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Departamento de Economía de la Empresa
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34)-916249607

GENERALIZED VECTOR RISK FUNCTIONS

A. Balbás¹ and P.J. Guerra²

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Keywords: Vector risk function, Representation theorem, Portfolio optimization.

¹ Universidad Carlos III. Departamento de Economía de la Empresa.

² UNED. Departamento de Matemáticas Fundamentales.

Generalized vector risk functions

Alejandro Balbás,^{*} and Pedro Jiménez-Guerra[†]

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The paper introduces a new notion of vector-valued risk function. Both deviations and expectation bounded coherent risk measures are defined and analyzed. The relationships with both scalar and vector risk functions of previous literature are discussed, and it is pointed out that this new approach seems to appropriately integrate several preceding points of view. The framework of the study is the general setting of Banach lattices and Bochner integrable vector-valued random variables. Sub-gradient linked representation theorems, as well as portfolio choice problems, are also addressed, and general optimization methods are presented. Finally, practical examples are provided.

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1. Introduction

The notion of coherent measure of risk was introduced in the seminal paper by Artzner et al. (1999), and since then their work has been extended in many directions. Jouini et al. (2004) justified the use of vector random variables to represent the final wealth provided by a given portfolio, as well as the use of “coherent vector risk measures” to reflect risk levels. Burgert and Rüschendorf (2006) also represented future pay-offs by using vector-valued random variables, although they measured the risk level with real-valued functions.

The interest of the approach of Jouini et al. (2004) seems to justify possible extensions of their discussion so as to incorporate much more practical situations. For instance, they deal with a L_∞ space, whereas many scalar coherent risk measures are defined on a larger L_p space (for example, L_1 is the natural space to introduce the Conditional Value at Risk). Moreover, while Artzner et al. (1999) understood their risk measures as initial capital requirements that investors and managers should

^{*}Corresponding Author. University Carlos III of Madrid. Department of Business Economics. CL. Madrid 126. 28903 Getafe (Madrid, Spain). alejandro.balbas@uc3m.es

[†]Spanish Open University. Department of Fundamental Mathematics. CL. Senda del Rey, 9. 28040 Madrid (Spain). pjimenez@mat.uned.es

provide in order to overcome negative evolutions of the market, recent literature has pointed out the interest of drawing on risk measures in order to address other classical topics, such as pricing and hedging issues (Nakano, 2004) or portfolio choice problems (Ogryczak and Ruszczyński, 1999, Benati, 2003, Konno et al., 2005, Rockafellar et al., 2006a, etc.). This fact has led to further studies concerning risk analysis, and the use of convex measures (Föllmer and Schied, 2002) or deviations and expectation bounded risk measures (Rockafellar et al., 2006b), amongst many other kinds of risk functions.

This paper aims to present a general framework of vector risk functions. We introduce a “generalized vector risk function” as a map $\rho : L_p(\mu, E) \rightarrow F$, with μ being a probability and E and F being general Banach lattices (Meyer-Nieberg, 1991). According to the properties of ρ , we define “coherent measures”, “deviations”, “expectation bounded measures”, etc. This approach retrieves suitable and natural properties; For instance, the simultaneous consideration of scalar deviations or coherent expectation bounded risk measures generates vector deviations or vector coherent expectation bounded risk functions.

The outline of the paper is as follows. Section 2 introduces the general setting and those previous concepts and properties that we will need throughout the article. Section 3 introduces the generalized vector risk functions, their properties and some important relationships. Section 4 presents Representation Theorems. We have followed the idea of Rockafellar et al. (2006b), in the sense that we represent the measure ρ “as an envelope of its sub-gradients”, which, as long as E satisfies the Radon-Nikodym property (Diestel and Uhl, 1977), are elements of $L_q(\mu, E^*)$; q being the conjugate of p and E^* denoting the dual space of E . Section 5 illustrates how the developed theory may apply in portfolio choice theory, and how the representation theorems may enable us to solve the resulting optimization problems. Section 6 presents some practical examples of vector risk functions and Section 7 concludes the article.

2. Preliminaries and notations

Throughout the paper, E , F , E_+ and F_+ will denote two Banach lattices and their non-negative cones respectively. Their dual Banach lattices and cones will be represented by E^* , F^* , E_+^* and F_+^* , and $\langle e^*, e \rangle$ will be “the usual product” of $e^* \in E^*$ and $e \in E$. If $e_1, e_2 \in E$, $e_1 - e_2 \in E_+$ and $e_1 - e_2 \neq 0$ then we will write $e_1 > e_2$. If $\langle e^*, e \rangle > 0$ for every non-null $e \in E_+$ we will say that e^* is strictly positive, and will denote $e^* \in E_{++}^*$. Similar ideas apply if F plays the role of E .

$L_+(E, F)$ will be the set of linear maps $T : E \rightarrow F$ that are non-negative (*i.e.*, $T(e) \geq 0$ whenever $e \geq 0$). Every $T \in L_+(E, F)$ is continuous (Meyer-Nieberg, 1991).

$(\Omega, \mathcal{F}, \mu)$ will be a probability space composed of the set Ω , the σ -algebra \mathcal{F} and the probability measure μ . $p \in [1, \infty]$ and $q \in [1, \infty]$ will be conjugate values, *i.e.*, $1/p + 1/q = 1$. If $p < \infty$ then $L_p(\mu, E)$ will represent the Banach space of those Bochner integrable (Diestel and Uhl, 1977) functions $y : \Omega \rightarrow E$ such that $\int_{\Omega} \|y(\omega)\|^p d\mu(\omega) < \infty$, endowed with the usual norm

$$\|y\|_p = \left(\int_{\Omega} \|y(\omega)\|^p d\mu(\omega) \right)^{\frac{1}{p}}.$$

Similarly, $L_{\infty}(\mu, E)$ will be the Banach space of E -valued essentially bounded and integrable functions, endowed with the norm

$$\|y\|_{\infty} = \text{ess - sup } \{ \|y(\omega)\| ; \omega \in \Omega \}$$

ess - sup denoting the essential supremum. It is well known that $L_{p_1}(\mu, E) \subset L_{p_2}(\mu, E)$ whenever $p_1 \geq p_2$ and the natural inclusion is continuous. If $p < \infty$ and E satisfies the Radon-Nikodym property then $L_q(\mu, E^*)$ is the dual space of $L_p(\mu, E)$ (see Diestel and Uhl, 1977, for further details on all of these topics).

If there is no confusion, then for every $y_0 \in E$ we will also represent by y_0 the constant element of $L_p(\mu, E)$ given by $y(\omega) = y_0$ *a.s.* Furthermore, almost surely constant functions will also be identified with elements of E . In particular, $L_p(\mu, E) \setminus E$ will be the set of non-constant (out of null-sets) elements of $L_p(\mu, E)$.

In order to simplify some analytical expressions we will denote

$$I(y) = \int_{\Omega} y d\mu$$

for every $y \in L_1(\mu, E)$.

Suppose that $T \in L_+(E, F)$ and $y \in L_1(\mu, E)$. Then it is easy to prove that $T \circ y \in L_1(\mu, F)$ and

$$T \left(\int_{\Omega} y d\mu \right) = \int_{\Omega} (T \circ y) d\mu.$$

3. Generalized risk functions

Definition 1. Every

$$\rho : L_p(\mu, E) \rightarrow F$$

will be called Vector Risk Function (*VRF*). Furthermore, ρ is said to be:

a) T -Translation invariant, if $T \in L_+(E, F)$ and $\rho(y + y_0) = \rho(y) - T(y_0)$ holds for every $y \in L_p(\mu, E)$ and every $y_0 \in E$.¹

b) Positively homogeneous, if $\rho(\alpha y) = \alpha \rho(y)$ holds for every real number $\alpha > 0$ and every $y \in L_p(\mu, E)$.

c) Sub-additive, if $\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2)$ holds for every $y_1, y_2 \in L_p(\mu, E)$.

d) Decreasing, if $\rho(y_2) \leq \rho(y_1)$ whenever $y_1, y_2 \in L_p(\mu, E)$ and $y_2 \geq y_1$ a.s.

e) T -Strongly decreasing if $T \in L_+(E, F)$ and $f^* \circ \rho(y_2) \leq f^* \circ \rho(y_1)$ for every $y_1, y_2 \in L_p(\mu, E)$ and every $f^* \in F_+^*$ with $f^* \circ T \circ y_2 \geq f^* \circ T \circ y_1$ a.s.

f) T -Mean dominating, if $T \in L_+(E, F)$ and $\rho(y) > -T \circ I(y)$ holds for every $y \in L_p(\mu, E) \setminus E$.

g) Strictly positive, if $\rho(y) > 0$ holds for every $y \in L_p(\mu, E) \setminus E$ and $\rho(y_0) = 0$ holds for every $y_0 \in E$.

h) T -Lower range dominated, if $T \in L_+(E, F)$ and

$$\langle f^*, \rho(y) \rangle \leq \langle f^*, T \circ I(y) \rangle - (\text{ess} - \inf) \{ \langle f^*, T \circ y(\omega) \rangle ; \omega \in \Omega \}$$

holds for every $f^* \in F_+^*$ and every $y \in L_p(\mu, E)$, *ess - inf* denoting the essential infimum (that may equal $-\infty$). □

Definition 2. The VRF ρ is said to be:

a) A deviation (or deviation measure), if it is 0-translation invariant, positively homogeneous, sub-additive, and strictly positive.

b) A T -expectation bounded risk measure, if $T \in L_+(E, F)$ and ρ is T -translation invariant, positively homogeneous, sub-additive, and T -mean dominating.

c) A T -coherent risk measure, if $T \in L_+(E, F)$ and ρ is T -translation invariant, positively homogeneous, sub-additive, and decreasing.

d) A T -strongly coherent risk measure if $T \in L_+(E, F)$ and ρ is T -translation invariant, positively homogeneous, sub-additive, and T -strongly decreasing. □

Proposition 3. Let ρ be a VRF .

a) If ρ is positively homogeneous then $\rho(0) = 0$.

b) If ρ is T -expectation bounded or T -coherent for some $T \in L_+(E, F)$ then $\rho(y_0) = -T(y_0)$ for every $y_0 \in E$.

c) If ρ is T -strongly coherent for some $T \in L_+(E, F)$ then ρ is T -coherent.

d) If $E = F = \mathbb{R}$ and T is the identity map on \mathbb{R} , then ρ is T -strongly coherent if and only if ρ is T -coherent.

¹Following Artzner et al. (1999), if T is onto we can consider that, given $y \in L_p(\mu, E)$, any $y_0 \in E$ such that $T(y_0) = \rho(y)$ may be understood as a final wealth or pay-off that must be guaranteed by the initial capital requirements. Indeed, one has that

$$\rho(y + y_0) = \rho(y) - T(y_0) = 0.$$

Proof. To prove *a*) notice that $\rho(0) = \rho(\alpha 0) = \alpha \rho(0)$, so $\rho(0) \neq 0$ would lead to $\alpha = 1$ for every positive α .

To prove *b*) notice that $\rho(y_0) = \rho(0 + y_0) = \rho(0) - T(y_0) = -T(y_0)$.

To prove *c*), take $y_1, y_2 \in L_p(\mu, E)$ with $y_2 \geq y_1$ *a.s.* Then $f^* \circ T \circ y_2 \geq f^* \circ T \circ y_1$ *a.s.* for every $f^* \in F_+^*$, and therefore $f^* \circ \rho(y_2) \leq f^* \circ \rho(y_1)$ for every $f^* \in F^*$. F being a Banach lattice the inequality above implies that $\rho(y_2) \leq \rho(y_1)$ (Meyer-Nieberg, 1991).

Finally, the proof of *d*) is trivial and therefore omitted. □

The following results establish the existence of one to one correspondences between deviations and expectation bounded risk measures on the one hand, and lower range dominated deviations and strongly coherent expectation bounded risk measures on the other hand.

Proposition 4. Let $T \in L_+(E, F)$. The relationship

$$R \rightarrow D = R + T \circ I$$

establishes a one to one correspondence between the set of T -expectation bounded risk measures and the set of deviations.

Proof. If R is a T -expectation bounded risk measure then set $D = R + T \circ I$ and D is trivially 0-translation invariant, positively homogeneous and sub-additive. To show that D is strictly positive, take $y_0 \in E$ and $y \in L_p(\mu, E) \setminus E$. Then

$$D(y_0) = R(y_0) + T\left(\int_{\Omega} y_0 d\mu\right) = R(y_0) + T(y_0) = 0,$$

since $R(y_0) = -T(y_0)$ due to Proposition 3. Besides

$$D(y) = R(y) + T\left(\int_{\Omega} y d\mu\right) > 0$$

because R is T -mean dominating.

Conversely, suppose that D is a deviation and set $R = D - T \circ I$. R is clearly T -translation invariant, positively homogeneous and sub-additive. To show that R is T -mean dominating, take $y \in L_p(\mu, E) \setminus E$, and one has that

$$R(y) = D(y) - T\left(\int_{\Omega} y d\mu\right) > -T\left(\int_{\Omega} y d\mu\right)$$

because $D(y) > 0$. □

Remark 1. If one takes $T = 0$ in Proposition 4 then one concludes that 0–expectation bounded VRF and vector deviations coincide. \square

Proposition 5. Under the conditions of Proposition 4, R is T –strongly coherent if D is T –lower range dominated, and the converse also holds if T is onto.

Proof. Suppose that D is T –lower rang dominated and take $f^* \in F_+^*$ and $y_1, y_2 \in L_p(\mu, E)$ such that $f^* \circ T \circ y_2 \geq f^* \circ T \circ y_1$ a.s. Since

$$R(y_2) \leq R(y_1) + R(y_2 - y_1)$$

we have that

$$f^* \circ R(y_2) \leq f^* \circ R(y_1) + f^* \circ R(y_2 - y_1),$$

and it is sufficient to see that

$$\langle f^*, R(y_2 - y_1) \rangle \leq 0.$$

D Being T –lower range dominated we have that

$$\begin{aligned} \langle f^*, R(y_2 - y_1) + T \circ I(y_2 - y_1) \rangle &\leq \\ \langle f^*, T \circ I(y_2 - y_1) \rangle - (ess - \inf) \{ \langle f^*, T(y_2(\omega) - y_1(\omega)) \rangle : \omega \in \Omega \}. \end{aligned}$$

Thus,

$$\langle f^*, R(y_2 - y_1) \rangle \leq - (ess - \inf) \{ \langle f^*, T(y_2(\omega) - y_1(\omega)) \rangle : \omega \in \Omega \} \leq 0.$$

Conversely, assume that R is T –strongly coherent. Set $f^* \in F_+^*$ and $y \in L_p(\mu, E)$. We must prove the inequality

$$\langle f^*, D(y) \rangle \leq \langle f^*, T \circ I(y) \rangle - (ess - \inf) \{ \langle f^*, T(y(\omega)) \rangle : \omega \in \Omega \},$$

i.e.,

$$\langle f^*, R(y) \rangle \leq - (ess - \inf) \{ \langle f^*, T(y(\omega)) \rangle : \omega \in \Omega \}.$$

The inequality is obvious if the essential infimum is not finite, so let us consider the existence of $\omega_0 \in \Omega$ where the infimum above is achieved (recall that T is onto). Then,

$$\begin{aligned} \langle f^*, R(y) \rangle &= \langle f^*, R(y - y(\omega_0)) \rangle - \langle f^*, T(y(\omega_0)) \rangle \\ &\leq - (ess - \inf) \{ \langle f^*, T(y(\omega)) \rangle : \omega \in \Omega \} \end{aligned}$$

since $\langle f^*, R(y - y(\omega_0)) \rangle \leq 0$ because $f^* \circ T \circ y(\omega) \geq f^* \circ T \circ y(\omega_0)$ and R is T –strongly coherent, and $\langle f^*, R(y(\omega_0)) \rangle = - \langle f^*, T(y(\omega_0)) \rangle$ owing to Proposition 3b. \square

4. Representation theorems

Artzner et al. (1999) and Jouini et al. (2004) stated Representation Theorems of “their coherent risk measures” (scalar and vector, respectively) by using duality properties and μ -continuous finitely or σ -finitely additive measures on the measurable space (Ω, \mathcal{F}) . Later, Rockafellar et al. (2006b) represented “their expectation bounded risk measures” by using $L_2(\mu, \mathbb{R})$, which may be identified with its dual space. Here we draw on the duality $(L_q(\mu, E^*), L_p(\mu, E))$ and follow the ideas of the authors above in order to represent the VRF by “some kind of envelope generated by its sub-gradients”.

Lemma 6. Suppose that $p < \infty$ and $F = \mathbb{R}$. Assume that E has the Radon-Nikodym property. If $D : L_p(\mu, E) \rightarrow \mathbb{R}$ is a real valued lower semi-continuous deviation then there exists $\Delta \subset L_q(\mu, E^*)$ satisfying the following conditions:

- a) Δ is convex and $\sigma(L_q(\mu, E^*), L_p(\mu, E))$ -closed.
- b) The equality

$$D(y) = \text{Sup} \left\{ - \int_{\Omega} \langle z^*(\omega), y(\omega) \rangle d\mu(\omega); z^* \in \Delta \right\} \quad (1)$$

holds for every $y \in L_p(\mu, E)$.

Proof. Since E satisfies the Radon-Nikodym property we have that $L_q(\mu, E^*)$ is the dual space of $L_p(\mu, E)$. Besides, if

$$\Delta_1 = \left\{ z^* \in L_q(\mu, E^*); D(y) \geq \int_{\Omega} \langle z^*(\omega), y(\omega) \rangle d\mu(\omega), \forall y \in L_p(\mu, E) \right\}$$

then it may be easily proved that Δ_1 is convex and $\sigma(L_q(\mu, E^*), L_p(\mu, E))$ -closed. Furthermore, since $L_q(\mu, E^*)$ is the dual space of $L_p(\mu, E)$, Theorem 2.4.14 in Zalinescu (2002) implies that

$$D(y) = \text{Sup} \left\{ \int_{\Omega} \langle z^*(\omega), y(\omega) \rangle d\mu(\omega); z^* \in \Delta_1 \right\}$$

for every $y \in L_p(\mu, E)$. Thus, the result trivially follows if one takes $\Delta = -\Delta_1$. \square

Remark 2. It is worth pointing out that the conclusion in the latter lemma also holds if $D(y) = 0$ for some $y \in L_p(\mu, E) \setminus E$. Moreover, the proof is nearly identical. Furthermore, notice that every $z^* \in \Delta$ satisfies

$$\int_{\Omega} z^* d\mu = 0.$$

Indeed, the proof of Lemma 6 points out that

$$-\int_{\Omega} \langle z^*, y_0 \rangle d\mu = -\left(\int_{\Omega} z^* d\mu\right)(y_0) \leq D(y_0) = 0$$

holds for every $y_0 \in E$. Thus, if $-y_0$ replaces y_0 we have $(\int_{\Omega} z^* d\mu)(y_0) = 0$. \square

Lemma 7. Suppose that $p < \infty$ and $F = \mathbb{R}$. Assume that E has the Radon-Nikodym property and take $e^* \in E_+^*$. If $R : L_p(\mu, E) \rightarrow \mathbb{R}$ is a real valued e^* -expectation bounded lower semi-continuous risk measure then there exists $\Delta \subset L_q(\mu, E^*)$ satisfying Condition a) above and

$$R(y) = \text{Sup} \left\{ -\int_{\Omega} \langle z^*(\omega), y(\omega) \rangle d\mu(\omega); z^* \in \Delta \right\} - \int_{\Omega} (e^* \circ y) d\mu \quad (2)$$

holds for every $y \in L_p(\mu, E)$. Furthermore, $z^*(\omega) + e^* \geq 0$ a.s. for every $z^* \in \Delta$ if and only if R is e^* -coherent.

Proof. The existence of Δ trivially follows from (1) and Proposition 4. Suppose that $z^*(\omega) + e^* \geq 0$ a.s. for every $z^* \in \Delta$. Then, take $y_1, y_2 \in L_p(\mu, E)$ with $y_2 \geq y_1$ a.s. and

$$-\int_{\Omega} \langle z^*(\omega) + e^*, y_1(\omega) \rangle d\mu(\omega) \geq -\int_{\Omega} \langle z^*(\omega) + e^*, y_2(\omega) \rangle d\mu(\omega)$$

trivially holds. Thus, R is e^* -coherent due to (2).

Conversely, suppose that $\mu(\{\omega \in \Omega; z^*(\omega) + e^* \notin E_+^*\}) > 0$ for some $z^* \in \Delta$. Then, there exists $y \in L_p(\mu, E)$, $y \geq 0$ a.s., such that

$$\int_{\Omega} \langle z^*(\omega) + e^*, y(\omega) \rangle d\mu(\omega) < 0.$$

Hence, (2) leads to $R(y) > 0$, i.e., according to Proposition 3, $R(y) > R(0)$, and R is neither decreasing nor e^* -coherent. \square

Remark 3. Notice that $\{e^*\} + \Delta$ may play the role of Δ in the latter lemma, in which case, we would obtain the existence of Δ , convex and $\sigma(L_q(\mu, E^*), L_p(\mu, E))$ -closed, satisfying ²

$$R(y) = \text{Sup} \left\{ -\int_{\Omega} \langle z^*(\omega), y(\omega) \rangle d\mu(\omega); z^* \in \Delta \right\} \quad (3)$$

² $\{e^*\} + \Delta$ is used to indicate that we add the elements in Δ plus the almost surely constant function $z^*(\omega) = e^*$.

for every $y \in L_p(\mu, E)$, and R is e^* -coherent if and only if $z^*(\omega) \geq 0$ a.s. for every $z^* \in \Delta$.

As in the preceding case, the statement remains true if $R(y) = -e^* \circ I(y)$ for some $y \in L_p(\mu, E) \setminus E$. Finally, note that elements in Δ will satisfy the equality $\int_{\Omega} z^* d\mu = e^*$. \square

Theorem 8. (Representation Theorem for Deviations). Suppose that $p < \infty$ and that E satisfies the Radon-Nikodym property. If $D : L_p(\mu, E) \rightarrow F$ is a deviation such that $f^* \circ D$ is lower semi-continuous for every $f^* \in F_+^*$, then for every $f^* \in F_+^*$ there exists $\Delta_{f^*} \subset L_q(\mu, E^*)$ satisfying the following conditions:

- a) Δ_{f^*} is convex and $\sigma(L_q(\mu, E^*), L_p(\mu, E))$ -closed.
- b) The equality

$$f^* \circ D(y) = \text{Sup} \left\{ - \int_{\Omega} \langle z^*(\omega), y(\omega) \rangle d\mu(\omega); z^* \in \Delta_{f^*} \right\}$$

holds for every $y \in L_p(\mu, E)$.³

Proof. It is a trivial consequence of Lemma 6 and Remark 2, if one takes into consideration that $f^* \circ D$ satisfies the properties of an \mathbb{R} -valued deviation, with the only exception that it might vanish on some elements of $L_p(\mu, E) \setminus E$. \square

Lemma 9. Suppose that $T \in L_+(E, F)$ and $R : L_p(\mu, E) \rightarrow F$ is a T -expectation bounded VRF . Then R is T -coherent if and only if $f^* \circ R$ is decreasing for every $f^* \in F_+^*$.

Proof. The result is clear if one bears in mind that F is a Banach lattice and, thus, for $f_1, f_2 \in F$ we have that $f_1 \leq f_2$ if and only if $\langle f^*, f_1 \rangle \leq \langle f^*, f_2 \rangle$ for every $f^* \in F_+^*$. \square

Theorem 10. (Representation Theorem for Expectation Bounded VRF). Suppose that $p < \infty$ and that E satisfies the Radon-Nikodym property. If $T \in L_+(E, F)$ and $R : L_p(\mu, E) \rightarrow F$ is a T -expectation bounded VRF such that $f^* \circ R$ is lower semi-continuous for every $f^* \in F_+^*$, then for every $f^* \in F_+^*$ there exists $\Delta_{f^*} \subset L_q(\mu, E^*)$ satisfying the following conditions:

- a) Δ_{f^*} is convex and $\sigma(L_q(\mu, E^*), L_p(\mu, E))$ -closed.

³Notice that $f^* \circ D$ is lower semi-continuous for every $f^* \in F_+^*$ if so is D , i.e., if for every $\tilde{y} \in L_p(\mu, E)$ and every open set $V \subset F$ with $0 \in V$ there exists $\delta > 0$ such that

$$\|y - \tilde{y}\|_p < \delta \Rightarrow D(y) \in D(\tilde{y}) + V + F_+.$$

b) The equality

$$f^* \circ R(y) = \text{Sup} \left\{ - \int_{\Omega} \langle z^*(\omega), y(\omega) \rangle d\mu(\omega); z^* \in \Delta_{f^*} \right\}$$

holds for every $y \in L_p(\mu, E)$. Moreover, R is T -coherent if and only if $z^* \geq 0$ a.s. for every $f^* \in F_+^*$ and every $z^* \in \Delta_{f^*}$.

Proof. It is a trivial consequence of Lemma 7, Lemma 9 and Remark 3, if one takes into consideration that $f^* \circ R$ satisfies the properties of an \mathbb{R} -valued $f^* \circ T$ -expectation bounded risk measure, with the only exception being that it might equal $-f^* \circ T \circ I(y)$ on some elements $y \in L_p(\mu, E) \setminus E$. \square

5. Optimizing vector risk functions

This section is devoted to dealing with the general portfolio choice problem

$$\left. \begin{array}{l} \text{Min} \quad \rho \left(\sum_{j=1}^n x_j y_j \right) \\ x \in X \end{array} \right\}, \quad (4)$$

which will be analyzed by drawing on the previous Representation Theorems. Here we assume that E satisfies the Radon-Nikodym property and $\{y_j\}_{j=1}^n \subset L_p(\mu, E)$ is the set of pay-offs of $n \in \mathbb{R}$ securities available in the market. These pay-offs are E -valued functions if there is an aggregation problem, in the line of Jouini et al. (2004), or if those situations presented in Burgert and Rüschendorf (2006) apply. Besides, $\rho : L_p(\mu, E) \rightarrow F$ is a generalized vector risk function, and $x = (x_1, x_2, \dots, x_n)$ denotes an arbitrary portfolio. The feasible set $X \subset \mathbb{R}^n$ of (4) will be given in practice with usual constraints (minimum required expected return, capital to invest, short-selling restrictions etc.).

Since (4) is a vector optimization problem we can apply both the scalarization method and the balance set approach. The scalarization method consists in solving

$$\left. \begin{array}{l} \text{Min} \quad f^* \circ \rho \left(\sum_{j=1}^n x_j y_j \right) \\ x \in X \end{array} \right\}, \quad (5)$$

$f^* \in F_+^*$ being arbitrary. Every solution of (5) also solves (4) if $f^* \in F_{++}^*$, and this kind of solution is usually called “proper”. Conversely, if X is convex and F_+^* satisfies some adequate assumptions (for instance, it has a non-empty interior), then for every solution \tilde{x} of (4) there exists $f^* \in F_+^*$ such that \tilde{x} solves (5). In practice, it is assumed that f^* contains “the weights” compatible with the decision maker utility function.

Problem (5) is non-differentiable but convex. Hence, there are several methods allowing us to deal with it. For instance, one can attempt to adapt some results from

Rockafellar et al. (2006a), though we propose here an alternative procedure based on the Representation Theorems stated in the section above.

The following result is straightforward and therefore its proof is omitted.

Proposition 11. Suppose that there exists a $\sigma(L_q(\mu, E^*), L_p(\mu, E))$ –compact and convex set $\Delta_{f^*} \subset L_q(\mu, E^*)$ such that

$$f^* \circ \rho(y) = \text{Sup} \left\{ - \int_{\Omega} \langle z^*(\omega), y(\omega) \rangle d\mu(\omega); z^* \in \Delta_{f^*} \right\},$$

holds for every $y \in L_p(\mu, E)$. Consider Problem

$$\left. \begin{array}{l} \text{Min } \theta \\ \theta + \sum_{j=1}^n \left(\int_{\Omega} \langle z^*, y_j \rangle d\mu \right) x_j \geq 0, \quad \forall z^* \in \Delta_{f^*} \\ x \in X \\ \theta \in \mathbb{R} \end{array} \right\} \quad (6)$$

(θ, x) being the decision variable. Then, \tilde{x} solves (5) if and only if there exists $\tilde{\theta}$ such that $(\tilde{\theta}, \tilde{x})$ solves (6), in which case $\tilde{\theta} = f^* \circ \rho \left(\sum_{j=1}^n \tilde{x}_j y_j \right)$. \square

Notice that Problem (6) can be dealt with more appropriately than Problem (5), since it is differentiable, or linear, if constraints generating X are as well. Moreover, the second constraint of (6) involves the Banach space $\mathcal{C}(\Delta_{f^*})$ of continuous and real-valued functions on the compact topological space Δ_{f^*} . Therefore, according to the Riesz Representation Theorem (Meyer-Nieberg, 1991), the Karush-Kuhn-Tucker optimality conditions of Problem (6), as well as the algorithms enabling us to solve this problem, involve the dual space $\mathcal{M}(\Delta_{f^*})$ of σ –additive and inner regular measures on the Borel σ –algebra of Δ_{f^*} . Actually, it may be established that a finite linear combination of Dirac Deltas provides us with the multipliers of (6). We will not address this issue here since analogous problems are studied in Balbás and Romera (2006) and Balbás et al. (2006), where they present a simplex-like method solving the dual problem of (6) as long as it is linear.

Notice that Proposition 11 requires Δ_{f^*} to be $\sigma(L_q(\mu, E^*), L_p(\mu, E))$ –compact, whereas the Representation Theorems only guarantee that this set must actually be $\sigma(L_q(\mu, E^*), L_p(\mu, E))$ –closed. Thus, according to Alaoglu’s Theorem, we need Δ_{f^*} to be bounded. Although Δ_{f^*} might be boundless, this fact hardly holds in the scalar case (see Rockafellar et al., 2006b for a significant set of examples) and we will see in the next section that many vectorial cases constructed from the usual scalar examples also have a bounded sub-gradient.

The vector Problem (4) may also be dealt with using the “balance space approach” of Galperin (1997), later extended in Balbás et al. (2002) for infinite-dimensional problems. Accordingly, let F be Dedekind complete (Meyer-Nieberg, 1991), let $F_0 \in F$ be the infimum value of Problem (4) and let $d \in F_+$ be a “direction of preferential deviations” (Galperin, 1997).⁴ Consider Problem

$$\left. \begin{array}{l} \text{Min } \theta \\ \rho \left(\sum_{j=1}^n x_j y_j \right) - \theta d \leq F_0 \\ x \in X, \theta \in \mathbb{R} \end{array} \right\}, \quad (7)$$

(θ, x) being the decision variable. Then, as stated in Balbás et al. (2002), the convexity of ρ guarantees that the set of efficient solutions of (4) coincides with the set of solutions of (7) as d covers $F_+ \setminus \{0\}$. Once again (7) is non-differentiable but, bearing in mind that its second constraint is equivalent to

$$f^* \circ \rho \left(\sum_{j=1}^n x_j y_j \right) - \theta f^*(d) \leq f^*(F_0), \forall f^* \in F_+^* \cap B^*,$$

B^* being the closed unit ball of F^* ,⁵ under the assumptions of Proposition 11 Problem (7) becomes

$$\left. \begin{array}{l} \text{Min } \theta \\ - \sum_{j=1}^n \left(\int_{\Omega} \langle z^*, y_j \rangle d\mu \right) x_j - \theta f^*(d) \leq f^*(F_0), \forall f^* \in F_+^* \cap B^*, \forall z^* \in \Delta_{f^*} \\ x \in X, \theta \in \mathbb{R} \end{array} \right\}. \quad (8)$$

Once again Problem (8) involves the spaces $\mathcal{C}(F_+^* \cap B^*)$, $\mathcal{C}(\Delta_{f^*})$, $\mathcal{M}(F_+^* \cap B^*)$ and $\mathcal{M}(\Delta_{f^*})$.

6. Examples

Example 1. First of all let us point out that one can obtain vector risk functions by simultaneously considering several scalar risk functions. In particular one can combine the Conditional Value at Risk (see, for instance, Rockafellar et al.,2006b), the interesting (coherent and expectation bounded) measure of Wang and the (also coherent and expectation bounded) Dual Power Transform (see Wang, 2000 and Whirch and Hardy, 2001) with several levels of confidence and one obtains a new vector risk function. Furthermore, the vector measure becomes coherent or expectation bounded

⁴The preferential deviation may be often understood as the vector indicating the ratio of losses among the diferent objectives that the decision maker is accepting.

⁵which is $\sigma(F^*, F)$ –compact.

if the components are as well. Similarly, if one combines several scalar deviations one obtains a vector deviation. For instance, if

$$\sigma_p(y) = \left(\int_{\Omega} \left| y - \int_{\Omega} y d\mu \right|^p d\mu \right)^{\frac{1}{p}}$$

is the usual p -deviation of $y \in L_p(\mu, \mathbb{R})$, $p < \infty$, and

$$\sigma_p^+(y) = \left(\int_{\Omega} \left(y - \int_{\Omega} y d\mu \right)_+^p d\mu \right)^{\frac{1}{p}}$$

and

$$\sigma_p^-(y) = \left(\int_{\Omega} \left(y - \int_{\Omega} y d\mu \right)_-^p d\mu \right)^{\frac{1}{p}}$$

are the positive and negative p -semi-deviations, then one can combine them all. In particular, if we are interested in standard deviation, asymmetry, kurtosis and absolute semi-deviation we can consider the vector deviation

$$D(y) = (\sigma_2(y), \sigma_3(y), \sigma_4(y), \sigma_1^-(y))$$

for every $y \in L_4(\mu, \mathbb{R})$.

Example 2. More generally, consider the set of Banach lattices E and $\{F_j\}_{j=1}^n$, take the family of non-negative linear maps $T_j : E \rightarrow F_j$, and the risk functions $\rho_j : L_p(\mu, E) \rightarrow F_j$, $j = 1, 2, \dots, n$. Take

$$\rho : L_p(\mu, E) \rightarrow \prod_{j=1}^n F_j$$

as usual. Then, if each ρ_j is a deviation so is ρ , and if each ρ_j is T_j -expectation bounded (respectively, T_j -coherent) then ρ is T -expectation bounded (respectively, T -coherent), where $T(y_0) = (T_j(y_0))_{j=1}^n \in \prod_{j=1}^n F_j$ for every $y_0 \in E$.

Example 3. Conditional Value at Risk. Consider $\alpha \in (0, 1)$, the Banach lattice E and $e^* \in E_+^*$. Take

$$\Delta_{e^*} = \left\{ z^* \in L_{\infty}(\mu, E^*); 0 \leq z^*(\omega) \leq \frac{e^*}{\alpha} \text{ a.s.}, \int_{\Omega} z^* d\mu = e^* \right\}.$$

Δ_{e^*} is clearly $\sigma(L_{\infty}(\mu, E^*), L_1(\mu, E))$ -closed and convex. Moreover, since $\|z^*\|_{\infty} \leq \|e^*\|/\alpha$ we have that Δ_{e^*} is clearly $\sigma(L_{\infty}(\mu, E^*), L_1(\mu, E))$ -compact (Alaoglu's Theorem). Define

$$CVaR_{(\alpha, e^*)}(y) = \text{Max} \left\{ - \int_{\Omega} \langle z^*, y \rangle d\mu; z^* \in \Delta_{e^*} \right\},$$

$y \in L_1(\mu, E)$. It is easy to see that $CVaR_{(\alpha, e^*)}$ is an e^* -expectation bounded and e^* -coherent risk measure. Moreover, bearing in mind the results of Rockafellar et al. (2006b) about the representation of the standard $CVaR_\alpha$, it is clear that $CVaR_{(\alpha, e^*)}$ is its genuine generalization.

Example 4. Extending scalar Risk Functions. More generally, if $\rho : L_p(\mu, \mathbb{R}) \rightarrow \mathbb{R}$ is a scalar coherent risk measure satisfying the expression

$$\rho_0(y) = \text{Max} \left\{ - \int_{\Omega} yz^* d\mu; z^* \in \Delta \right\}, \tag{9}$$

$\Delta \subset L_{q+}(\mu, \mathbb{R})$ being convex and $\sigma(L_q(\mu, \mathbb{R}), L_p(\mu, \mathbb{R}))$ -compact,⁶ then, for each non-null $e^* \in E_+^*$ one can consider

$$\Delta_{e^*} = \{z^* \in L_{q+}(\mu, E^*); \|z^*\| \in (e^*)\Delta\}.$$

Δ_{e^*} is clearly $\sigma(L_q(\mu, E^*), L_p(\mu, E))$ -compact and convex. Define

$$\rho_{e^*}(y) = \text{Max} \left\{ - \int_{\Omega} \langle z^*, y \rangle d\mu; z^* \in \Delta_{e^*} \right\},$$

$y \in L_p(\mu, E)$. It is easy to see that ρ_{e^*} is an e^* -coherent risk measure.

Example 5. Extending scalar Risk Functions II. Let E, F be Banach lattices and $T \in L_+(E, F)$. The ideas above may be used to define a T -coherent risk measure under some assumptions. Indeed, suppose that for every $f^* \in F_+^*$ the construction of Example 4 leads to the $f^* \circ T$ -coherent risk measure

$$\rho_{f^* \circ T} : L_p(\mu, E) \rightarrow \mathbb{R}.$$

Then, $\rho : L_p(\mu, E) \rightarrow F$ may be easily defined as long as for every $y \in L_p(\mu, E)$ and every $f^* \in F_+^*$ there exists $\rho(y) \in F$ with $f^*(\rho(y)) = \rho_{f^* \circ T}(y)$.

7. Conclusions

The paper has introduced a new notion of vector risk function and has extended concepts such as vector deviation, vector expectation bounded risk measure or vector coherent risk measure. Relationships amongst them have been analyzed. In this sense, the generalized vector risk functions may be used to provide initial capital requirements as well as to deal with most of the classical topics (pricing, hedging,

⁶Balbás et al. (2006) showed that the dual power transform and the measure of Wang satisfy Expression (9).

portfolio choice, etc.). The link with both scalar and vector risk functions studied in previous literature has been discussed, and it has been pointed out that this new approach seems to appropriately integrate several preceding points of view. Sub-gradient linked representation theorems, and portfolio selection problems, have been addressed as well, and practical examples have been illustrated.

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References

- [1] Artzner, P., F. Delbaen, J.M. Eber and D. Heath, 1999. “Coherent measures of risk”. *Mathematical Finance*, 9, 203-228.
- [2] Balbás, A., R. Balbás and S. Mayoral, 2006. “Optimizing measures of risk: A simplex-like algorithm”. University of Navarra, Working Paper n°11/06 (available at <http://www.gloriamundi.org>).
- [3] Balbás, A., P. Jiménez-Guerra, and M.J. Muñoz-Bouzo, 2002. “The balance space approach in optimization with Riesz spaces valued objectives. An application to financial markets”. *Computers & Mathematics with Applications*, 44, 7, 887-897.
- [4] Balbás, A. and R. Romera, 2006. “Hedging bond portfolios by optimization in Banach spaces”. *Journal of Optimization Theory and Applications*, 131, 1, forthcoming.
- [5] Benati, S., 2003. “The optimal portfolio problem with coherent risk measure constraints”. *European Journal of Operational Research*, 150, 572-584.
- [6] Burgert, C. and L. Rüschendorf, 2006. “Consistent risk measures for portfolio vectors”. *Insurance: Mathematics and Economics*, 38, 2, 289-297.
- [7] Diestel, J. and J.J. Uhl, 1977. “Vector measures”. American Mathematical Society, Providence, Rhode Island.
- [8] Föllmer, H. and A. Schied, 2002. “Convex measures of risk and trading constraints”. *Finance & Stochastics*, 6, 429-447.
- [9] Galperin E.A., 1997. “Pareto analysis vis-à-vis balance space approach in multiobjective global optimization”. *Journal of Optimization Theory and Applications*, 93, 3, 533-545.

- [10] Jouini, E., M. Meddeb and N. Touci, 2004. "Vector-valued coherent risk measures". *Finance & Stochastics*, 8, 531-552
- [11] Konno, H., K. Akishino and R. Yamamoto, 2005. "Optimization of a long-short portfolio under non-convex transaction costs". *Computational Optimization and Applications*, 32, 115-132.
- [12] Meyer-Nieberg P., 1991. "Banach lattices". Springer-Verlag. New York.
- [13] Nakano, Y., 2004. "Efficient hedging with coherent risk measure". *Journal of Mathematical Analysis and Applications*, 293, 345-354.
- [14] Ogryczak, W. and A. Ruszczyński, 1999. "From stochastic dominance to mean risk models: Semideviations and risk measures". *European Journal of Operational Research*, 116, 33-50.
- [15] Rockafellar, R.T., S. Uryasev and M. Zabarankin, 2006a. "Optimality conditions in portfolio analysis with general deviations measures". *Mathematical Programming, Ser. B*, 108, 515-540.
- [16] Rockafellar, R.T., S. Uryasev and M. Zabarankin, 2006b. "Generalized deviations in risk analysis". *Finance & Stochastics*, 10, 51-74.
- [17] Wang, S.S., 2000. "A class of distortion operators for pricing financial and insurance risks". *Journal of Risk and Insurance*, 67, 15-36.
- [18] Whirch, J.L., and M.R. Hardy, 2001. "Distortion risk measures. Coherence and stochastic dominance". Unpublished Working Paper (available at <http://www.gloriamundi.org>).
- [19] Zalinescu, C., 2002. "Convex analysis in general vector spaces". World Scientific Publishing Co.