0 \qquad \text{if } \lambda_4 > 0$$

$$c_{12}^{1-\gamma} + \rho_2 c_{22}^{1-\gamma} - \left[\rho_2^{\frac{-1}{\gamma}} \frac{b_{12}^{\frac{-1}{\gamma}} c_{21} + b_{12}^g c_{11}}{a} \right]^{1-\gamma} - \rho_2 \left[\frac{c_{21} + b_{12}c_{11}}{a} \right]^{1-\gamma} = 0 \qquad \text{if } \lambda_5 > 0$$

Second best allocation

Input Data
t, $\rho 1$, $\rho 2$ bo1, bo2, b1, Eb_{12}, $b12_{min}$, $b12_{max}$

Definition of the density function
$L = \min (Eb_{12} - b_{12}min, b_{12}max - Eb_{12})$; $y_{bmed} = 1/L$

For $\gamma = \gamma_1$ to γ_2 step $\Delta\gamma$ $Kol_{max} = 0$, $OFmedia_{Max} = 0$

For ko1 = 0.01 to 0.99 step 0.01   OFmedia = 0

For $b_{12} = b_{12}$-L to b12+L in 40 steps   OFmax = 0

Determine: Δarea

Case   Calculate Consumptions   Solut. OK ?   yes   OF>OFmax ?   yes

no

OFmax=OF

no

OFmedia=OFmedia + OFmax*Δarea

OFmedia > $OFmedia_{Max}$?   yes $OFmedia_{Max}$= OFmedia
Kol_{max} = Ko1

no

Print optimal Ko1 (Kol_{max})

$Ko1 = Kol_{max}$

For $b_{12} = b_{12}$-L to b12+L   OFmax = 0

Case   Calculate Consumptions   Solut. OK ?   yes   OF>OFmax ?   yes

no

OFmax=OF

no

Print b_{12} & optimal consumptions

End

Newton- Rapshon tecnique

$Mat4 = Mat3 - [J]^{-1} \cdot Mat2$

$Mat1 = Jacobian = [J]$
$Mat2 = [g_i (x)]$
$Mat3 = [x_i]$ iteration n
$Mat4 = [x_i]$ iteration $n + 1$

Calculate consumptions (Case)

Input initial data for the case
Cij (v[1] to v[4]), μ v[5], λi (v[6] to v[10])

do ciclo until a solution is reached

Row = 0

For Rest = 1 to Num. of restrictions

no   v[Rest1] $\Leftrightarrow$ 0 ?   yes

Row = Row + 1

First ciclo iteration ?   yes   Mat3 (Row) = First (Row)
no

Mat2 (Row) = First (Row)

Col = 0

for Rest2 = 1 to Num. of restrictions

no   v[Rest2] $\Leftrightarrow$ 0 ?   yes

Col = Col + 1

Mat1 (Row, Col) = DerivateFirst (Row, Col)

Inverse:	Mat1 in Mat5
Multiply:	Mat5 . Mat2 in Mat6
Substract:	Mat3 - Mat6 in Mat4
Check:	Mat4 - Mat3 = Matrix of [0]
Substract:	Mat3 - Mat6 in Mat3
Update:	Cij, μ, λi $\Leftrightarrow$ 0 with values in Mat4

until Check = Matrix of [0] or Iterations > 300

Iterations > 300   yes   Solution does not converge
no

Check Restrictions   no
(Kunh - Tucker conditions)   satisfies

satisfies

Print Results
Cij (v[1] to v[4]), μ v[5], λi (v[6] to v[10])

End

MAIN PROGRAM

```
┌─────────────────────────────────────────┐
│                  START                   │
└─────────────────────────────────────────┘
                     │
┌─────────────────────────────────────────┐
│             OPT-CONS.BAS                 │
├─────────────────────────────────────────┤
│ Optimal consumption levels and liquidation policy, │
│      for fixed values of the intial investment.    │
│        (ko1, ko2 considered exogenous)             │
└─────────────────────────────────────────┘
                     │
┌─────────────────────────────────────────┐
│             OPT-INVE.BAS                 │
├─────────────────────────────────────────┤
│      Optimal initial investments levels.           │
│  Maximization of the indirect utility function     │
│               derived above.                       │
└─────────────────────────────────────────┘
                     │
┌─────────────────────────────────────────┐
│             OPT-CONS.BAS                 │
├─────────────────────────────────────────┤
│ Optimal consumption levels and liquidation policy, │
│   for the optimal values of the intial investment. │
└─────────────────────────────────────────┘
                     │
            ┌────────────────┐
            │      END       │
            └────────────────┘
```

```
                    ┌─────────────────────┐
                    │   OPT-CONS.BAS      │
                    └─────────────────────┘
                              │
                              ▼
        ┌──────────────────────────────────────────────────────┐
        │                   Input Data                         │
        │ t, δ₁, δ₂, α−δ₁, α−δ₂, bo1, bo2, b1, ko1, ko2, b12a, b12b │
        └──────────────────────────────────────────────────────┘
                              │
                              ▼
                    ┌─────────────────────────┐
                    │ For b₁₂ = b₁₂ₐ to b₁₂ᵦ   ⟩
                    └─────────────────────────┘
                              │
                              ▼
                         ┌──────────────────────┐
                         │ For  Case = A to B    ⟩
                         └──────────────────────┘
                              │
                              ▼
                         ┌──────────────┐
                         │ Select Case  │
                         └──────────────┘
```

$$\boxed{\textbf{OPT-INVE.BAS}}$$

$$\boxed{\begin{array}{c}\text{Input Data}\\ t, \delta_1, \delta_2, \alpha-\delta_1, \alpha-\delta_2, \text{bo1, bo2, b1, } Eb_{12}, b12_{min}, b12_{max}\end{array}}$$

$$\boxed{\begin{array}{c}\text{Definition of the density function}\\ L = \min(Eb_{12} - b_{12}min, b_{12}max - Eb_{12}); y_{bmed} = 1/L\end{array}}$$

$$\boxed{\text{For ko1 = 0.01 to 0.99 step 0.01}}$$

$$\boxed{\text{ko1} < \text{ko1crit}} \xrightarrow{\text{no}} \boxed{\text{Calculation of } b_{lim}}$$

yes

$$\boxed{\text{Area A: Area of the distribution function up to } b_{lim}}$$

$$\boxed{\begin{array}{c}\text{Calculation of:}\\ c_{11}, c_{21}, c_{12}, c_{22,} \mu\\ \text{OF for Case C}\end{array}} \qquad \boxed{\begin{array}{c}\text{Calculation of:}\\ c_{11}, c_{21}, c_{12}, c_{22,} \mu\\ \text{OF for Case A}\end{array}}$$

$$\boxed{\text{OFtotal} = \text{OF(C)}} \qquad \boxed{\text{OFtotal} = \text{OF(A)} \cdot \text{Area A}}$$

$$\boxed{\text{For } b_{12} = b_{lim} \text{ to } Eb_{12} + L \text{ step } db_{12}}$$

$$\boxed{\Delta \text{area B: } [f(b_{12}) + f(b_{12} + db_{12})] / 2 * db_{12}}$$

$$\boxed{\begin{array}{l}\text{Calculation of : cij, } \mu\\ \text{OF for Case B with}\\ b_{12} = b_{12} + db_{12} / 2\end{array}}$$

$$\boxed{\text{OFtotal} = \text{OFtotal} + \text{OF(B)}*\Delta\text{area B}}$$

$$\boxed{\text{if OFtotal} > \text{OFmax}} \xrightarrow{\text{yes}} \boxed{\begin{array}{c}\text{OFmax} = \text{OFtotal:}\\ \text{Kmax} = \text{ko1}\end{array}}$$

no

$$\boxed{\begin{array}{c}\text{Print Results}\\ \text{Kmax}\end{array}}$$

$$\boxed{\text{End}}$$

Case A ◄—|—► Case B

Area A

Yb_{med}

diferential of Area B

b_{min} Eb_{med} b_{lim} db_{12} b_{max}

L   L

OF

OF associated
with the
differential of b_{12}

0,900
0,860
0,820
0,780
0,740
0,700

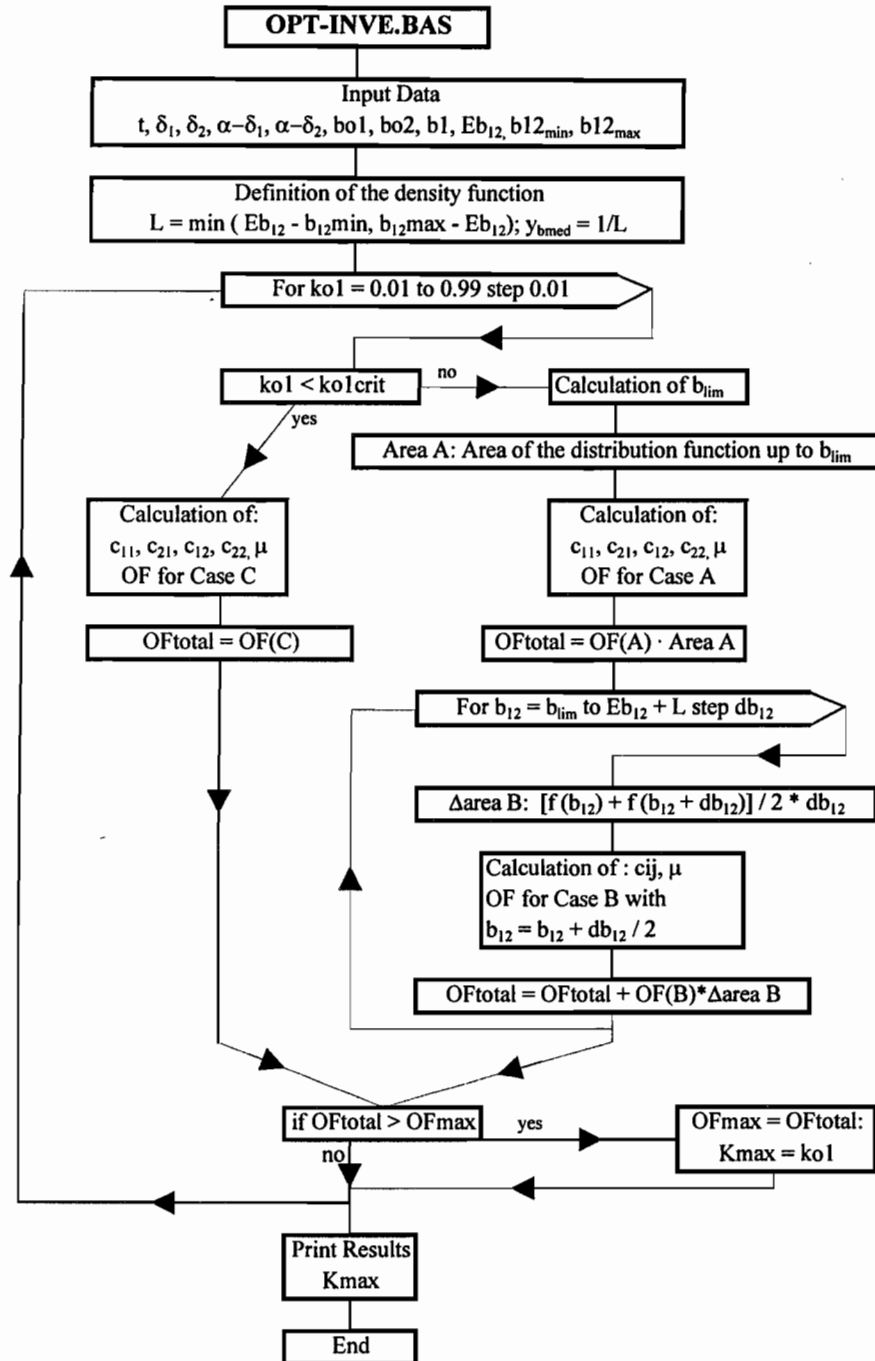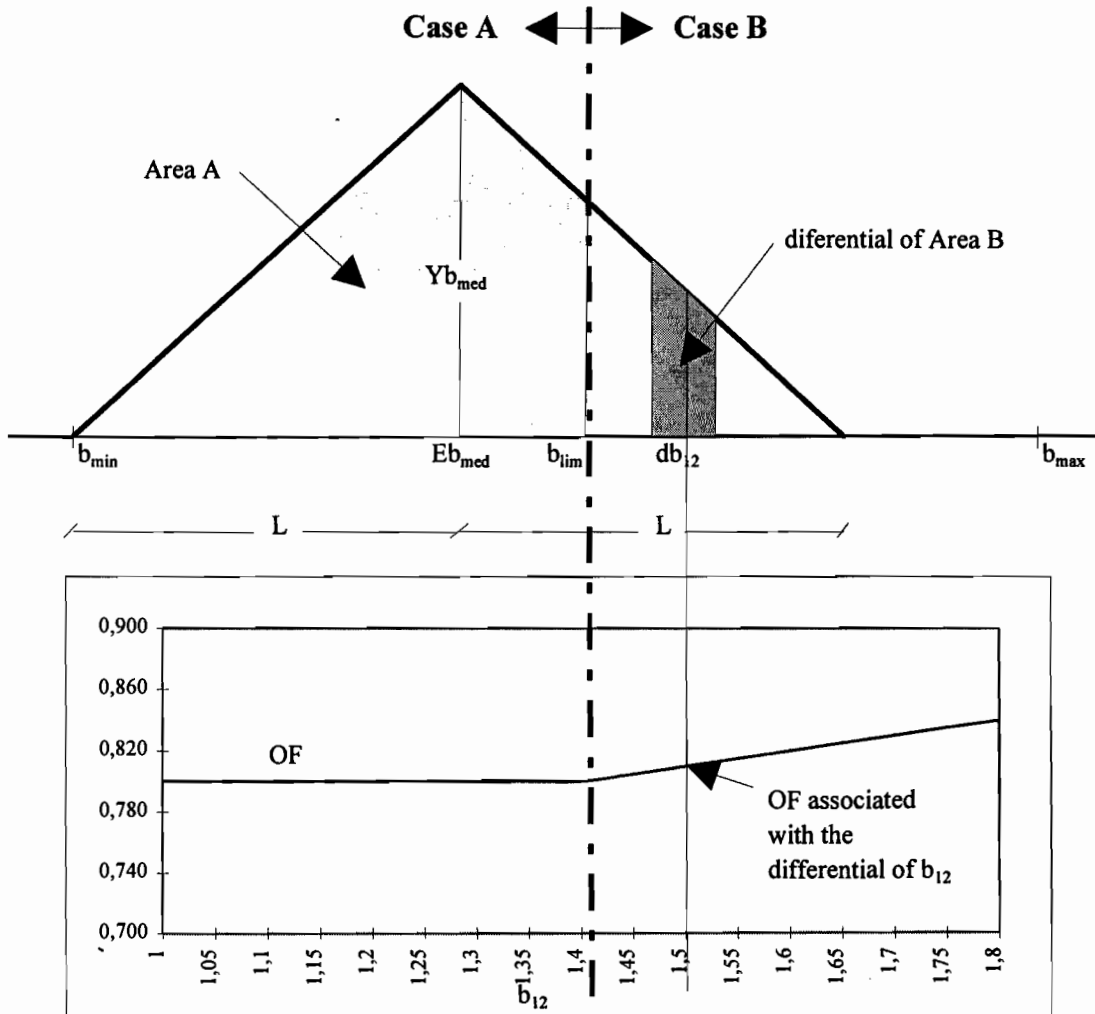1   1,05   1,1   1,15   1,2   1,25   1,3   1,35   1,4   1,45   1,5   1,55   1,6   1,65   1,7   1,75   1,8

b_{12}

Appendix B: Elasticities

A.- Multiplicative Utility function

A.1.- First-Best (=Second-Best) solution (CASE B)

Type-1=Type-2 elasticities at $T=1$

$$\varepsilon_{b_{12}} = -\frac{b_{o2}k_{o2}}{b_{o1}k_{o1}b_{12}+b_{o2}k_{o2}} \qquad [1]$$

Type-1=Type-2 elasticities at $T=2$

$$\varepsilon_{b_{12}} = 1 - \frac{b_{o2}k_{o2}}{b_{o1}k_{o1}b_{12}+b_{o2}k_{o2}} \qquad [2]$$

B.- Additive Utility function

B.1.- Elasticities in the First-Best solution (CASE B)

Type-1=Type-2 elasticities at $T=1$:

$$\varepsilon_{b_{12}} = 1 - \frac{1}{\gamma} - \frac{b_{o2}k_{o2}}{b_{o1}k_{o1}b_{12}+b_{o2}k_{o2}} - g\left\{ 1 - \frac{t+(1-t)\left[\frac{\rho_1}{\rho_2}\right]^{-1/\gamma}}{t+(1-t)\left[\frac{\rho_1}{\rho_2}\right]^{-1/\gamma}+b_{12}{}^g\rho_1{}^{-1/\gamma}} \right\} \qquad [3]$$

Type-1=Type-2 elasticities at $T=2$:

$$\varepsilon_{b_{12}} = 1 - \frac{b_{o2}k_{o2}}{b_{o1}k_{o1}b_{12}+b_{o2}k_{o2}} - g\left\{ 1 - \frac{t+(1-t)\left[\frac{\rho_1}{\rho_2}\right]^{-1/\gamma}}{t+(1-t)\left[\frac{\rho_1}{\rho_2}\right]^{-1/\gamma}+b_{12}{}^g\rho_1{}^{-1/\gamma}} \right\} \qquad [4]$$

B.2.- Elasticities in the Second-Best solution (CASE B)

Type-1 elasticity at $T=1$:

$$\varepsilon_{b_{12}} = 1 - \frac{1}{\gamma} - \frac{b_{o2}k_{o2}}{b_{o1}k_{o1}b_{12} + b_{o2}k_{o2}} - g\left\{1 - \frac{1}{1 + b_{12}{}^g \rho_1{}^{-1/\gamma}}\right\} \qquad [5]$$

Type-2 elasticity at $T=1$:

$$\varepsilon_{b_{12}} = 1 - \frac{1}{\gamma} - \frac{b_{o2}k_{o2}}{b_{o1}k_{o1}b_{12} + b_{o2}k_{o2}} - g\left\{1 - \frac{1}{1 + b_{12}{}^g \rho_2{}^{-1/\gamma}}\right\} \qquad [6]$$

Type-1 elasticity at $T=2$:

$$\varepsilon_{b_{12}} = 1 - \frac{b_{o2}k_{o2}}{b_{o1}k_{o1}b_{12} + b_{o2}k_{o2}} - g\left\{1 - \frac{1}{1 + b_{12}{}^g \rho_1{}^{-1/\gamma}}\right\} \qquad [7]$$

Type-2 elasticity at $T=2$:

$$\varepsilon_{b_{12}} = 1 - \frac{b_{o2}k_{o2}}{b_{o1}k_{o1}b_{12} + b_{o2}k_{o2}} - g\left\{1 - \frac{1}{1 + b_{12}{}^g \rho_2{}^{-1/\gamma}}\right\} \qquad [8]$$

and the value of $g = \dfrac{\gamma - 1}{\gamma}$

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