Distinguished Limits of Lévy-Stable Processes, and Applications to Option Pricing

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Abstract

In this paper we derive analytic expressions for the value of European Put and Call options when the stock process follows an exponential Lévy-Stable process. It is shown that the generalised Black-Scholes operator for the Lévy-Stable case can be obtained as an asymptotic approximation of a process where the random variable follows a Damped-Lévy process. Finally, it is also shown that option prices under the Lévy-Stable case generate the volatility smile encountered in the financial markets when the Black-Scholes framework is employed.

Keywords: Lévy-Stable processes, Stable Paretian hypothesis, Damped Lévy-Stable, option pricing.

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1 Introduction

Up until the early 1990's most of the stochastic processes used in the financial literature were based on a combination of Brownian Motion and Poisson processes. One of the most fundamental assumptions throughout has been that financial asset returns are the cumulative outcome of many small events that happen at a 'microscopic level' and occur very often in time; so often that may be regarded as continuous. If these microscopic events are considered statistically independent with finite variance it is straightforward to characterise their cumulative behaviour by invoking the Central Limit Theorem (CLT). Hence, Gaussian-based financial models have been proposed as a plausible choice.

But are there any other limiting distributions that characterise the behaviour of the sum of many 'microscopic' events? The answer is yes. The sum of many iid events always has, after appropriate scaling and shifting, a limiting distribution (by the generalised version of the Central Limit Theorem), namely a Lévy-Stable law. The Gaussian distribution is one particular case of the class of Lévy-Stable distributions. Therefore, based on this fundamental result it is plausible to assume that the 'formation' of prices in the market is the sum of many stochastic events with a Lévy-Stable limiting distribution; as shown by the Generalised Central Limit Theorem (GCLT).

The Lévy-Stable process is a particular class of the family of Lévy processes (infinitely divisible distributions). These provide a much richer and versatile environment to model the behaviour of financial markets than those purely based on Brownian Motion. For example, Gaussian[1] models perform poorly when modelling extreme events, because the probability of a substantial change in the underlying process is considerably smaller than the frequency observed in financial markets. On the other hand, a large class of infinitely divisible distributions is better suited to model such extreme events.

The use of Lévy-Stable models has been objected to for a number of reasons. First, the non-existence of moments of second or higher order has been seen as a major drawback from an empirical point of view. Second, with the exception of a few cases, the probability density function (pdf) is not known in closed form; therefore the models are relatively less

[1] The Gaussian distribution is a particular case of the Lévy-Stable distribution; however it does not exhibit skewness or heavy tails.

tractable and one must make use of their characteristic functions instead. A great number of recent developments in derivatives pricing, such as [28], [12], [1], [26] and [5], based on the characteristic function of the stochastic processes, have lead to more efficient and versatile numerical techniques as well as being the only way of obtaining analytic solutions to more complex problems.

On the other hand Lévy-Stable models share characteristics unique to their class such as 'stability', ie linear combinations of different random variables have again a Lévy-Stable distribution, or equivalently the distribution of the sum of Lévy-Stable random variables has the same shape as the individual random variables up to scale and shift. Other important features are that they can easily accommodate heavy tails and skewness of stock returns, a much desired property in empirical finance [23]. This motivates us to revisit alternative distributions such as the Lévy-Stable as the driving stochastic components in our models.

For these reasons, the use of Lévy processes in the modelling of stock returns [20], stochastic volatility [2] and other financial phenomena [4] has recently become substantially more popular. However, already in the early 1960's, models driven by Lévy-Stable[2] distributions had been proposed by Mandelbrot, Taylor and Fama [18], as an alternative to the log-normal assumption.

The next section reviews the literature on option pricing with Lévy-Stable processes. Section 3 develops the background theory of Lévy-Stable and infinitely divisible distributions upon which our financial modelling is built. Section 4 looks at process intimately related to the Lévy-Stable process. Section 5 calculates the value of American perpetual options when the stock price follows a totally skewed Lévy-Stable process. Section 6 derives a generalised Black-Scholes PDE for the Lévy-Stable case and numerical results are presented in Section 7. Finally, Section 8 looks at a particular model in which innovations are Damped-Lévy-Stable.

2 Option pricing with Lévy-Stable processes

The Lévy-Stable hypothesis in a financial context was first proposed by Mandelbrot [15], [16], [8], [18], [17]. In its early stages the Lévy-Stable hypothesis was supported by empirical

[2]In the literature these distributions have also been labelled Stable, Pareto-Lévy and Stable-Paretian.

evidence in Fama [9] and Roll [24] and Fama extended the results of the Capital Asset Pricing Model to include the Lévy-Stable hypothesis [10]. Moreover, Ziemba [29] looks at optimal portfolio decisions when returns have stable distributions.

There is relatively little literature on option pricing with Lévy-Stable processes. One of the early attempts to price options using Lévy-Stable processes was by McCulloch [22] using a utility maximisation setting. Hurst, Platen, and Rachev [13] based on the Mandelbrot and Taylor [18] subordinated process were able to price European options with Lévy-Stable symmetric returns.

The most recent study is that of Carr and Wu [6]. The authors are able to price European options when stock returns follow a totally skewed Lévy-Stable process but are unable to provide a solution to the general case when the skewness parameter is allowed to take all possible values in the interval $[-1, 1]$.

Our approach provides a solution without restricting the values of the parameters that characterise the Lévy-Stable process.

3 Lévy-Stable processes

In this section we will present properties of Lévy-Stable processes. Where proofs are omitted, they can be found in the references [3], [11] and [25]. We shall start by giving a general definition of a Lévy process.

Definition 1 ***Lévy process.***

Let $X(t)$ be a random variable dependent on time t. Then the stochastic process

$$X(t), \text{ for } 0 < t < \infty \text{ and } X(0) = 0,$$

is a Lévy process if and only if it has independent and stationary increments.

By stationary increments we mean that for each $s > 0$ the random variable $X(t + s) - X(t)$ has the same distribution as the random variable $X(t' + s) - X(t')$ for all t, $t' \geq 0$. Some well known examples are the Gaussian and the Poisson processes, as well as the Lévy-Stable and Damped-Lévy processes which we use later.

A characterisation of Lévy processes is given by the Lévy-Khintchine representation [2].

Theorem 1 ***Lévy-Khintchine representation.***

Let X(t) be a Lévy process. Then the natural logarithm of the characteristic function can be written as

$$
\begin{aligned}
\ln \mathbb{E}[e^{i\theta X(t)}] &= ait\theta - \frac{1}{2}\sigma^2 t\theta^2 + t\int_{\{|x|\geq 1\}} \left(1 - e^{i\theta x}\right) W(dx) \\
&+ \ t\int_{\{|x|<1\}} \left(1 - e^{i\theta x} + i\theta x\right) W(dx), \qquad (1)
\end{aligned}
$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ *and the Lévy measure* W *must satisfy*

$$\int_{\mathbb{R}} \min\{1, x^2\} W(dx) < \infty$$

and have no mass at 0.

A Lévy process can be seen as a combination of a drift component, a Brownian Motion (Gaussian) component and a jump component. These three components are completely determined by the Lévy-Khintchine triplet (a, σ^2, W). The parameter a parametrises the 'trend' component which is responsible for the development of the process $X(t)$ on the average. The parameter σ^2 defines the variance of the continuous Gaussian component of $X(t)$. Finally, the Lévy measure W is responsible for the behaviour of the jump component of $X(t)$ and determines the frequency and magnitude of jumps.

Remark 1 *One important implication of the independence and stationarity of increments is that the distributions of a Lévy process are completely determined by their distribution over unit time. In other words,*

$$\ln \mathbb{E}[e^{i\theta X(t)}] = t \ln \mathbb{E}[e^{i\theta X(1)}].$$

Alternatively, if we denote the characteristic triplet of a Lévy process $X(t)$ *by* (a_t, σ_t^2, W_t) *and the characteristic triplet of its unit-time distribution by* (a, σ^2, W) *we have that* $a_t = ta$, $\sigma_t^2 = t\sigma^2$ *and* $W_t = tW$. *See [2], [27].*

In this paper we will focus on two closely related Lévy processes: the Lévy-Stable and the Damped-Lévy processes. We will show that by choosing a particular triplet (a, σ^2, W) we obtain these two processes. The definitions given below are all for the one-dimensional case; however they can all be extended to higher dimensions [25], [27].

Definition 2 ***Stable random variable.***

Let X be a random variable. X has a stable distribution if, for any positive numbers A, B, there is a positive number C and a real number D such that (equality in distribution)

$$AX_1 + BX_2 \stackrel{d}{=} CX + D, \tag{2}$$

where X_1 and X_2 are independent copies of X, and where $\stackrel{d}{=}$ denotes equality in distribution.

In other words, we are saying that the shape of the distribution of $AX_1 + BX_2$ is the same as the shape of the distribution of X up to scale and shift.

Another definition for stable random variables, equivalent to Definition 2, is the following.

Definition 3 ***Stable random variable.***

A random variable X is said to have a stable distribution if for any $n \geq 2$, there is a positive number C_n and a real number D_n such that

$$X_1 + X_2 + \cdots + X_n \stackrel{d}{=} C_n X + D_n, \tag{3}$$

where $X_1, X_2, ..., X_n$ are independent copies of X.

Remark 2 *It can be shown that $C_n = n^{1/\nu}$ with $0 < \nu \leq 2$, [11], which reveals the role of ν as a time-scaling parameter. We discuss the the interpretation of this paramater below.*

Definition 4 ***Lévy-Stable process I.***

Let $X(t)$ be a random variable dependent on time t. Then the stochastic process

$$X(t), \textit{ for } 0 < t < \infty,$$

is a Lévy-Stable process if the finite-dimensional distribution of X is stable.

Alternatively we can define a Lévy-Stable process in the following way.

Definition 5 ***Lévy-Stable process II.***

A non-degenerate one-dimensional Lévy process $X(t)$ is a Lévy-Stable process if and only if for each $c > 0$ there exists a number D (depending on c in general) and $\nu \in (0, 2]$ such that

$$X(ct) \overset{d}{=} c^{1/\nu} X(t) + Dt. \tag{4}$$

The characteristic function of Lévy-Stable process is given in the following proposition. It will be seen that it is a class of Lévy process with a particular choice of a Lévy measure and with no Gaussian component (ie unless $\nu = 2$).

Proposition 1 ***Characteristic Function of Lévy-Stable Process.***

Let X be a Lévy-Stable random variable. Then the natural logarithm of its characteristic function is given in terms of certain parameters ν, κ, η and m by

$$\ln \mathbb{E}[e^{iX\theta}] \equiv [\Psi(\theta)] = \begin{cases} -\kappa^{\nu}|\theta|^{\nu} \left\{1 - i\eta \; sign(\theta) \tan(\nu\pi/2)\right\} + im\theta & for\ \nu \neq 1, \\ -\kappa|\theta| \left\{1 + \frac{2i\eta}{\pi} \; sign(\theta) \ln|\theta|\right\} + im\theta & for\ \nu = 1. \end{cases} \tag{5}$$

Proof

Following [11] we have that the logarithm of the characteristic function for Lévy-Stable random variables is given by the Lévy-Khintchine representation (1)

$$\Psi(\theta) = \int_{-\infty}^{\infty} \left(e^{i\theta x} - 1 - i\theta\tau_{\nu}(x)\right) W(dx), \tag{6}$$

with triplet $(0, 0, W)$, where the scaling requirement forces

$$W(x) = \begin{cases} Cq\,|x|^{-1-\nu} & \text{for } x < 0, \\ Cpx^{-1-\nu} & \text{for } x > 0, \end{cases}$$

and

$$\tau_{\nu}(x) = \begin{cases} x & \text{for } \nu > 1, \\ \sin x & \text{for } \nu = 1, \\ 0 & \text{for } \nu < 1; \end{cases}$$

here $C > 0$ is a scale constant, $p \geq 0$ and $q \geq 0$, with $p + q = 1$.

Straightforward integration yields the result.

If the random variable X belongs to a stable distribution with parameters ν, κ, η, m we write $X \sim S_\nu(\kappa, \eta, m)$. The parameter ν is known as the stability index or characteristic exponent; κ is a scaling parameter; η is a skewness parameter and m is a location parameter. These parameters can be interpreted as follows.

- The parameter ν is called the characteristic exponent. It takes values $\nu \in (0, 2]$ and in particular when $\nu = 2$ we get the Normal distribution. Thus, it can be seen as a "departure" from the Gaussian case as ν moves away from $\nu = 2$. Intuitively this parameter can also be interpreted as the shape parameter or fatness of tails (see Figure 1 below).
- The parameter $\kappa \geq 0$ is the scale parameter. It cannot be interpreted as the standard deviation of the process since this exists only for the Gaussian case. However, the larger κ is the 'wider' is the pdf of the random variable.
- The parameter $\eta \in [-1, 1]$ refers to the skewness of the density function. When $\eta = 1$ the stable distribution is "totally skewed" to the right and similarly when $\eta = -1$ it is "totally skewed" to the left. When $\eta = 0$ we have a symmetric pdf and a symmetric cumulative density function (cdf). As above, we shall sometimes write $\eta = p - q$ where p and q are two non-negative real numbers such that $p + q = 1$. In the cases where $p = 1$ and $\nu < 1$ the distribution has support on $[0, \infty)$ and similarly when $q = 1$ and $\nu < 1$ the distribution has support on $(-\infty, 0]$.
- The shift or location parameter is $m \in \mathbb{R}$. When the first moment exists it is equal to the location parameter: $\mathbb{E}[X] = m$.

Remark 3 ***Characterising the Moments.***

Let X be a Lévy-Stable random variable with characteristic exponent $0 < \nu < 2$. Then for the case $0 < \nu \leq 1$ the random variable X does not have any integer moments; and for the case $1 < \nu < 2$ only the first integer moment exists.

From an intuitive point of view the best way to develop a feel for Lévy-Stable distributions is to imagine the Gaussian distribution but with a series of continuous deformations

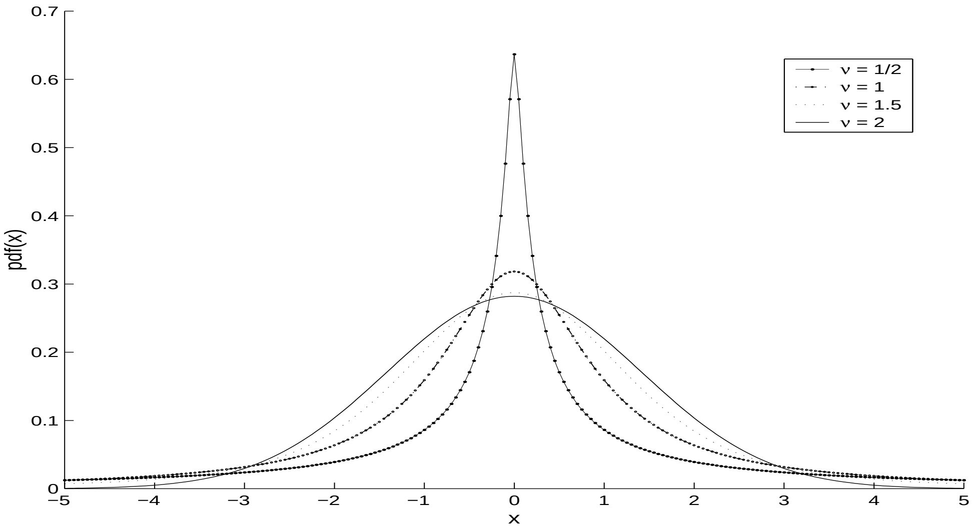

Figure 1: **The effect of the characteristic exponent ν.** The figure shows probability density functions for $\eta = 0$, $\kappa = 1$, $m = 0$ and different characteristic exponents. The dotted line corresponds to $\nu = 0.5$; the dash-dotted line corresponds to $\nu = 1$; the dashed line corresponds to $\nu = 1.5$ and the solid line corresponds to $\nu = 2$, ie the Gaussian distribution.

such as

- **Fatter tails**. The Gaussian distribution has tails that decay very quickly whereas Lévy-Stable distributions exhibit much slower decay.
- **Skewness**. Again one of the properties of the Gaussian distribution is its symmetry. However, Lévy-Stable distributions can exhibit skewness, that is, the distribution is not symmetric.
- **Peaks**. As the parameter ν moves away from the Gaussian case the peak becomes 'thiner' and 'taller'.

This intuitive picture of Lévy-Stable distributions is illustrated in Figure 1 below. Four symmetric probability density functions are depicted when the characteristic parameter ν takes the values 1/2 (the Lévy-Smirnov distribution), 1 (the Cauchy distribution), 3/2 and 2 (the Gaussian distribution).

The use of Lévy-Stable distributions has faced two major practical obstacles that have slowed their applicability. In first place, second and higher moments are infinite for the whole family with the exception of the Gaussian distribution. Second, the probability density functions of Lévy-Stable variables are only known in closed form in three cases: Gaussian $S_2(\kappa, 0, m) = N(m, 2\kappa^2)$, Cauchy $S_1(\kappa, 0, m)$ and Lévy-Smirnov $S_{\frac{1}{2}}(\kappa, 1, m)$.

3.1 Definitions and Properties

Property 1 ***Tails of the Lévy-Stable distributions: asymptotic behaviour.***

Let $X \sim S_\nu(\kappa, \eta, m)$ *with* $0 < \nu < 2$. *Then*

$$\text{as } x \to \infty \quad \mathbb{P}(X > x) \sim \begin{cases} x^{-\nu} \frac{1+\eta}{2} \kappa^\nu \frac{1-\nu}{\Gamma(2-\nu)\cos \nu\pi/2} & \text{for } \nu \neq 1, \\ x^{-\nu} \frac{1+\eta}{2} \kappa^\nu \frac{2}{\pi} & \text{for } \nu = 1, \end{cases} \tag{7}$$

and

$$\text{as } x \to -\infty \quad \mathbb{P}(X < x) \sim \begin{cases} |x|^{-\nu} \frac{1-\eta}{2} \kappa^\nu \frac{1-\nu}{\Gamma(2-\nu)\cos \nu\pi/2} & \text{for } \nu \neq 1, \\ |x|^{-\nu} \frac{1-\eta}{2} \kappa^\nu \frac{2}{\pi} & \text{for } \nu = 1 \end{cases} \tag{8}$$

where the notation $a \sim b$ *is used to denote* $\lim_{x\to\infty} a/b = 1$.

Given the tails of the Lévy-Stable distribution it can be shown that unless the distribution is totally skewed to the left (ie $\eta = -1$) exponential moments do not exist. Moreover, the following proposition shows the Laplace transfrom of a totally skewed Lévy-Stable random variable.

Proposition 2 ***The Laplace Transform [25].***

The Laplace Transform $\mathbb{E}[e^{-\tau X}]$ *with* $\tau \geq 0$ *of the Lévy-Stable variable* $X \sim S_\nu(\kappa, 1, 0)$ *with* $0 < \nu \leq 2$ *and scale parameter* $\kappa > 0$ *satisfies*

$$\ln \mathbb{E}[e^{-\tau X}] = \begin{cases} -\frac{\kappa^\nu}{\cos \frac{\pi\nu}{2}} \tau^\nu & \text{for } \nu \neq 1, \\ \frac{2\kappa}{\pi} \tau \ln \tau & \text{for } \nu = 1. \end{cases} \tag{9}$$

Remark 4 *If the random variable* $X \sim S_\nu(\kappa, -\eta, 0)$ *then* $-X \sim S_\nu(\kappa, \eta, 0)$. *We note this trivial statement since at times we refer to the Laplace Transform (or the moment generating function) of a random variable with either* $\eta = -1$ *or* $\eta = 1$ *with the appropriate sign of* τ.

The fact that only exponential moments of Lévy-Stable random variables exist for the totally skewed case and the fact that for $0 < \nu < 2$ the variance does not exist makes it very difficult to derive pricing formulas for derivatives written on underlyings that follow a Lévy-Stable process. One possible way to get around this problem is to look for stochastic models that preserve some of the interesting properties of the Lévy-Stable distributions such as fat tails and skewness but at the same time exhibit finite moments for a more general class than the totally skewed Lévy-Stable. This motivates the use of Damped-Lévy distributions which can be thought of as Lévy-Stable distributions with an exponential cut-off of the tails so that all moments exist.

4 Damped-Lévy Processes

As mentioned above, the infinite moments of Lévy-Stable random variables are due to the fact that the 'fat' tails do not allow finiteness of moments $\mathbb{E}[X^p]$ when $p > \nu$. damping of the tails is one obvious choice to ensure finite moments. Mantegna and Stanley [19] were the first to propose a damping or cut-off of the tails at some arbitrary point. A different damping was proposed by Koponen [14] who introduced a smooth exponential cut-off of the tails. Koponen's family of Damped distributions lead to a mathematical expression for the characteristic function suitable for our purposes of option pricing.

In this section we will show how to derive the characteristic function of a Damped-Lévy distribution when an exponential cut-off of the tails is introduced in the Lévy-Stable distribution. One immediate consequence is that Damped-Lévy processes are defined in a similar way as above for the Lévy-Stable process.

4.1 Damped-Lévy distributions and processes

Proposition 3 ***The Damped-Lévy characteristic function.***

Let

$$\Psi(\theta) = \int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta\tau_\nu(x))W(x)dx, \tag{10}$$

be the natural logarithm of the Lévy-Khintchine representation of a Lévy-Stable random variable where the Lévy measure W is given above. Introduce an exponential cut-off $e^{-\lambda|x|}$ to obtain the Damped Lévy measure W_{tl}

$$W_{tl}(x) = \begin{cases} Cq\,|x|^{-1-\nu}\, e^{-\lambda|x|} & \text{for } x < 0, \\ Cpx^{-1-\nu}e^{-\lambda x} & \text{for } x > 0. \end{cases} \tag{11}$$

Then the natural logarithm of the characteristic function, for the Damped-Lévy distribution with shift (location) parameter $m = 0$ is

$$\begin{aligned} \Psi_{TL}(\theta) &= \kappa^{\nu}\left\{p(\lambda - i\theta)^{\nu} + q(\lambda + i\theta)^{\nu} - \lambda^{\nu}\right\}, \\ \Psi_{TL}(\theta) &= \kappa^{\nu}\left\{p(\lambda - i\theta)^{\nu} + q(\lambda + i\theta)^{\nu} - \lambda^{\nu} - i\nu\lambda^{\nu-1}(q-p)\theta\right\}, \end{aligned} \tag{12}$$

for $0 < \nu < 1$ and for $1 < \nu \leq 2$ respectively.[3]

Proof

The proof is similar to the derivation of the Lévy-Stable case above. Note that the only difference is the inclusion of the exponential cut-off at the origin in the Lévy measure and the case $\nu = 1$ is excluded.

It is straightforward to see that if the damping parameter $\lambda \to 0$, the characteristic function of the Damped-Lévy distribution becomes Lévy-Stable. We emphasise that, only as $\lambda \to 0$ does the damped distribution asymptotically approach the Lévy-Stable distributions studied above in Proposition 1. Moreover, one can see that by choosing the Lévy triplet $(0, 0, W_{tl})$ with W_{tl} given by (11) the characteristic function of the Damped-Lévy distribution belongs to the family of infinitely divisible distributions as characterised by the Lévy-Khintchine representation.

We write that X belongs to a Damped-Lévy distribution with parameters ν, κ, η, m and λ: $X \sim DL_{\nu}(\kappa, \eta, m, \lambda)$. The parameters have the same interpretation as in the Lévy-Stable case and as explained above the cut-off parameter is given by λ.

Remark 5 *We note that the relationship between the scaling parameter in the Lévy-Stable notation, say $\acute{\kappa}$, and the scaling parameter for the Damped-Lévy, say κ, are related to each other in the following way*

$$\kappa^{\nu} = \frac{\acute{\kappa}^{\nu}}{-\cos \nu\pi/2} \qquad \text{as} \qquad \lambda \to 0.$$

[3]Note that in the original derivation in Koponen [14] the scaling constant is incorrect.

Definition 6 ***Damped-Lévy process.***

A non-degenerate one-dimensional Lévy process $X(t)$ is a Damped-Lévy process if the natural logarithm of its characteristic function is given by

$$\begin{aligned}
\Psi_{TL}(\theta) &= t\kappa^{\nu}\left\{p(\lambda - i\theta)^{\nu} + q(\lambda + i\theta)^{\nu} - \lambda^{\nu}\right\}, \\
\Psi_{TL}(\theta) &= t\kappa^{\nu}\left\{p(\lambda - i\theta)^{\nu} + q(\lambda + i\theta)^{\nu} - \lambda^{\nu} - i\nu\lambda^{\nu-1}(q-p)\theta\right\},
\end{aligned} \tag{13}$$

for $0 < \nu < 1$ and for $1 < \nu \leq 2$ respectively.

Proposition 4 ***Characterising the moments of Damped-Lévy random variables.***

Let X be a Damped-Lévy random variable. Then all moments $\mathbb{E}[X^n]$ of X are finite with $\mathbb{E}[X] = 0$ and for $n > 1$ moments are given by

$$\begin{aligned}
\mathbb{E}[X^n] &= (q-p)\kappa^{\nu}\nu(\nu-1)\cdots(\nu-n+1)\lambda^{\nu-n} \quad \text{for } n \text{ odd,} \\
\mathbb{E}[X^n] &= \kappa^{\nu}\nu(\nu-1)\cdots(\nu-n+1)\lambda^{\nu-n} \quad \text{for } n \text{ even.}
\end{aligned} \tag{14}$$

Proof

Direct evaluation of $\mathbb{E}[X^n] = i^{-n}\frac{d^n\Psi(\theta)}{d\theta^n}$ at $\theta = 0$ yields the result.

Proposition 5 ***Tails of the Damped-Lévy distribution [25] and [21].***

The tails of the Damped-Lévy distribution are given by

$$\text{as } x \to \infty \quad \mathbb{P}(X > x) \sim \begin{cases} x^{-\nu}e^{-\lambda x}\frac{1+\eta}{2}\kappa^{\nu}\frac{1-\nu}{\Gamma(2-\nu)\cos\nu\pi/2} & \text{for } \nu \neq 1, \\ x^{-\nu}e^{-\lambda x}\frac{1+\eta}{2}\kappa^{\nu}\frac{2}{\pi} & \text{for } \nu = 1, \end{cases} \tag{15}$$

and

$$\text{as } x \to -\infty \quad \mathbb{P}(X < x) \sim \begin{cases} |x|^{-\nu}e^{-\lambda|x|}\frac{1-\eta}{2}\kappa^{\nu}\frac{1-\nu}{\Gamma(2-\nu)\cos\nu\pi/2} & \text{for } \nu \neq 1, \\ |x|^{-\nu}e^{-\lambda|x|}\frac{1-\eta}{2}\kappa^{\nu}\frac{2}{\pi} & \text{for } \nu = 1. \end{cases} \tag{16}$$

One important implication of the proposition above is the existence of exponential moments for Damped-Lévy random variables.

Proposition 6 ***Existence of Exponential Moments.***

Let X be a Damped-Lévy random variable i.e. $X \sim DL_{\nu}(\kappa, \eta, m, \lambda)$. Then, provided that $|\tau| < \lambda$, the Laplace Transform $\mathbb{E}[e^{\tau X}]$ exists.

This is straightforward given the tails of the distribution and leads us to the following interesting result.

Proposition 7 ***The Laplace Transform for Damped-Lévy Random Variables [21].***

Let $X \sim DL_\nu(\kappa, \eta, m, \lambda)$ be a Damped-Lévy random variable. Then, if $\lambda > |\tau|$, the Laplace Transform satisfies

$$\ln \mathbb{E}[e^{\tau X}] = \kappa^\nu \left\{p(\lambda - \tau)^\nu + q(\lambda + \tau)^\nu - \lambda^\nu\right\} \quad \textit{for } 0 < \nu < 1,$$
$$\ln \mathbb{E}[e^{\tau X}] = \kappa^\nu \left\{p(\lambda - \tau)^\nu + q(\lambda + \tau)^\nu - \lambda^\nu - \nu\lambda^{\nu-1}(q-p)\tau\right\} \quad \textit{for } 1 < \nu < 2.$$

With this last proposition it is possible to take the first step in the derivation of a generalisation of the 'Black-Scholess PDE' for the Lévy-Stable case. We will do so in two steps. Firstly, we will derive an expression for the Black-Scholes perpetual option when the random shocks to the stock price process are Damped-Lévy. Secondly, with the information given by the perpetual option solution it will be straightforward to derive a generalisation of the Black-Scholes PDE which can then be solved to value financial claims. The derivation of the results rely on a particular choice of scaling and limit, to be discussed below.

5 Perpetual Options under Damped-Lévy and Lévy-Stable Shocks

In this section we derive the ODE for perpetual options and its solution for perpetual calls with $\eta = -1$ and perpetual puts with $\eta = 1$. The key is to set the problem for a small time interval Δt and then, under both a suitable scaling and in an appropriate limit, to approximate the Lévy-Stable case as a limit of the Damped-Lévy regime.

5.1 The price process

We will assume that the natural logarithm of the stock price process, under the risk-neutral measure, is a Damped-Lévy process, so

$$S_{t+\Delta t} = S_t e^{(r-D_0)\Delta t - \Psi_{TL}(-i\sigma) + \sigma\phi} \tag{17}$$

where $r > 0$ is the risk-free rate, $\sigma > 0$, $1 < \nu \leq 2$ and the variable $\phi \sim DL_\nu(\kappa, \eta, 0, \lambda)$ is a Damped-Lévy random variable. The stock pays a known dividend D_0 per unit of time. The general interpretation of the parameters is as usual but we recall that σ is not the standard deviation of $\log S$ unless $\nu = 2$. Note that the exponential moment $\mathbb{E}[e^{\sigma\phi}] = e^{\Psi_{TL}(-i\sigma)\Delta t}$ and for the particular case $\nu = 2$ we get $\mathbb{E}[e^{\sigma\phi}] = e^{\kappa^2\sigma^2\Delta t}$.

We recall that, by construction, the Damped-Lévy-Stable process is discontinuous, in other words it is a pure jump process. Hence, the path followed by the stock process above will be discontinuous given the nature of the random variable ϕ unless $\nu = 2$. We also point out that we restrict the choice of ν to the interval $\nu \in (1, 2]$ since empirical studies suggest this interval as a plausible range in financial applications [9].

In the price process above the scaling parameter of the random variable ϕ is denoted by κ. We now turn to the question of how Δt should enter the form of κ in the stock price process. We already know the answer to this question when $\nu = 2$, ie Brownian Motion drives the stochastic shocks, κ scales with $\Delta t^{1/2}$ but we note that $\mathbb{E}[\Delta \ln S] = (r - D_0 - \frac{1}{2}\sigma^2)\Delta t + O(\Delta t)$ so that the square of the volatility term contributes at the same order as the drift.[4] For $1 < \nu < 2$ one possible choice for the scaling is to require the same property. This leads to the following proposition, after a remark, which shows that there are only two feasible and financially plausible ways in which Δt scales the distribution of ϕ for a given ν.

Remark 6 *Note that with the parametrisation we have chosen to specify the characteristic function of the Lévy-Stable process we have that $\mathbb{E}[e^{\sigma\phi}] = e^{\sigma^2}$ when $\kappa = 1$ instead of the usual $\mathbb{E}[e^{\sigma\phi}] = e^{\frac{1}{2}\sigma^2}$. Hence, for simplicity, whenever we refer to the case $\nu = 2$ we let the standarised $\phi \sim DL_2(1/2, \eta, 0, \lambda)$ instead of $\phi \sim DL_2(1, \eta, 0, \lambda)$.*

Proposition 8 ***Time-scaling of parameters.***

Let the stock price process be as above, $S_{t+\Delta t} = S_t e^{(r-D_0)\Delta t - \Psi(-i\sigma) + \sigma\phi}$ with $\phi \sim DL_\nu(\kappa, \eta, 0, \lambda)$. Then there are two ways of scaling the distribution of ϕ.

1. *Fix the damping parameter λ and scale the diffusion coefficient σ with time so that $\sigma \sim \Delta t^{1/2}\tilde{\sigma}$ as $\Delta t \to 0$, or*

[4]For simplicity we will assume that when $\nu = 2$ we set $\kappa^2 = 1/2$ instead of $\kappa = 1$ for the standarised Damped-Lévy and Lévy-Stable random variables. See Remark 6.

2. scale both the damping parameter λ and the diffusion coefficient σ with time so that $\lambda \sim \Delta t^{1/\nu}\tilde{\lambda}$ and $\sigma \sim \Delta t^{1/\nu}\tilde{\sigma}$ where $\tilde{\lambda}/\tilde{\sigma} \equiv a$ is fixed at $O(1)$.

Before proceeding with the proof we note that the first of these alternatives will clearly (by a Central Limit Theorem argument) lead to the standard Gaussian model; its implications for option pricing are discussed briefly at the end of the paper in Section 8. The second is less straightforward and is the one we will be interested in; we discuss it in detail below.

Proof

First consider the expected value $\mathbb{E}[e^{(r-D_0)\Delta t - \Psi(-i\sigma)+\sigma\phi}]$ without scaling σ. This is given in proposition (7) above; for $1 < \nu < 2$,we have

$$\kappa^\nu \left\{ p(\lambda-\sigma)^\nu + q(\lambda+\sigma)^\nu - \lambda^\nu - \nu\lambda^{\nu-1}(q-p)\sigma \right\} = O(\Delta t) \qquad \text{as } \Delta t \to 0,$$

$$\begin{aligned} \ln \mathbb{E}[e^{(r-D_0)\Delta t - \Psi(-i\sigma)+\sigma\phi}] \quad &= \quad (r-D_0)\Delta t - \Psi(-i\sigma) \\ &\quad + \kappa^\nu \left\{ p(\lambda-\sigma)^\nu + q(\lambda+\sigma)^\nu - \lambda^\nu - \nu\lambda^{\nu-1}(q-p)\sigma \right\}. \end{aligned} \tag{18}$$

We require that

$$\kappa^\nu \left\{ p(\lambda-\sigma)^\nu + q(\lambda+\sigma)^\nu - \lambda^\nu - \nu\lambda^{\nu-1}(q-p)\sigma \right\} = O(\Delta t) \qquad \text{as } \Delta t \to 0,$$

to balance the component $(r - D_0)\,\Delta t$.

<u>Case 1, $\sigma = (\Delta t)^{1/2}\tilde{\sigma}$</u>

In equation (18) above one possibility is to scale the diffusion coefficient σ with time $(\Delta t)^\alpha \tilde{\sigma}$ where $\alpha \in \mathbb{R}$.

Now we expand $(\lambda \pm (\Delta t)^\alpha \tilde{\sigma})^\nu$ for small Δt, giving

$$\begin{aligned} \left(1 \pm \frac{(\Delta t)^\alpha \tilde{\sigma}}{\lambda}\right)^\nu \quad &\sim \quad 1 \pm \nu \frac{(\Delta t)^\alpha \tilde{\sigma}}{\lambda} + \frac{1}{2}\nu(\nu-1)\left(\frac{(\Delta t)^\alpha \tilde{\sigma}}{\lambda}\right)^2 \\ &\pm \quad \frac{1}{3!}\nu(\nu-1)(\nu-2)\left(\frac{(\Delta t)^\alpha \tilde{\sigma}}{\lambda}\right)^3 + \cdots . \end{aligned}$$

Now we substitute in (18) above, giving

$$\begin{aligned}\ln \mathbb{E}[e^{(\Delta t)^\alpha \tilde{\sigma} X}] &= (r - D_0 - \Psi(-i\tilde{\sigma}))\Delta t + \kappa^\nu \frac{1}{2}\nu(\nu-1)\left(\frac{(\Delta t)^\alpha \tilde{\sigma}}{\lambda}\right)^2 \\ &+ \kappa^\nu (q-p)p\left(\frac{1}{3!}\nu(\nu-1)(\nu-2)\left(\frac{(\Delta t)^\alpha \tilde{\sigma}}{\lambda}\right)^3 + ...\right).\end{aligned}$$

We need to balance $(r - D_0 - \Psi(-i\tilde{\sigma}))\Delta t$ with at least one term from the remainder, and the only feasible choice is $\alpha = 1/2$.

Case 2, $\lambda = \Delta t^{1/\nu}\tilde{\lambda}$ and $\tilde{\sigma} = \Delta t^{1/\nu}\tilde{\sigma}$

This case is straightforward since

$$\begin{aligned}\ln \mathbb{E}[e^{(r-D_0)\Delta t - \Psi(-i\sigma) + \sigma\phi}] &= (r - D_0)\Delta t - \Psi(-i\sigma) + \kappa^\nu\{p(\Delta t^{1/\nu}\tilde{\lambda} - \Delta t^{1/\nu}\tilde{\sigma})^\nu \\ &\quad + q(\Delta t^{1/\nu}\tilde{\lambda} + \Delta t^{1/\nu}\tilde{\sigma})^\nu - \left(\Delta t^{1/\nu}\tilde{\lambda}\right)^\nu - \nu\left(\Delta t^{1/\nu}\tilde{\lambda}\right)^{\nu-1}(p-q)\Delta t^{1/\nu}\tilde{\sigma}\} \\ &= (r - D_0 - \Psi(-i\tilde{\sigma}))\Delta t \\ &\quad + \kappa^\nu\{p(\tilde{\lambda} - \tilde{\sigma})^\nu + q(\tilde{\lambda} + \tilde{\sigma})^\nu - \tilde{\lambda}^\nu - \nu\tilde{\lambda}^{\nu-1}(p-q)\tilde{\sigma}\}\Delta t \\ &= (r - D_0 - \Psi(-i\tilde{\sigma}))\Delta t \\ &\quad + \kappa^\nu\tilde{\sigma}^\nu\{p(a-1)^\nu + q(a+1)^\nu - a^\nu - \nu a^{\nu-1}(p-q)\}\Delta t,\end{aligned}$$

where $a = \tilde{\lambda}/\tilde{\sigma}$.

5.2 Perpetual Options under Lévy-Stable Shocks

The following proposition derives the value of the perpetual American Call and American Put for the case where the shocks are Lévy-Stable where shocks are totally skewed to the left and totally skewed to the right respetively. The proof is straightforward: first we price the perpetual option for a time interval Δt between asset price innovations, and once we have found the result in terms of Δt we simply let $\Delta t \to 0$. The most important step is that we scale both the damping parameter λ and the volatility parameter by $\Delta t^{1/\nu}$ as above, so as to guarantee the existence of the Laplace Transform of the stock price process. We recall from above that when the damping parameter is sent to zero, departing from the Damped-Lévy case, we converge to the Lévy-Stable case.

Proposition 9 ***Perpetual Option under Totally Skewed Lévy-Stable Shocks.***

Let the stock price process be $S_{t+\Delta t} = S_t e^{(r-D_0)\Delta t - \Psi_{TL}(-i\sigma)+\sigma\phi}$ *where the shocks are* $\phi \sim DL_\nu(\kappa,\eta,0,\lambda)$, $\eta = \pm 1$ *and* $1 < \nu \leq 2$. *Moreover,* r *is the risk-free rate and the stock pays a known dividend* D_0 *per unit of time. Let* $\lambda = \tilde{\lambda}\Delta t^{1/\nu}$, $\sigma = \tilde{\sigma}\Delta t^{1/\nu}$ *with* $\tilde{\lambda}$, $\tilde{\sigma} > 0$. *Then, the value of the Lévy-Stable perpetual American call, with* $\eta = -1$, *and put, with* $\eta = 1$, *struck at* K *are given in the limit* $\Delta t \to 0$ *by*

$$C(S,t) = \begin{cases} \left(\frac{\beta_{tl}-1}{\beta_{tl}K}\right)^{\beta_{tl}} \frac{K}{\beta_{tl}-1} S^{\beta_{tl}} & \text{for} \quad S < S_c^\star, \\ \\ S - K & \text{for} \quad S \geq S_c^\star, \end{cases} \tag{19}$$

$$P(S,t) = \begin{cases} \left(\frac{\beta_{tl}^- -1}{\beta_{tl}^- K}\right)^{\beta_{tl}^-} \frac{K}{1-\beta_{tl}^-} S^{\beta_{tl}^-} & \text{for} \quad S > S_p^\star, \\ \\ K - S & \text{for} \quad S \leq S_c^\star, \end{cases} \tag{20}$$

where β_{tl} *and* β_{tl}^- *are the the positive and negative roots, when they exist, of the characteristic equation*

$$\kappa^\nu \left\{ p(\tilde{\lambda} - \beta\tilde{\sigma})^\nu + q(\tilde{\lambda} + \beta\tilde{\sigma})^\nu - \tilde{\lambda}^\nu - \nu\tilde{\lambda}^{\nu-1}(q-p)\beta\tilde{\sigma} \right\} + \beta\overline{\mu} - r = 0,$$

where $\overline{\mu} = r - D_0 - \Psi_{TL}(-i\tilde{\sigma})$, *and* $p = 0$ *for the call and* $p = 1$ *for the put and* $S_c^\star$ *and* $S_p^\star$ *are the optimal exercise boundaries given by*

$$S_c^\star = \frac{\beta_{tl}}{\beta_{tl}-1} K, \tag{21}$$

and

$$S_p^\star = \frac{\beta_{tl}^-}{\beta_{tl}^- - 1} K. \tag{22}$$

Proof

The value of an instrument $V(S,t)$ with a payoff $\Pi(S)$ at time T is given by

$$V(S,t) = \mathbb{E}^Q \left[e^{-r(T-t)} \Pi(S) \right].$$

The pricing problem can also be formulated as a Bellman equation.[5] Write $[t,T] = [t, t+\Delta t) \cup [t+\Delta t, T]$; in the continuation region we have the homogeneous Bellman equation

$$\begin{aligned} V(S,t) &= e^{-r\Delta t}\mathbb{E}^Q \left[e^{-r(T-t-\Delta t)} \Pi(S) \right] \\ &= e^{-r\Delta t}\mathbb{E}^Q \left[V(S_{t+\Delta t}, t+\Delta t) \right]. \end{aligned}$$

[5] See [7] pages 121-123 for an example using Brownian Motion.

Now let $U_t = e^{\overline{\mu}\Delta t+\sigma\phi_t}$ and expand the value function $V(S_{t+\Delta t}, t+\Delta t) = V(SU_t)$ about $U_t = 1$:

$$\begin{aligned} V(S_t) &= e^{-r\Delta t}\mathbb{E}^Q[V(S_t) + \frac{\partial V}{\partial S}(U_t-1) + \frac{1}{2!}\frac{\partial^2 V}{\partial S^2}S_t^2(U_t-1)^2 \\ &\quad + \frac{1}{3!}\frac{\partial^3 V}{\partial S^3}S_t^3(U_t-1)^3 + \cdots]. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} V(S_t)e^{r\Delta t} &= V(S_t) + \frac{\partial V}{\partial S}S\mathbb{E}^Q\left[(U_t-1)\right] + \frac{1}{2!}\frac{\partial^2 V}{\partial S^2}S_t^2\mathbb{E}^Q\left[(U_t-1)^2\right] \\ &\quad + \frac{1}{3!}\frac{\partial^3 V}{\partial S^3}S_t^3\mathbb{E}^Q\left[(U_t-1)^3\right] + \cdots. \end{aligned} \tag{23}$$

First note that since this in an 'infinite horizon' problem the value function $V(S_t)$ is independent of time; therefore $\partial V/\partial t$ does not appear in the expansion above and we can also drop the subscript t. Thus (23) reduces to an (infinite order) ODE which we solve using $V(S) = S^\beta$ as a trial solution. Hence

$$\begin{aligned} e^{r\Delta t} &= 1 + \beta\mathbb{E}^Q\left[(U-1)\right] + \beta(\beta-1)\mathbb{E}^Q\left[\frac{1}{2!}(U-1)^2\right] \\ &\quad + \beta(\beta-1)(\beta-2)\mathbb{E}^Q\left[\frac{1}{3!}(U-1)^3\right] + \cdots \\ &= \mathbb{E}^Q\left[1 + \beta(U-1) + \beta(\beta-1)\frac{1}{2!}(U-1)^2 + \beta(\beta-1)(\beta-2)\frac{1}{3!}(U-1)^3 + \cdots\right] \\ &= \mathbb{E}^Q\left[U^\beta\right], \end{aligned}$$

by summation of the binomial series.

Now, using the Laplace Transform presented above and requiring that $|\beta\tilde{\sigma}| < \tilde{\lambda}$,

$$\mathbb{E}^Q\left[U^\beta\right] = e^{\kappa^\nu\left\{p(\tilde{\lambda}-\beta\tilde{\sigma})^\nu + q(\tilde{\lambda}+\beta\tilde{\sigma})^\nu - \tilde{\lambda}^\nu - \nu\tilde{\lambda}^{\nu-1}(q-p)\beta\tilde{\sigma}\right\}\Delta t + \overline{\mu}\Delta t}. \tag{24}$$

Equating the exponents gives the characteristic equation

$$\kappa^\nu\left\{p(\tilde{\lambda}-\beta\tilde{\sigma})^\nu + q(\tilde{\lambda}+\beta\tilde{\sigma})^\nu - \tilde{\lambda}^\nu - \nu\tilde{\lambda}^{\nu-1}(q-p)\beta\tilde{\sigma}\right\} + \beta\overline{\mu} - r = 0. \tag{25}$$

Note that this procedure will only make sense as $\Delta t \to 0$ if the scaling for both the damping parameter and the diffusion coefficient is chosen as discussed above.

Now we are interested in the roots of the characteristic equation above. In fact, only positive roots greater than unity make financial sense (for the perpetual call), and it can be shown that under certain conditions it is straightforward to show that only one such root β_{tl} exists (see the Appendix). Therefore the general solution is given by

$$V(S,t) = AS^{\beta_{tl}},$$

where A is a constant to be determined from the usual value-matching and value-maximising (equivalent to smooth-pasting) condition; recalling that $\Pi(S) = \max(S-K,0)$ for a perpetual call option, this gives

$$C(S,t) = \left(\frac{\beta_{tl}-1}{\beta_{tl}K}\right)^{\beta_{tl}} \frac{K}{\beta_{tl}-1} S^{\beta_{tl}}.$$

The perpetual put is calculated in a similar way.

Remark 7 *Note that as $\Delta t \to 0$ the damping parameter $\lambda = \tilde{\lambda}\Delta t^{1/\nu}$ also goes to zero. Therefore the underlying shocks to the price process belong to a Lévy-Stable distribution in the limit, i.e. $\phi \sim DL_\nu(\kappa,\eta,0,\lambda) \to \phi \sim S_\nu(\kappa,\eta,0)$ as $\Delta t \to 0$. Moreover, note that the restriction on the parameters required by the Laplace Transform above, for the existence of the exponential moments, is that*

$$|\Delta t^{1/\nu}\tilde{\sigma}\beta_{tl}| < \Delta t^{1/\nu}\tilde{\lambda} \qquad \textit{for all } \Delta t,$$

or equivalently,

$$|\tilde{\sigma}\beta_{tl}| < \tilde{\lambda} \qquad\qquad \textit{for all } \Delta t. \tag{26}$$

Therefore this condition will hold even when $\Delta t \to 0$.

Remark 8 *We point out that in the cases where the shocks to the stock price process are totally skewed to the left, $\eta = -1$, we can set the damping parameter $\lambda = 0$ and still find closed form solutions for the perpetual Call. The pricing formula can be derived as above and we get*

$$C(S,t) = \begin{cases} \left(\frac{\beta_*-1}{\beta_* K}\right)^{\beta_*} \frac{K}{\beta_*-1} S^{\beta_*} & \textit{for} \quad S < S_c^\star, \\ \\ S-K & \textit{for} \quad S \geq S_c^\star, \end{cases}$$

where β_* is the root of the characteristic equation

$$-\frac{\kappa^\nu \tilde{\sigma}^\nu \beta^\nu}{\cos(\pi\nu/2)} + \left(r - D_0 + \frac{(\kappa\tilde{\sigma})^\nu}{\cos(\pi\nu/2)}\right)\beta - r = 0.$$

6 The ‘Black-Scholes Formula’ under Lévy-Stable Shocks

We now extend the technique for the perpetual case described above to derive the ‘Black-Scholes PDE’ with a finite time-horizon.

Proposition 10 ***The ‘Black-Scholes PDE’ for the Lévy-Stable case.***

Let $1 < \nu \leq 2$. Let the stock price process, under the risk neutral measure, follow

$$S_{t+\Delta t} = S_t e^{\overline{\mu}\Delta t + \sigma\phi}$$

where $\phi \sim DL_\nu(\kappa, \eta, 0, \lambda)$ and $\overline{\mu} = r - D_0 - \Psi_{TL}(-i\tilde{\sigma})$. Let the damping parameter be $\lambda = \Delta t^{1/\nu}\tilde{\lambda}$ and $\sigma = \Delta t^{1/\nu}\tilde{\sigma}$ with $\tilde{\lambda} > 0$ and $\tilde{\sigma} > 0$. Then as $\Delta t \to 0$ the ‘Black-Scholes PDE’, given in Fourier space[6] for the Lévy-Stable case is

$$-\frac{\partial \hat{V}}{\partial t} = [\Psi_{TL}(-\tilde{\sigma}\xi) + i\xi\left(\Psi_{TL}(-i\tilde{\sigma}) + D_0\right) - r(1 + i\xi)]\hat{V}, \tag{27}$$

where Ψ_{TL} is the (log) characteristic function (13) and $\xi \in \mathbb{C}$ is the Complex Fourier Transform variable.

Proof

We start as in the perpetual option case above. The value of an instrument $V(S,t)$ with a payoff $\Pi(S,T)$ at time $t = T$ is given by

$$V(S,t) = \mathbb{E}^Q\left[e^{-r(T-t)}\Pi(S,T)\right].$$

The pricing problem can be formulated as a trivial Bellman equation. Writing $[t,T] = [t, t+\Delta t) \cup [t+\Delta t, T]$, we have the homogeneous Bellman Equation

$$\begin{aligned} V(S,t) &= e^{-r\Delta t}\mathbb{E}^Q\left[e^{-r(T-t-\Delta t)}\Pi(S,T)\right] \\ &= e^{-r\Delta t}\mathbb{E}^Q\left[V(S_{t+\Delta t}, t+\Delta t)\right]. \end{aligned}$$

This last equation can also be rewritten in the following form (note that we will only be interested in keeping terms of order $O(\Delta t)$):

$$rV(S,t)\Delta t = \mathbb{E}^Q\left[\Delta V(S,t)\right]$$

[6] We denote the Fourier Transform of a function f by $\hat{f}$.

where $\Delta V(S,t) \equiv V(S+\Delta S, t+\Delta t) - V(S,t)$. Its financial interpretation is straightforward. On the left-hand side we have that the value of the financial instrument grows at a rate r and using the risk neutral measure this should equal the expected gains in capital appreciation over the interval Δt.

Now we will focus on the term $\mathbb{E}^Q[\Delta V(S,t)]$ of the equation above. We will expand $\Delta V(S,t)$ and keep only those terms of order $O(\Delta t)$ after taking the expectation. It is this last step that is a crucial one since we shall see that, when the damping parameter and the volatility are both scaled with $\Delta t^{1/\nu}$, ie. $\lambda = \Delta t^{1/\nu}\tilde{\lambda}$ and $\sigma = \Delta t^{1/\nu}\tilde{\sigma}$, *all* the terms of the expansion of $\Delta V(S,t)$ with respect to the state variable S become of order $O(\Delta t)$; therefore we must keep them in the expansion. We have

$$\mathbb{E}^Q[\Delta V(S,t)] = \mathbb{E}^Q\left[\frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial S}\Delta S + \frac{1}{2!}\frac{\partial^2 V}{\partial S^2}(\Delta S)^2 + \frac{1}{3!}\frac{\partial^3 V}{\partial S^3}(\Delta S)^3 \cdots\right].$$

Now we proceed to calculate $\mathbb{E}^Q[\Delta S^n]$ for $n = 1,2,3,4,\ldots.$ First we write $\Delta S \equiv S_{t+\Delta t} - S_t$, hence

$$\Delta S_t = S_t\left(e^{\overline{\mu}\Delta t + \tilde{\sigma}\phi\Delta t^{1/\nu}} - 1\right).$$

Now we expand $e^{\overline{\mu}\Delta t + \Delta t^{1/\nu}\tilde{\sigma}\phi} - 1$, take expected values given in Proposition 4, and neglect terms of higher order than Δt to get

$$\begin{aligned}
\mathbb{E}^Q[\Delta S] &= S\mathbb{E}^Q\left[\overline{\mu}\Delta t + \tilde{\sigma}\phi\Delta t^{1/\nu} + \frac{1}{2}\left(\overline{\mu}\Delta t + \tilde{\sigma}\phi\Delta t^{1/\nu}\right)^2 + \cdots\right] \\
&= S\left(\overline{\mu} + \frac{1}{2}\tilde{\sigma}^2\kappa^\nu\nu(\nu-1)\lambda^{\nu-2}\right)\Delta t + o(\Delta t)
\end{aligned}$$

and in a similar way we can show that

$$\mathbb{E}^Q[(\Delta S)^n] = S^n K_n' \Delta t + \mathrm{o}(\Delta t),$$

where K_n''s are constants.

Now dividing through by Δt and taking $\Delta t \to 0$, we obtain an infinite-order PDE of the form

$$rV(S,t) = \frac{\partial V}{\partial t} + K_1' S\frac{\partial V}{\partial S} + K_2' S^2\frac{\partial^2 V}{\partial S^2} + K_3' S^3\frac{\partial^3 V}{\partial S^3} + K_4' S^4\frac{\partial^4 V}{\partial S^4} + \cdots.$$

i.e.

$$rV(S,t) = \frac{\partial V}{\partial t} + \sum_{j=1}^{\infty} K_j' S^j \frac{\partial^j V}{\partial S^j}, \tag{28}$$

where K'_j's are constants that we shall determine below.

It will be more convenient to perform the substitution $z = \ln S$ in equation (28), so that $\partial/\partial z = S\partial/\partial S$, which leads to

$$rV(z,t) = \frac{\partial V}{\partial t} + \sum_{j=1}^{\infty} K_j \frac{\partial^j V}{\partial z^j}, \tag{29}$$

where K_j's are constants to be determined and are related to K'_j.

The next step is to determine the constants K_j's above using the information we have from the perpetual option. This satisfies the ODE

$$rV(z,t) = \sum_{j=1}^{\infty} K_j \frac{\partial^j V}{\partial z^j}. \tag{30}$$

Substituting $V = e^{\beta z}$ we get the following characteristic equation:

$$r = \sum_{j=1}^{\infty} K_j \beta^j. \tag{31}$$

However, we know that this characteristic equation must be the same as the one above given by equation (25). We now convert (25) into a power series by expanding $(\lambda \pm \beta\tilde{\sigma})^\nu$, to yield

$$r = \overline{\mu}\beta + \kappa^\nu \left\{ \frac{1}{2!}\nu(\nu-1)\tilde{\lambda}^{\nu-2}\tilde{\sigma}^2\beta^2 + \frac{1}{3!}\nu(\nu-1)(\nu-2)\tilde{\lambda}^{\nu-3}\tilde{\sigma}^3\beta^3\eta + \cdots \right\}.$$

Now matching coefficients with (31) we have that

$$\begin{aligned} K_1 &= \overline{\mu}, \\ K_2 &= \frac{1}{2!}\kappa^\nu\nu(\nu-1)\tilde{\lambda}^{\nu-2}\tilde{\sigma}^2, \\ K_3 &= \frac{1}{3!}\kappa^\nu\nu(\nu-1)(\nu-2)\tilde{\lambda}^{\nu-3}\tilde{\sigma}^3\beta^3\eta, \end{aligned}$$

and recursively we get

$$K_n = \begin{cases} \frac{1}{n!}\kappa^\nu\nu(\nu-1)(\nu-2)...(\nu-n+1)\tilde{\lambda}^{\nu-n}\tilde{\sigma}^n\beta^n & \text{for } n \quad \text{even}, \\ \frac{1}{n!}\kappa^\nu\nu(\nu-1)(\nu-2)...(\nu-n+1)\tilde{\lambda}^{\nu-n}\tilde{\sigma}^n\beta^n\eta & \text{for } n \quad \text{odd}. \end{cases}$$

We note that although there is no power solution for the perpetual option case when the distribution is not totally skewed, we may still use the information provided by a trial solution of the type $V = e^{\beta z}$ to obtain the constants K_j's for any η.

Before putting these results together we will apply the Complex Fourier Transform to the ODE (30).[7] The Complex Fourier Transform of a function $g(x)$ is given by:

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} g(x) dx$$

and the inverse is given by

$$g(x) = \frac{1}{2\pi} \int_{i\xi_i-\infty}^{i\xi_i+\infty} e^{i\xi x} \hat{g}(\xi) d\xi,$$

where $\xi_i = \mathrm{Im}\ \xi$. Then, for a suitable range of ξ_i, the Complex Fourier Transform of (29) is

$$-\frac{\partial \hat{V}}{\partial t} = [\kappa^\nu \{p(\tilde{\lambda} + i\xi\tilde{\sigma})^\nu + q(\tilde{\lambda} - i\xi\tilde{\sigma})^\nu - \tilde{\lambda}^\nu + \nu\tilde{\lambda}^{\nu-1}\eta i\xi\tilde{\sigma}\} - i\overline{\mu}\theta - r]\hat{V}.$$

Finally, we write it in short form using the expression for the characteristic equation (13):

$$-\frac{\partial \hat{V}}{\partial t} = [\Psi_{TL}(-\tilde{\sigma}\xi) + i\xi\left(\Psi_{TL}(-i\tilde{\sigma}) + D_0\right) - r(1 + i\xi)]\hat{V}.$$

as required.

Note that we can recover the standard Black-Scholes equation, in Fourier Space, from our equation above by letting $\nu = 2$, $\tilde{\lambda} = 0$ and $\eta = 0$. We get

$$-\frac{\partial \hat{V}}{\partial t} = \left[-\frac{1}{2}\tilde{\sigma}^2\xi^2 + i\xi\left(\frac{1}{2}\tilde{\sigma}^2 + D_0\right) - r(1 + i\xi)\right]\hat{V};$$

inversion gives the standard Black-Scholes equation.

Remark 9 *We point out the importance of having obtained a genereralised Black-Scholes PDE which will allow us not only to price European instruments but other types of options such as American options for the Lévy-Stable case.*

6.1 Solving the Lévy-Stable Black-Scholes equation

Above we derived the analogue of the Black-Scholes PDE when the stock process followed an exponential Lévy-Stable process. In this subsection we will show how to obtain prices for instruments such as Put and Call options with finite expiry.

[7] We use the Complex Fourier Transform so that the transform of the payoff functions for vanilla options exist.

The first step is to also represent the payoff function for both the Put and Call in Fourier space applying the Complex Fourier Transform; note the restriction on ξ_i imposed by the exponential terms in the pay-off.

$$\begin{aligned}\hat{C}(\xi, T) &= \int_{-\infty}^{\infty} e^{i\xi z} \max(e^z - K, 0) dz \\ &= -\frac{K^{i\xi+1}}{\xi^2 - i\xi} \qquad \text{for } \xi_i > 1,\end{aligned} \tag{32}$$

$$\begin{aligned}\hat{P}(\xi, T) &= \int_{-\infty}^{\infty} e^{i\xi z} \max(K - e^z, 0) dz \\ &= -\frac{K^{i\xi+1}}{\xi^2 - i\xi} \qquad \text{for } \xi_i < 0.\end{aligned} \tag{33}$$

Note that both transformed payoffs have the same functional form but are defined in different strips in the complex plane ($K^{i\xi+1}$ has a branch point at $\xi = i$). Note also that when the Complex Fourier Transform is applied to the operator given by equation (29) the restriction on the Fourier variable ξ_i is the same restriction required by the transform of the payoffs above.

By inversion of the Complex Fourier Transform, the general solution of the Lévy-Stable Black-Scholes equation is therefore given by

$$V(z,t) = \frac{1}{2\pi} \int_{i\xi_i-\infty}^{i\xi_i+\infty} e^{-i\xi z} \hat{\Pi}(\xi, T) e^{(T-t)[\Psi_{TL}(-\tilde{\sigma}\xi)+i\xi(\Psi_{TL}(-i\tilde{\sigma})+D_0)-r(1+i\xi)]} d\xi. \tag{34}$$

In order to carry out the inversion we require that $e^{(T-t)[\Psi_{TL}(-\tilde{\sigma}\xi)-i\overline{\mu}\xi-r]}$ is analytic in certain strip $a < \xi_i < b$ with $a, b \in \mathbb{R}$. Furthermore, this strip must intersect with the strip where the transform of the payoff function, $\hat{\Pi}(\xi, T)$, exists. Then the inversion contour is taken along this strip.

By inspection of the function $e^{(T-t)[\Psi_{TL}(-\tilde{\sigma}\xi)-i\overline{\mu}\xi]}$ we see that there are two branch points located at

$$\xi_i = \frac{\tilde{\lambda}}{\tilde{\sigma}} > 1 \qquad \text{and} \qquad \xi_i = -\frac{\tilde{\lambda}}{\tilde{\sigma}} < -1.$$

This function is therefore analytic in the strip $\mid \xi_i \mid < \tilde{\lambda}/\tilde{\sigma}$ with $\tilde{\lambda}/\tilde{\sigma} > 1$, and so for a call option we can take the inversion contour to be along any line $0 < \xi_i < 1$, $\xi_i =$ constant, yielding the formula

$$C(z,t) = e^z e^{-D_0(T-t)} - \frac{1}{2\pi} e^{-r(T-t)} K \int_{i\xi_i-\infty}^{i\xi_i+\infty} e^{-i\xi z} \frac{K^{i\xi}}{\xi^2 - i\xi} e^{(T-t)[\Psi_{TL}(-\sigma\xi) - i\overline{\mu}\xi]} d\xi \tag{35}$$

with $0 < \xi_i < 1$. The dashed line in Figure 2 shows a contour in this strip and this is the contour we will use when numerical calculations of Call options are carried out in the following section.

For the put option we can take a similar contour but now with $-1 < \xi_i < 0$; however, noting that by Put-Call parity we have

$$\begin{aligned} (P(\xi,T) - K)\hat{} &= -\int_{-\infty}^{\ln K} e^z e^{i\xi z} dz - K \int_{\ln K}^{\infty} e^{i\xi z} dz \\ &= -\frac{K^{i\xi+1}}{\xi^2 - i\xi} \qquad \text{for } 0 < \xi_i < 1, \end{aligned} \tag{36}$$

we can move the contour up, picking up a contribution from the pole at $\xi - 0$, to obtain

$$P(z,t) = K e^{-r(T-t)} - \frac{1}{2\pi} e^{-r(T-t)} K \int_{i\xi_i-\infty}^{i\xi_i+\infty} e^{-i\xi z} \frac{K^{i\xi}}{\xi^2 - i\xi} e^{(T-t)[\Psi_{TL}(-\sigma\xi) - i\overline{\mu}\xi]} d\xi \tag{37}$$

with $0 < \xi_i < 1$.

7 Numerical results: Comparison with Black-Scholes Pricing and Smiles

In this section we calculate European option prices under different scenarios. The calculations are carried out in Matlab by inverting equation (35). We will focus on two important pieces of information when the thickness of the tails and the skewness of the distribution change, i.e. when ν and η vary. First, we compare the prices given by the Lévy-Stable case with those given by the classical Black-Scholes theory. Second, we calculate the implied volatility that Lévy-Stable prices induce through the classical Black-Scholes framework.

Although Lévy-Stable distributions do not have second moments, with the exception of the Gaussian case, the limiting approximation we found above does have moments of all

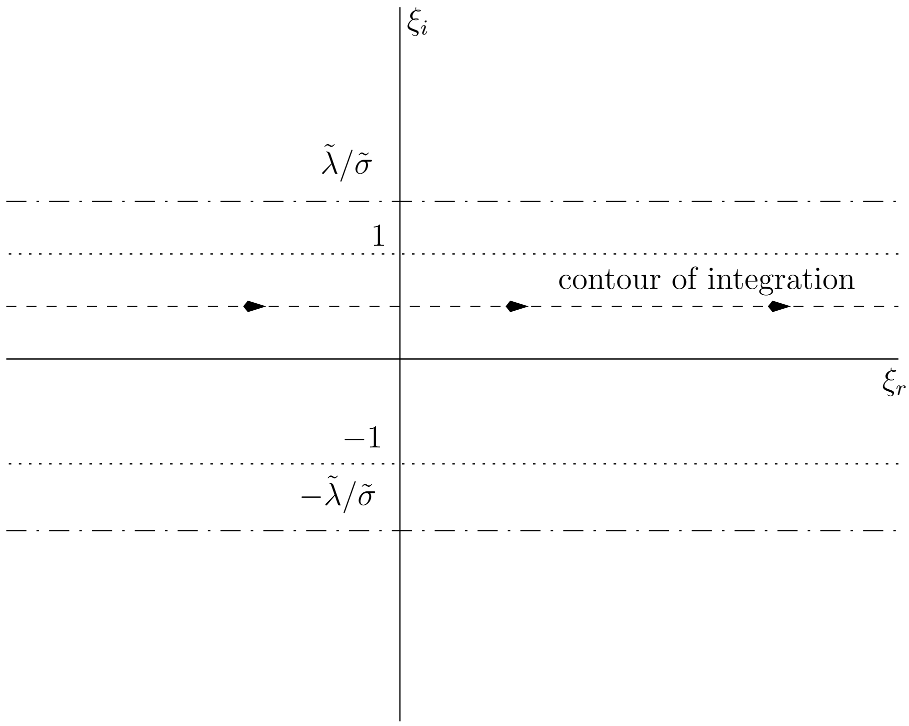

Figure 2: **Contour of integration**. The figure shows the contour of integration used to price Call options. In our calculations we choose Im $\xi = 1/2$.

orders. Therefore we can match some of the parameters of the characteristic function (1) to those of the Brownian motion in the Black-Scholes framework. The rationale is that since we obtained the Lévy-Stable pricing equation as the limiting distribution of the Damped-Lévy distribution we may match the first two moments of the Damped-Lévy to that of the Gaussian process (resulting from assuming Brownian motion). We use the expression for the moments given above in Proposition 4. In the Black-Scholes case the second moment of the Brownian motion is matched letting $\kappa^\nu \nu(\nu-1)a^{\nu-2}\tilde{\sigma}^\nu = \tilde{\sigma}_{BS}^2$, where $a = \tilde{\lambda}/\tilde{\sigma}$ and $\tilde{\sigma}_{BS}$ denotes the volatility in the Black-Scholes case. In other words, we choose

$$\kappa^\nu \tilde{\sigma}^\nu = \frac{a^{2-\nu}}{\nu(\nu-1)}\tilde{\sigma}_{BS}^2.$$

For all calculations we took $\tilde{\sigma}_{BS} = 0.20$ and $a = 1.1$. Moreover the initial stock price is $S = 100$ and for simplicity the interest rate used is $r = 0$ and no dividends are paid, $D_0 = 0$.

Our main findings show two interesting points that are depicted in the figures below. First, in comparison with the standard Black-Scholes the Lévy-Stable option prices are above the Black-Scholes outside a neighbourhood around the strike price $K = S = 100$.

Second, the existence of fat tails for the Lévy-Stable cases show that the Lévy-Stable case captures the volatility smile encountered in the financial data when the Black-Scholes framework is used to generate implied volatilities.

It is a well-known fact that the skew obtained in the implied volatility is a consequence of the absence of normality in the underlying stochastic process for stock returns. The downward slope of the implied volatility is a consequence of the asymmetry, determined in our model by the parameter $\eta \in [-1, 1]$, in the risk-neutral distribution of the underlying stock return. On the other hand, the convexity shown by the implied volatility is a consequence of the thickness of the tails of the distribution, which in our model is determined by the characteristic exponent $\nu \in (0, 2]$.

We consider three cases. The first case assumes $\nu = 1.5$ and $\eta = -0.5$. The second case assumes $\nu = 1.8$ and $\eta = 0$. We show results for three different expiry dates. The last case looks at the effect of skewness for $\nu = 1.5$ with $\eta = -0.5$, 0 and 0.5.

Figure 3 above shows that for European calls in the neighbourhood of at the money ($S = K = 100$ in this case) we see that the Black-Scholes model delivers a more expensive option. When the strike moves away from the at the money value the Lévy-Stable prices are above the Black-Scholes. The main reason why we see a 'dip' for at the money values is that the Gaussian distribution has more probability mass around this value than the Lévy-Stable distribution. Similarly, the Lévy-Stable distribution contains more mass than the Gaussian pdf around the tails (ie heavy tails) therefore we see that outside the at the money neighbourhood the Lévy-Stable prices are higher to reflect the likelihood of extreme movements of the underlying stock price. The figure also shows that the longer the expiry of the option the more accentuated is the difference in prices due to the fact that the pdf's scale with time. Moreover, Figure 4 shows the effect of heavy tails in the implied volatility.

The following two figures, 5 and 6, may be interpreted as the two above but using $\nu = 1.8$ and $\eta = 0$. The important message is that as ν approaches the Gaussian case, ie $\nu = 2$, the differences are less accentuated.

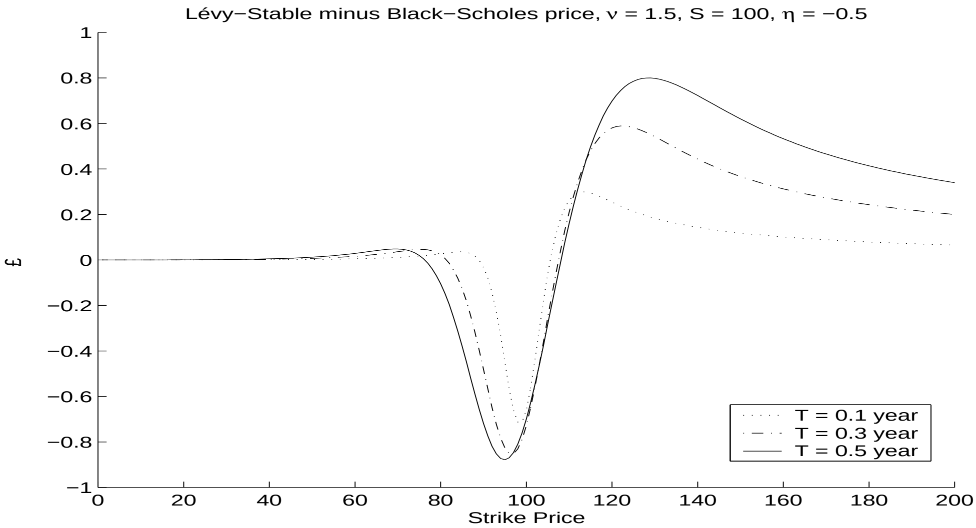

Figure 3: The figure above shows the price difference between a Lévy-Stable Call option and the Black-Scholes price. The parameters are $\nu = 1.5$, $\eta = -0.5$, $T = 0.1$, $T = 0.3$ and $T = 0.5$ years.

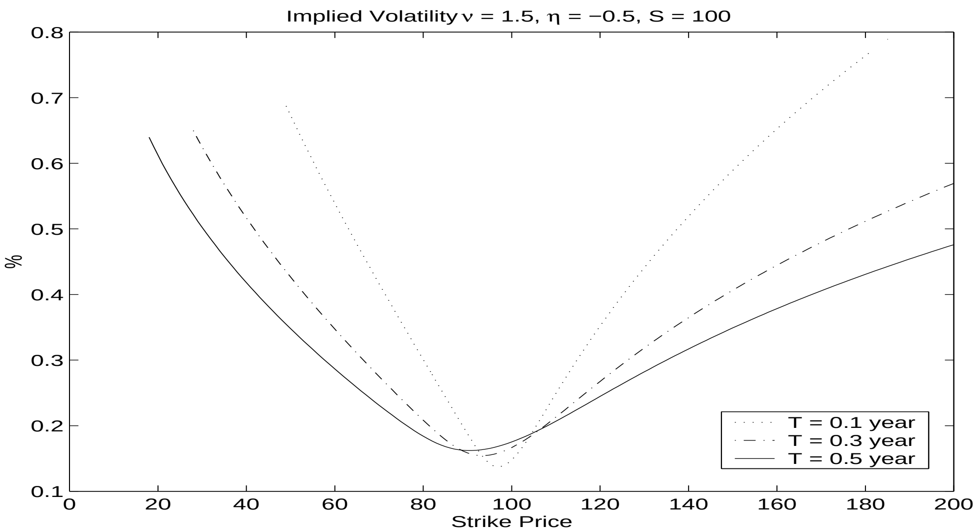

Figure 4: The Black-Scholes Implied volatility for the Lévy-Stable Call option derived above with parameters $\nu = 1.5$, $\eta = -0.5$, $T = 0.1$, $T = 0.3$ and $T = 0.5$ years.

Finally, we show how the skewness parameter affects the difference in prices. Figure 7 above shows for $\nu = 1.5$ and three different skewness, $\eta = -0.5$, $\eta = 0$ and $\eta = 0.5$; Figure 8 shows the corresponding (normalised) pdf. We know that when the distribution is skewed the left tail (high probability of an adverse large move for a call option) is higher but most

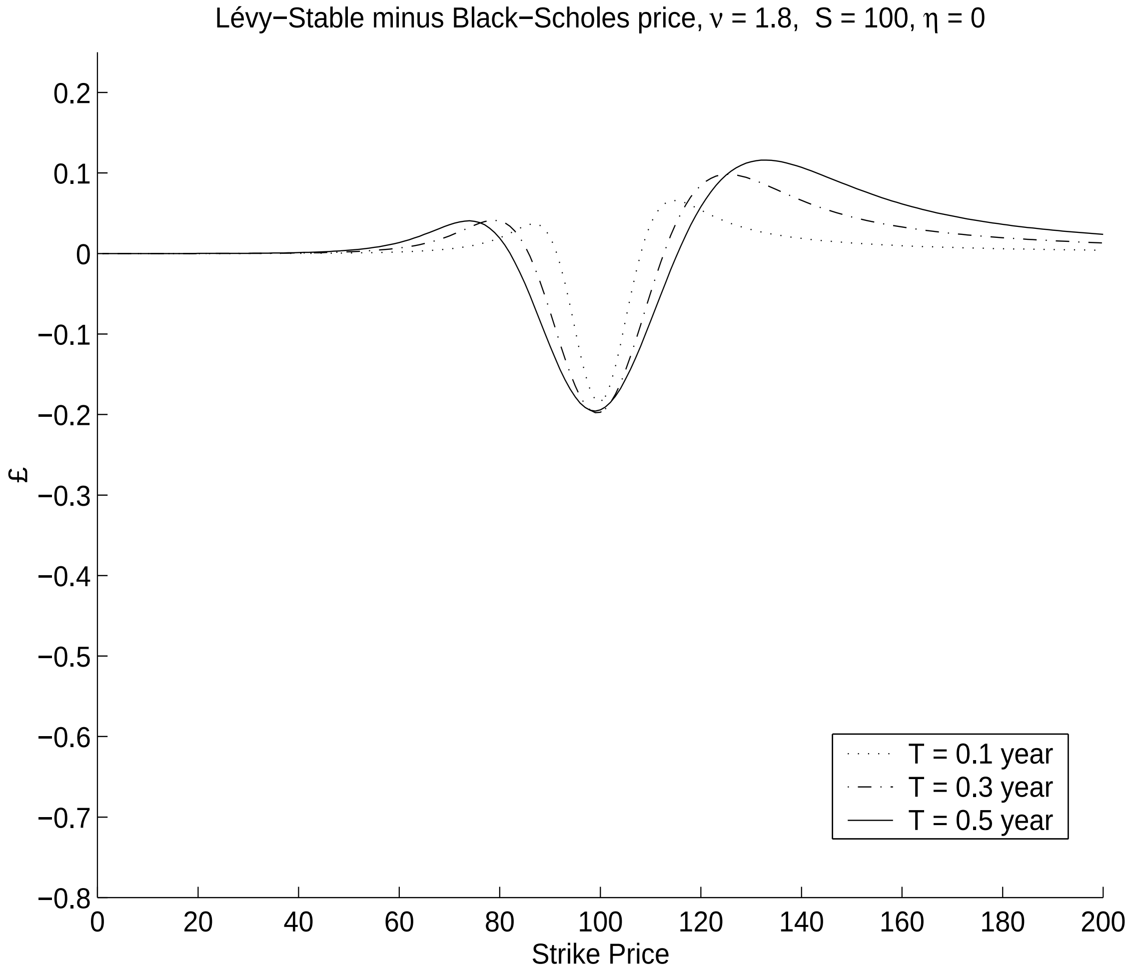

Figure 5: The figure above shows the price difference between a Lévy-Stable Call option and the Black-Scholes price. The parameters are $\nu = 1.8$, $\beta = -0.5$, $T = 0.1$, $T = 0.3$ and $T = 0.5$ years.

of the probability mass is shifted to the right as shown by Figure 8. If the distribution is skewed to the right the probability of favourable movements in the underlying is higher therefore the difference in prices is higher for in the money call options.

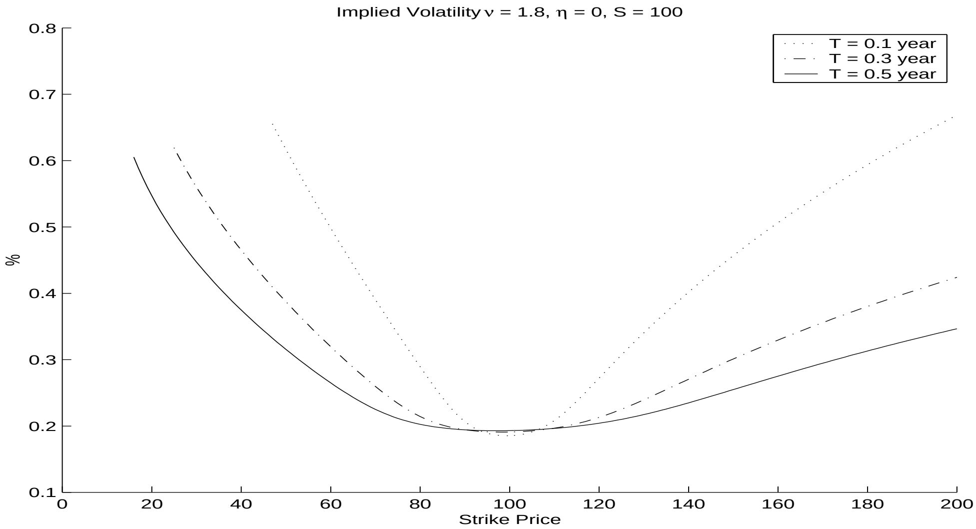

Figure 6: The Black-Scholes Implied volatility for the Lévy-Stable Call option derived above with parameters $\nu = 1.8$, $\eta = -0.5$, $T = 0.1$, $T = 0.3$ and $T = 0.5$ years.

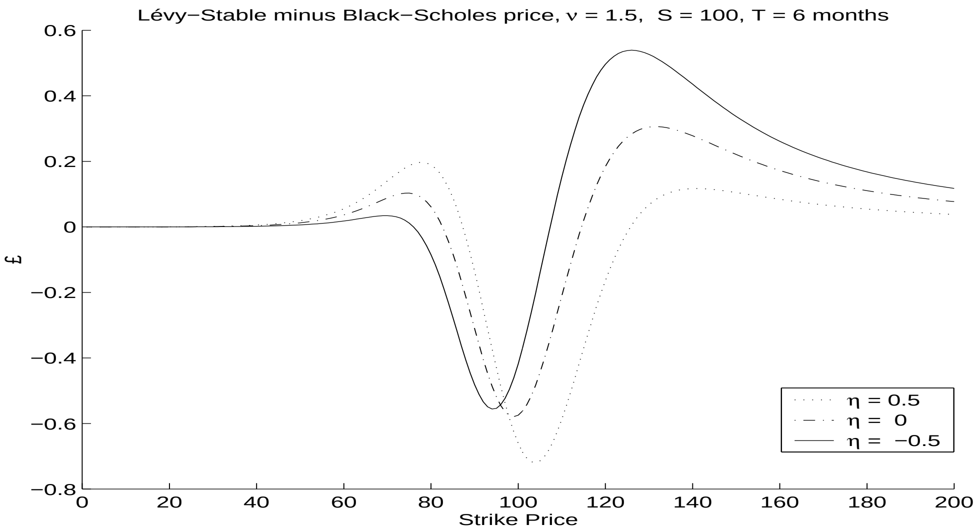

Figure 7: The figure above shows the price difference between a Lévy-Stable Call option and the Black-Scholes price. The parameters are $\nu = 1.5$, $\eta \in \{-0.5, 0, 0.5\}$, with expiry $T = 6$ months.

8 The 'Damped-Black-Scholes PDE': only the diffusion coefficient is scaled

Above we showed that to derive the pricing PDE as a distinguished limit of the Lévy-Stable process we had to scale both the damping and the volatility parameters with $\Delta t^{1/\nu}$. We

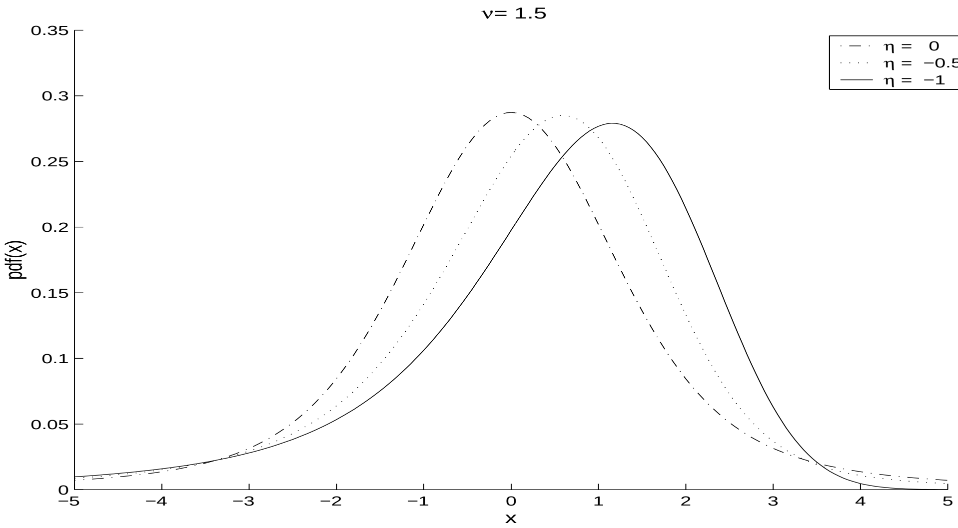

Figure 8: The figure above shows the pdf for Lévy-Stable distributions for $\nu = 1.5$ and $\eta = 0$, $\eta = -0.5$ $\eta = -1$.

also showed that if we chose not to scale the damping parameter, the other feasible scaling for volatility is $\sigma = (\Delta t)^{1/2}\tilde{\sigma}$. We now show that when the latter scaling is used the usual Black-Scholes formula can be retrieved. Note that since the damping parameter is not sent to zero the process will not converge to the Lévy-Stable case.

8.1 Derivation of the Damped-Lévy-Black-Scholes partial differential equation

The setting of our problem is as the one above. The only difference is that the damping parameter is not scaled with the time step.

Proposition 11 ***The Damped-Lévy-Black-Scholes formula.***

Let the stock price, under the risk neutral measure, follow the Damped-Lévy process

$$S_{t+\Delta t} = S_t e^{\overline{\mu}\Delta t + \sqrt{\Delta t}\tilde{\sigma}\phi}$$

where $\overline{\mu} = r - D_0 - \Psi_{TL}(-i\tilde{\sigma})$ is the drift, D_0 is a continuous dividend, $\tilde{\sigma}$ is a diffusion coefficient and the random shocks come through the Damped-Levy random variable

$$\phi \sim DL_\nu(\kappa, \eta, 0, \lambda).$$

Then the value $V(S,t)$ of a financial instrument with terminal payoff $\Pi(S,T)$ at time T satisfies

$$\begin{aligned} rV(S,t) &= \frac{\partial V}{\partial t} + \left(\overline{\mu} + \sigma^2\kappa^\nu\nu(\nu-1)\lambda^{\nu-2}\right) S\frac{\partial V}{\partial S} \\ &+ \sigma^2\kappa^\nu\nu(\nu-1)\lambda^{\nu-2}S^2\frac{\partial^2 V}{\partial S^2} \end{aligned}$$

with boundary condition $V(S,T) = \Pi(S,T)$.

The proof is very similar to those above and the solution of the above partial differential equation is the same as for the classical Black-Scholes formula, only the parameters vary. It is straightforward to check that when $\nu = 2$, we have that $\overline{\mu} = r - \frac{1}{2}\sigma^2$, $\kappa^2 = 1/2$ yielding the usual Black-Scholes equation:

$$rV = \frac{\partial V}{\partial t} + (r - D_0)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}.$$

The above result is not surprising since with this particular scaling the result is a direct consequence of the Central Limit Theorem. The same result was also derived in [21] using a different approach.

9 Conclusions

We have shown how to value claims where the underlying follows an exponential Lévy-Stable process. The way in which the pricing equations are derived is to start with a process whose shocks belong to a Damped-Lévy distribution. Then under a suitable scaling of the damping parameter we can get, in the limit $\Delta t \to 0$, pricing equations that converge to the Lévy-Stable case. We derive a generalised Black-Scholes partial differential equation that can be used to price for example European, American and perpetiual options. For perpetual claims we are able to derive analytic solutions and for European claims we also derive analytic solutions but in Fourier space. In this latter case we showed that there is a suitable contour on which to perform the numerical inversions of the pricing equations and show the volatility smile for different values of ν (thickness of tails), η (skewenss of the underlying stochastic innovations and for different expiry dates.

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A Appendix

Proposition 12 ***Roots of the Characteristic Equation.***

Let $a = \tilde{\lambda}/\tilde{\sigma}$. If $r\left(\frac{a-1}{a}\right) > D_0$ the characteristic equation

$$\begin{aligned} Y(\beta) &= \kappa^\nu \left\{ p(\tilde{\lambda} - \tilde{\sigma}\beta)^\nu + q(\tilde{\lambda} + \tilde{\sigma}\beta)^\nu - \tilde{\lambda}^\nu - \nu\tilde{\lambda}^{\nu-1}(q-p)\tilde{\sigma}\beta \right\} \\ &\quad + (r - D_0 - \Psi_{TL}(-i\tilde{\sigma}))\beta - r \end{aligned} \tag{38}$$

has a unique positive real root, denoted β_{tl}. Moreover, if $D_0 + \Psi_{TL}(-i\tilde{\sigma}) > r\left(\frac{a+1}{a}\right)$ there is a unique negative root β_{tl}^-.

Proof

First we let $\frac{\tilde{\lambda}}{\tilde{\sigma}} = a > 1$ and consider the more relevant cases $\nu \in (1, 2]$. Then we can rewrite the characteristic equation (38) as

$$Y(\beta) = \kappa^\nu \tilde{\sigma}^\nu \left\{ p(a-\beta)^\nu + q(a+\beta)^\nu - a^\nu - \nu a^{\nu-1}(q-p)\beta \right\} + (r - D_0 - \Psi_{TL}(-i\tilde{\sigma}))\beta - r.$$

To find a real positive root we focus on the interval $\beta \in [0, a]$. Note that if $\beta > a$ the characteristic equation $Y(\beta) \in \mathbb{C}$, ie it is complex, due to the component $(a-\beta)^\nu$, for $\nu < 2$. Note also that outside the interval $[-a, a]$ the Laplace Transform of the stock price process (17) does not exist. Now we ask how should the parameters r, D_0, p and q be so that there exists a positive root. From Figure 9 below it can be seen that one way to guarantee the existence of such a root is to require that the characteristic equation is real and positive at $\beta = a$, ie that $Y(a) > 0$. Therefore we must require that

$$\kappa^\nu \tilde{\sigma}^\nu a^\nu \left\{ q2^\nu - \nu(2q-1) - 1 \right\} + (r - D_0 - \Psi_{TL}(-i\tilde{\sigma}))a - r > 0. \tag{39}$$

Note that for $\nu \in (1, 2]$ and $q \in [0, 1]$

$$q2^\nu - \nu(2q-1) - 1 \geq 0.$$

We can also show that the function $\Psi_{TL}(\theta)$ is increasing in θ; then we have that

$$\kappa^\nu \tilde{\sigma}^\nu a^\nu \left\{ q2^\nu - \nu(2q-1) - 1 \right\} - a\Psi_{TL}(-i\tilde{\sigma}) > 0.$$

Hence for the inequality (39) to hold we must require that

$$r\left(\frac{a-1}{a}\right) > D_0.$$

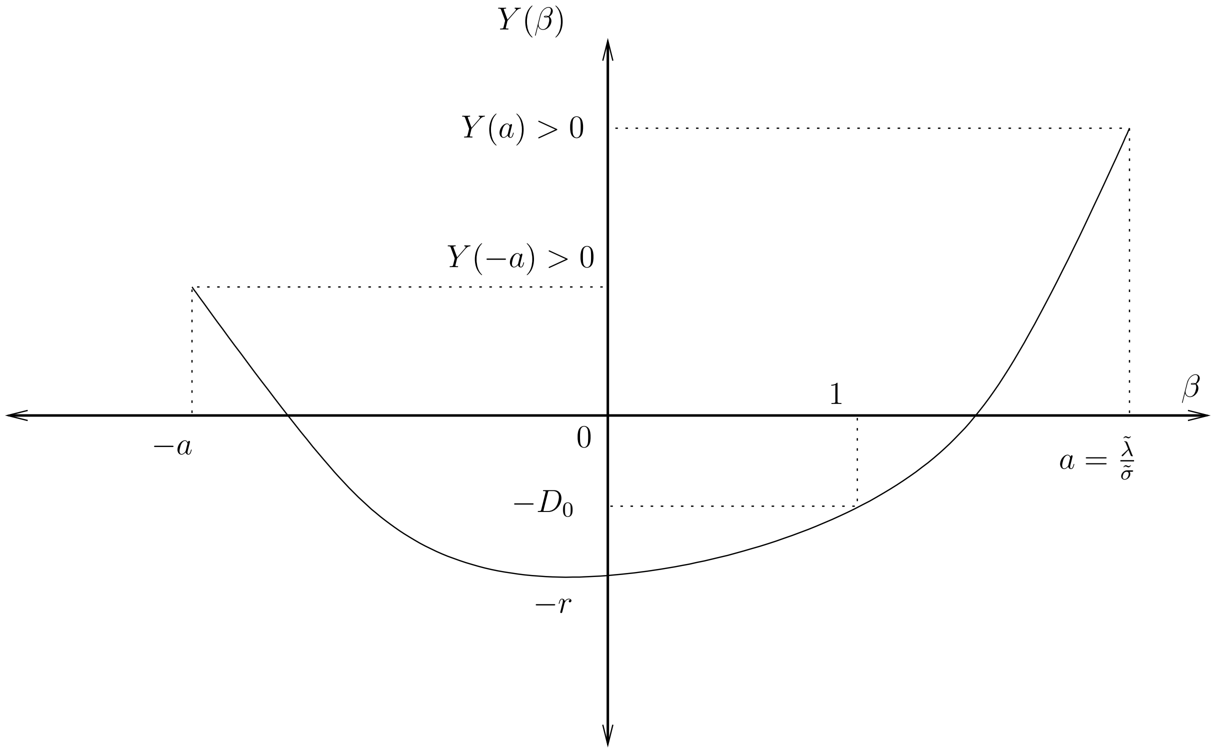

Figure 9: The figure shows the motivation for the proof of existence of a real positive root β_{tl} to price a perpetual Call. The approach is to focus on the interval $\beta \in [0, a]$ to guarantee that the characteristic function $Y(\beta) \in \mathbb{R}$ and require that $Y(a) > 0$. Therefore, by the Intermediate Value Theorem, we know that there is $\beta_{tl} \in (1, a)$ such that $Y(\beta_{tl}) = 0$. And in a similar way we provide sufficient conditions for a negative root β_{tl}^{-} to exist.

Hence, by the Intermediate Value Theorem, we have that there exists $\beta_{tl} \in (1, a)$ such that $Y(\beta_{tl}) = 0$.

An alternative way to see this result is to observe that $Y(0) < 0$, $Y(1) < 0$ therefore since $Y(\beta)$ is strictly convex it must be increasing and by choosing $a > 1$, such that $Y(a) \in \mathbb{R}$, we have that $Y(a) > 0$ hence there exists a positive root.

Now, to show that there is a negative root β_{tl}^{-} we proceed in a similar way by looking at the behaviour of $Y(\beta)$ in the interval $\beta \in [-a, 0]$. Note that at $\beta = -a$

$$Y(-a) = \kappa^{\nu} \tilde{\sigma}^{\nu} a^{\nu} \left\{(1-q)2^{\nu} + \nu(2q-1) - 1\right\} - (r - D_0 - \Psi_{TL}(-i\tilde{\sigma}))a - r,$$

and for $\nu \in (1, 2]$ and $q \in [0, 1]$

$$(1-q)2^{\nu} + \nu(2q-1) - 1 \geq 0.$$

Then if we require that

$$D_0 + \Psi_{TL}(-i\tilde{\sigma}) > r\left(\frac{a+1}{a}\right)$$

there is $\beta_{tl}^{-} \in (-a, 0)$ such that $Y(\beta_{tl}^{-}) = 0$.

QED