Asymmetric long memory GARCH: a reply to Hwang's model

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Abstract

Hwang (Econom. Lett. 71 (2001) 1) proposes the FIFGARCH model to represent long memory asymmetric conditional variances. However, the model is badly specifie and does not nest some fractionally integrated heteroskedastic models previously proposed. We suggest an alternative specificatio and illustrate the results with simulated data.

Keywords: EGARCH; FGARCH; FIGARCH; FIEGARCH

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1. Introduction

The GARCH model was originally proposed by Engle (1982) and Bollerslev (1986) to represent the dynamic evolution of conditional variances and has been mainly applied to represent time series of high frequency financia returns. In the simplest framework, the returns series, ε_t, is given by $\varepsilon_t = z_t\sigma_t$, where z_t is an independent white noise process with unit variance and symmetric distribution and σ_t is the volatility, specifie as a linear function of past squared returns. Two of the most popular generalizations of the GARCH model are related with the asymmetric response of σ_t^2 to positive and negative returns and with the persistence often observed in the autocorrelations of squared returns.

To represent asymmetric volatilities, Hentschel (1995) proposed the Family GARCH (FGARCH) model where the conditional variance of returns is given by:

$$\frac{\sigma_t^{\lambda} - 1}{\lambda} = \omega + \alpha\sigma_{t-1}^{\lambda} f^{\nu}(z_{t-1}) + \delta\frac{\sigma_{t-1}^{\lambda} - 1}{\lambda} \qquad (1a)$$

$$f(z_t) = |z_t - b| - c(z_t - b) \tag{1b}$$

where the parameter λ represents the Box and Cox (1964) transformation for σ_t so that the limit when λ goes to zero is the logarithmic transformation. The FGARCH model encompasses, among many others, the asymmetric EGARCH model of Nelson (1991).

With respect to the persistence of volatility, Baillie et al. (1996) proposed the Fractionally Integrated GARCH (FIGARCH) model. In the FIGARCH (1, d, 1) model, the conditional variance is given by:

$$\sigma_t^2 = (1-\delta)^{-1}\kappa + [1 - (1-\delta L)^{-1}(1-\phi L)(1-L)^d]\varepsilon_t^2 \tag{2}$$

where d is the long memory parameter such that $0 \leq d < 1$.

Finally, Bollerslev and Mikkelsen (1996) proposed the Fractionally Integrated EGARCH (FIEGARCH) model that represents simultaneously long memory and asymmetric volatilities. In the FIEGARCH (1, d, 1) model, the volatility equation is given by:

$$(1-\phi L)(1-L)^d \log(\sigma_t^2) = \omega' + (1+\psi L)g(z_{t-1}) \tag{3}$$

where $g(z_t) = \theta z_t + \gamma[|z_t| - E(|z_t|)]$.

Recently, Hwang (2001) proposed an alternative asymmetric long memory model, the Asymmetric FIFGARCH model, that is supposed to nest both the FIGARCH and FIEGARCH models. The objective of this paper is to show that the FIFGARCH model is badly specifie and does not encompass the FIEGARCH model.

The rest of the paper is organized as follows. Section 2 describes the FIFGARCH model and shows that its specificatio is not adequate. An alternative specification that seems to fullfi the objectives stated by Hwang (2001) is proposed. Section 3 illustrates, with simulated series, the problems faced by the FIFGARCH model to generate series with long memory and asymmetric conditional variances. Finally, Section 4 concludes the paper.

2. The asymmetric long memory GARCH model

Hwang (2001) proposes the FIFGARCH (1, d, 1) model to represent both the long memory and asymmetry properties of conditional standard deviations. In this model, the specificatio of the conditional variance is given by:

$$\sigma_t^\lambda = (1-\delta)^{-1}\kappa + \left[1 - (1-\delta L)^{-1}(1-\phi L)(1-L)^d\right] f^\nu(z_t)\sigma_t^\lambda \tag{4}$$

where $f(\cdot)$ is the function define in (1b). Looking at Eq. (4), it seems that Hwang (2001) has tried to generalize the FGARCH model in (1) to allow for long memory, by introducing the fractional root, $(1-L)^d$, in the volatility equation. Alternatively, model (4) could be seen as a generalization of the FIGARCH model in (2) using a transformation of the conditional variance similar to the one proposed by Hentschel (1995). Anyhow, Hwang (2001) does not refer to any of the previous related papers in this area so it is difficul to know which model he is exactly generalizing. Actually, we guess that the acronym FIFGARCH stands for Fractionally Integrated Family-GARCH model.

In any case, model (4) does not seem to be able to represent adequately the simultaneous presence of long memory and asymmetry in the conditional variance. The main problem is that Hwang (2001) is not considering a proper Box–Cox transformation of the conditional standard deviation and, as a consequence, model (4) is misspecifie when $\lambda = 0$. In this case, the model becomes:

$$1 = (1-\delta)^{-1}\kappa + \left[1-(1-\delta L)^{-1}(1-\phi L)(1-L)^{d}\right]f^{\nu}(z_t)$$

and the conditional standard deviation is not defined Therefore, the FIEGARCH model in (3) cannot be obtained from model (4) when $\lambda = 0$, as Hwang claims. Moreover, Tables 1 and 2 in Hwang (2001) set out the FIEGARCH model as a special case of the FIFGARCH model when $\lambda = 0$, $\nu = 1$ and $c = b = 0$. However, taking $c = 0$ removes the asymmetry that characterizes the FIEGARCH model. But even if model (4) were define in such a way that, when $\lambda = 0$, the logarithmic transformation would be obtained, and considering the appropriate constraints on the parameters, namely $\lambda = 0$, $\nu = 1$ and $b = 0$, the model will become:

$$\log(\sigma_t) = (1-\delta)^{-1}\kappa + \left[1-(1-\delta L)^{-1}(1-\phi L)(1-L)^{d}\right](|z_t| - cz_t)\log(\sigma_t)$$

which is not a FIEGARCH model either. Notice that the logarithmic transformation has often been used in the empirical analysis of financia returns and, consequently, it is an important particular case of the Box–Cox transformation of volatilities.

On the other hand, notice that the FIGARCH model in (2) can be obtained as a particular case of model (4) when $\lambda = \nu = 2$ and $c = b = 0$. Actually, this is the only previous fractionally integrated model that is encompassed by Hwang's model. However, the FIGARCH model is not able to represent asymmetries in the response of conditional volatilities to negative and positive returns. Furthermore, if $d = 0$ in Eq. (4) and the proper Box–Cox transformation is applied to σ_t in the left-hand side of such equation, then the short memory Hentschel's FGARCH model will be obtained. However, it is not obvious that the asymmetric FIFGARCH model is able to encompass adequately models with conditional variances that simultaneously are asymmetric and have long memory.

Alternatively, we propose a direct generalization of the FGARCH model in (1) based on introducing a fractional unit root in the volatility equation as follows:

$$(1-\phi L)(1-L)^{d}\frac{\sigma_t^{\lambda}-1}{\lambda} = \omega^* + \alpha(1+\psi L)\sigma_{t-1}^{\lambda}[f^{\nu}(z_{t-1})-1] \tag{5}$$

Imposing appropriate restrictions on the parameters and reparametrizing this model, the following specification can be obtained: if $\lambda = 0$, $\nu = 1$ and $b = 0$, we obtain the FIEGARCH model in (3) with $\theta = -2\alpha c$ and $\gamma = 2\alpha$; if $\lambda = \nu = 2$ and $b = c = 0$, the alternative FIGARCH model in Eq. (A.7) in Appendix A is obtained with $\alpha_1 = 2\alpha$; and finally if $d = \psi = 0$, the FGARCH model in (1) is obtained with $\delta = \phi - \lambda\alpha$.

Nevertheless, it will be necessary to analyze in detail the statistical properties and stationary conditions of model (5) as well as the properties of the estimators of the parameters before the model could be applied to the analysis of real time series.

3. Simulation experiments

In this section, we carry out simulation experiments to illustrate the lack of adequacy of the model

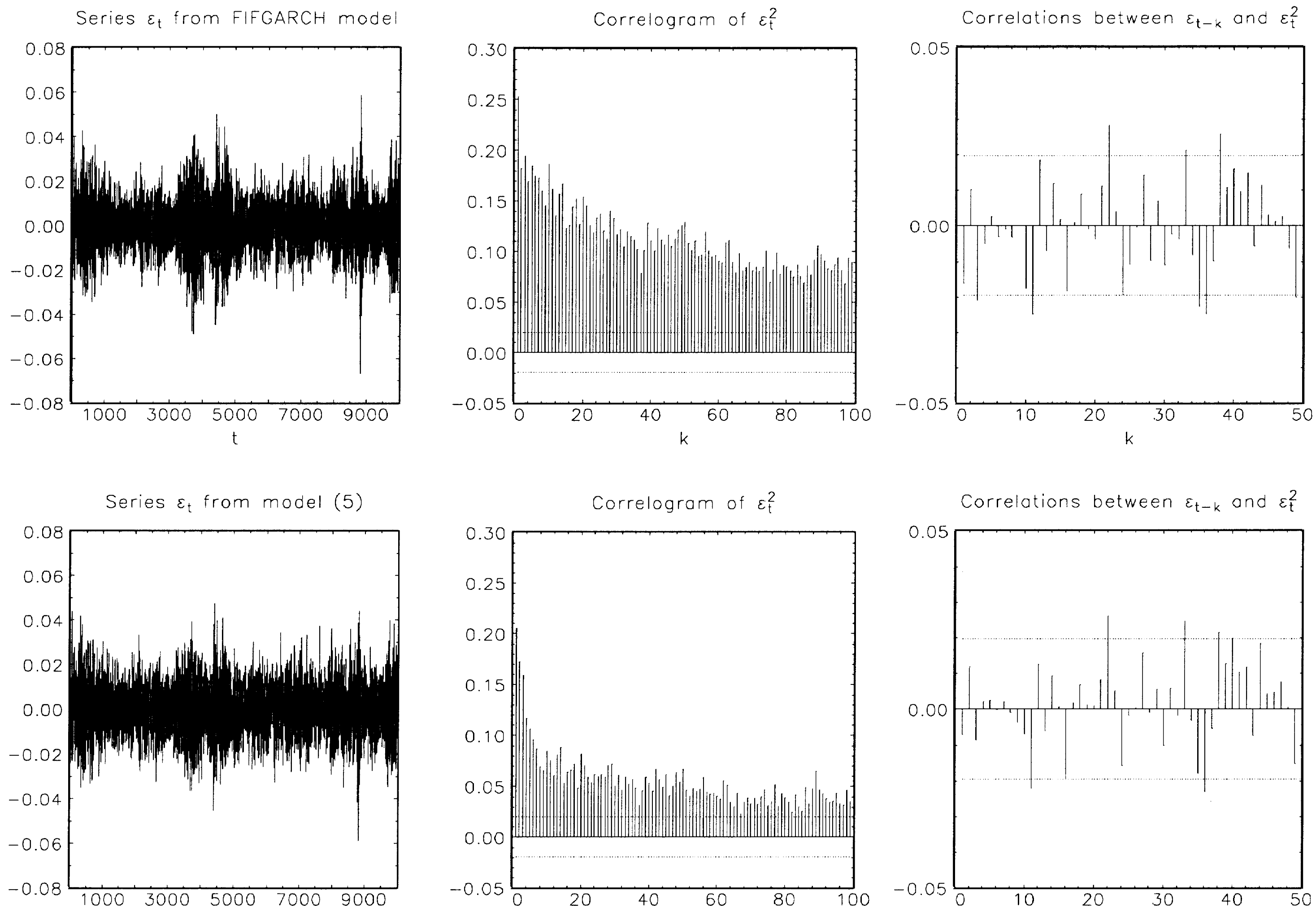

Fig. 1. Series generated by Asymmetric FIFGARCH model and by model (5) with $\lambda = \nu = 2$ and $b = c = 0$.

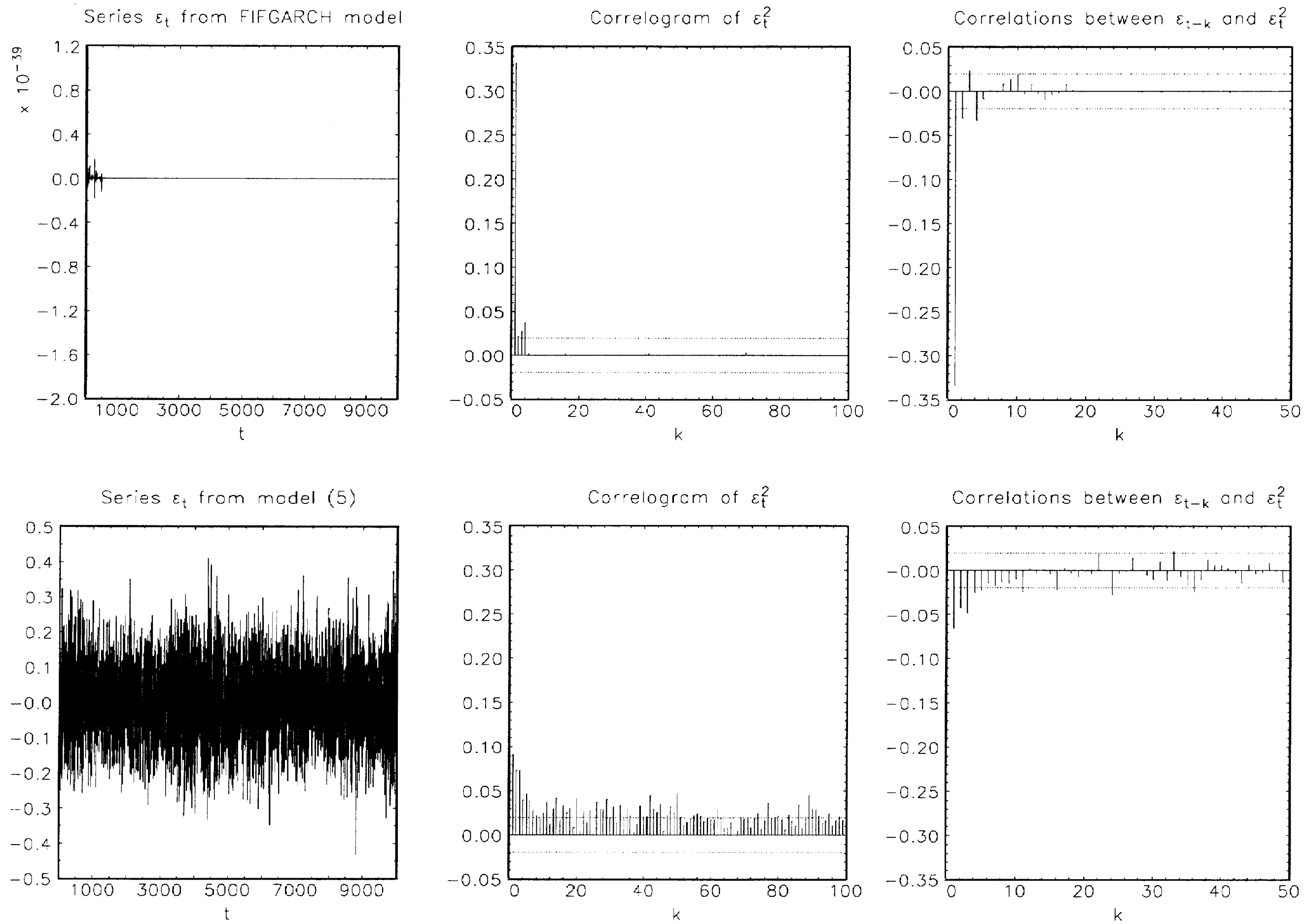

Fig. 2. Series generated by Asymmetric FIFGARCH model and by model (5) with $\lambda = 0$, $\nu = 1$ and $b = 0$.

proposed by Hwang (2001) to represent conditionally heteroskedastic time series with asymmetric volatilities and long memory in squared observations. In order to identify the long memory property, we use the correlogram of the squared observations. The asymmetry of volatility is identifie using the correlations between ε_t^2 and ε_{t-k}, $k \geq 1$. Observations of the series $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T\}$, with $T = 10\,000$, are generated by the FIFGARCH model in (4) with two different sets of parameter values, namely $\Theta_1 = \{\lambda = \nu = 2,\ b = c = 0,\ \kappa = 2.07 \times 10^{-6},\ \delta = 0.627,\ d = 0.393,\ \phi = 0.381\}$ and $\Theta_2 = \{\lambda = 0.1,\ \nu = 1, b = 0,\ c = 0.525,\ \kappa = 2.07 \times 10^{-6},\ \delta = 0.627,\ d = 0.393,\ \phi = 0.381\}$. The firs set of parameter values has been chosen to replicate the estimates obtained by Hwang (2001) when the FIGARCH (1, d, 1) model is fitte to a series of daily exchange returns of the ECU against the Dollar. The second parameter set is chosen to resemble a FIEGARCH (1, d, 1) model. Recall that Hwang (2001) claims that the FIFGARCH model encompasses the FIEGARCH model when $\lambda = 0$ and $\nu = 1$. However, given that the conditional variance of the FIFGARCH model with $\lambda = 0$ is not defined we choose a strictly positive although very small value of λ. The other parameter values are not taken from the corresponding estimated FIEGARCH model in Hwang (2001) because there is an additional parameter α in that model which is not defined

For the sake of comparison, we also generate two series by model (5) with the following parameter values: $\Theta_3 = \{\lambda = \nu = 2,\ b = c = 0,\ \omega^* = 1.035 \times 10^{-6},\ \alpha = 0.0735,\ d = 0.393,\ \phi = 0.381,\ \psi = 0.257\}$ and $\Theta_4 = \{\lambda = 0, \nu = 1, b = 0,\ c = 0.525,\ \omega^* = -0.019,\ \alpha = 0.0735,\ d = 0.393,\ \phi = 0.381,\ \psi = 0.257\}$. These parameter values have been chosen to parallel the parameters of the two previous FIFGARCH models.

The simulation of fractional processes requires truncating the infinit lag polynomial $(1 - L)^d$ in Eqs. (4) and (5). Since the fractional differencing operator is designed to capture the long memory features of the process, truncating at too low lag may destroy important long-run dependencies. To mitigate these effects, the truncation lag is set to $J = 3000$. In order to avoid start-up problems, the firs 7000 realizations are discarded. For each of the series, the noise z_t is generated by a standard Normal variable.

Fig. 1 plots two series generated with $\lambda = \nu = 2$ and $b = c = 0$, together with the corresponding correlogram of squared observations and the sample correlations between ε_{t-k} and ε_t^2, for $k = 1,2,\ldots,50$. The top panels represent the FIFGARCH process with parameters Θ_1 and the bottom panels represent the analogous series from model (5) with parameters Θ_3. It can be observed that, in this case, both models generate long memory in the series of squared returns, which exhibits slowly decaying and significan correlations up to very high lags. As it can be expected, the correlations between ε_{t-k} and ε_t^2 are not significan for any of the models.

Similarly, Fig. 2 displays the same plots for the FIFGARCH process with parameters Θ_2 and the series from model (5) with parameters Θ_4. This figur clearly shows that when λ goes to zero, the conditional variance of the FIFGARCH process collapses to zero producing unrealistic values of the series ε_t and no long memory in squared returns. However, model (5) generates series with properties similar to the ones observed in real time series, in particular, long memory in the squares, with small but very persistent correlations of ε_t^2, and the asymmetric response of σ_t^2 to positive or negative returns.

4. Conclusions

The asymmetric FIFGARCH model has been proposed by Hwang (2001) to represent the asymmetry and persistence often observed in the conditional variances of high frequency series of returns. It seems that the FIFGARCH model is trying to generalize the model proposed by Hentschel (1995) in the same way the GARCH model has been generalized to the FIGARCH model. However, contrary to the claims made by Hwang (2001), we show that the asymmetric FIFGARCH is not able to encompass previous fractionally integrated heteroskedastic models proposed in the literature with the exception of the FIGARCH model. The lack of adequacy of the FIFGARCH model when the logarithm of volatility is modelled, is illustrated with simulated data.

We propose an alternative specificatio that encompasses the most popular long memory and asymmetric models for conditional variances and show, with simulated series, that their sample properties are close to the properties usually observed in the empirical analysis of real time series of returns. At the moment, the statistical properties of this model are unknown but their analysis is beyond the objectives of this paper. However, it is important to mention that, before the model can be fitte to real time series, these properties as well as the properties of the corresponding estimators should be analyzed.

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Appendix A

In this appendix, we derive an alternative expression to the FIGARCH model originally proposed by Baillie et al. (1996).

Consider the following GARCH (2,1) model:

$$\begin{aligned} \varepsilon_t &= z_t \sigma_t \\ \sigma_t^2 &= \kappa + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \delta \sigma_{t-1}^2 \end{aligned} \tag{A.1}$$

This model can be written as an ARMA (2, 1) for ε_t^2 as follows:

$$[1 - (\alpha_1 + \delta)L - \alpha_2 L^2]\varepsilon_t^2 = \kappa + (1 - \delta L)\nu_t \tag{A.2}$$

where $\nu_t = \varepsilon_t^2 - \sigma_t^2$ is a zero mean serially uncorrelated noise. When the autoregressive lag polynomial in (A.2) contains a unit root, the corresponding Integrated GARCH model is obtained:

$$(1 - \phi L)(1 - L)\varepsilon_t^2 = \kappa + (1 - \delta L)\nu_t \tag{A.3}$$

where $[1 - (\alpha_1 + \delta)L - \alpha_2 L^2] = (1 - \phi L)(1 - L)$.

The FIGARCH model is obtained by replacing the firs difference operator in (A.3) with the fractional differencing operator, so that the following expression for ε_t^2 is obtained:

$$(1 - \phi L)(1 - L)^d \varepsilon_t^2 = \kappa + (1 - \delta L)\nu_t \tag{A.4}$$

see Baillie et al. (1996). Finally, substituting $\nu_t = \varepsilon_t^2 - \sigma_t^2$ in (A.4) and rearranging terms, the FIGARCH (1, d, 1) model in Eq. (2) comes up.

An alternative representation of the FIGARCH (1, d, 1) model can be obtained by considering the following expression of the conditional variance in (A.1):

$$\sigma_t^2 = \kappa + (\alpha_1 + \alpha_2 L)\sigma_{t-1}^2(z_{t-1}^2 - 1) + (\delta + \alpha_1 + \alpha_2 L)\sigma_{t-1}^2 \tag{A.5}$$

Reorganizing terms in (A.5), the following expression for σ_t^2 is obtained:

$$[1 - (\alpha_1 + \delta)L - \alpha_2 L^2]\sigma_t^2 = \kappa + (\alpha_1 + \alpha_2 L)\sigma_{t-1}^2(z_{t-1}^2 - 1) \tag{A.6}$$

If we now proceed as we did to work out expression (A.4) and factorize the autoregressive polynomial in the left-hand side of (A.6) allowing for a fractional root, the following alternative representation for the FIGARCH (1, d, 1) model turns out:

$$(1 - \phi L)(1 - L)^d \sigma_t^2 = \kappa + \alpha_1(1 + \psi L)\sigma_{t-1}^2(z_{t-1}^2 - 1) \tag{A.7}$$

where $\psi = \alpha_2/\alpha_1$.

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