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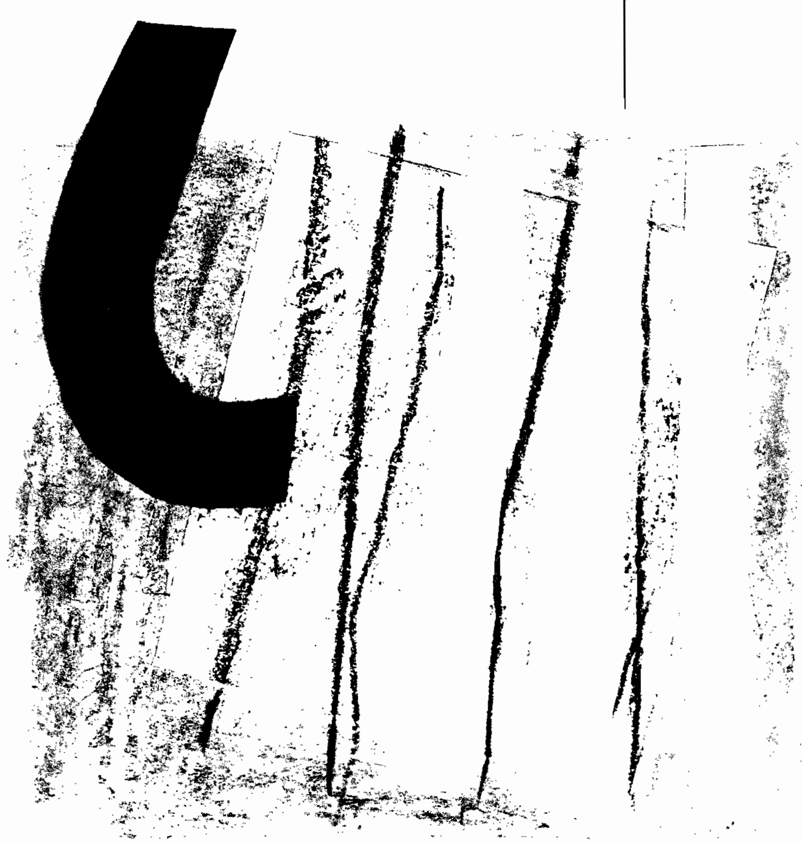

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Key Words

Autocorrelation function; Ljung-Box statistic; Normalized spectral distribution function; Residuals.

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ON THE CUMULATED PERIODOGRAM GOODNESS-OF-FIT TEST IN ARMA MODELS

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Abstract. The asymptotic distribution of the cumulated periodogram goodness-of-fit test statistic for ARMA models is obtained, and is shown to be different from the limiting distribution of the standard Kolmogorov-Smirnov test statistic for probability distributions. The implications of this anomaly for inference purposes are analyzed. A modified cumulated periodogram goodness-of-fit test statistic is suggested, and its properties are studied and compared with other goodness-of-fit criteria proposed in the literature.

Keywords. Autocorrelation function; Ljung-Box statistic; normalized spectral distribution function; residuals.

1. INTRODUCTION

Consider an ARMA(p,q) model

$$\phi(B)X_t = \theta(B)\varepsilon_t, \qquad (1.1)$$

where $\{\varepsilon_t: t = 0, \pm1, \pm2, \dots, \}$ is a zero mean white noise sequence with variance σ^2, B is the backward shift operator $BX_t = X_{t-1}$, and the polynomials $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$ and $\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$ have all their roots outside the unit circle. After a model of the form (1.1) has been fitted to a finite observed series $X_1, \dots, X_n$, it is useful to investigate the adequacy of the fit by examining the autocorrelations

$$\hat{r}_k = \hat{g}_k/\hat{g}_0 = [\sum_{t=1}^{n-k} \hat{\varepsilon}_t\hat{\varepsilon}_{t+k}/n]/[\sum_{t=1}^{n} \hat{\varepsilon}_t^2/n], \qquad 0\le k\le n-1, \qquad (1.2)$$

of the residuals

$$\hat{\varepsilon}_t = \hat{\theta}(B)^{-1}\hat{\phi}(B)X_t, \; t = 1, \ldots, n, \tag{1.3}$$

where the polynomials $\hat{\phi}(z) = 1 - \hat{\phi}_1 z - \ldots - \hat{\phi}_p z^p$ and $\hat{\theta}(z) = 1 + \hat{\theta}_1 z + \ldots + \hat{\theta}_q z^q$ are constructed on efficient estimates $\hat{\phi} = (\hat{\phi}_1, \ldots, \hat{\phi}_p)'$ and $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_q)'$ of the parameters $\phi = (\phi_1, \ldots, \phi_p)'$ and $\theta = (\theta_1, \ldots, \theta_q)'$. By convention, $X_t \equiv 0$ for $t \le 0$. A standard goodness-of-fit method is the Ljung and Box (1978) statistic,

$$\hat{Q}_n = n(n+2) \sum_{k=1}^{m} (n-k)^{-1}\hat{r}_k^2, \tag{1.4}$$

where $m = m_n$ is a suitable function of the sample size n. The approximate null distribution of $\hat{Q}_n$ is chi squared with m-(p+q) degrees of freedom.

An alternative to (1.4) is to test for goodness-of-fit in the frequency domain. Define, for $0 \le \lambda \le \pi$, the periodogram ordinates

$$\hat{I}_n(\lambda) = [\hat{g}_0 + 2 \sum_{k=1}^{n-1} \hat{g}_k \cos(k\lambda)]/2\pi, \tag{1.5}$$

put $h = [n/2]$, and consider the quantities $\hat{C}_j = \sum_{k=1}^{j} \hat{I}_n(2\pi k/n)$, and $\hat{U}_j = \hat{C}_j/\hat{C}_h$, $j = 1, \ldots, h$. A plot of $\hat{U}_j$ against j/h is called the cumulative periodogram of the residuals. The statistic

$$\hat{D}_n = h^{1/2} \max_j |\hat{U}_j - j/h|, \tag{1.6}$$

can be used to test the adequacy of model (1.1). In practice, $\hat{D}_n$ is calibrated superimposing on the cumulated periodogram two parallel lines to the left and to the right of the line y = x at a distance $h^{-1/2}k_\alpha$, where k_α is the appropriate $(1-\alpha)$x100% upper quantile of the Kolmogorov-Smirnov distribution (Diggle, 1990, p. 55). Box and Jenkins (1976, p. 297) comment that this distribution gives only approximate significance levels for $\hat{D}_n$. In fact, Durbin (1975) proofs that the asymptotic distribution of (1.6) is different from the Kolmogorov-Smirnov distribution.

The paper reviews briefly, in section 2, the work of Durbin (1975) on the cumulated periodogram goodness-of-fit test. An alternative characterization of the asymptotic distribution of $\hat{D}_n$ is obtained that leads, in section 3, to a modification of the test statistic (1.6) that can be shown to be asymptotically Kolmogorov-Smirnov distributed. Section 4 is devoted to simulations and comparisons and section 5 to some final comments.

2. THE ASYMPTOTIC DISTRIBUTION OF $\hat{D}_n$

$\hat{D}_n$ is obtained replacing the integrals by Riemann sums evaluated at the Fourier frequencies $\lambda_j = 2\pi j/n$, $j = 1, \ldots, h$, in

$$\hat{T}_n = (n/2)^{1/2} \sup_{0\le\lambda\le\pi} |\hat{F}_n(\lambda) - F_0(\lambda)|, \tag{2.1}$$

where $\hat{F}_n(\lambda) = 2\int_0^\lambda \hat{I}_n(t)dt/\hat{g}_0$, $F_0(\lambda) = 2\int_0^\lambda f_0(t)dt = \lambda/\pi$, and $f_0(\lambda) = (2\pi)^{-1}$ is the normalized spectral density of a white noise process. It is clear that the asymptotic distributions of $\hat{D}_n$ and $\hat{T}_n$ are the same. Consider the errorwise version of (2.1), namely,

$$T_n = (n/2)^{1/2} \sup_{0\le\lambda\le\pi} |F_n(\lambda) - F_0(\lambda)|, \tag{2.2}$$

where $F_n(\lambda) = 2\int_0^\lambda I_n(t)dt/g_0$, and $I_n(\lambda)$ are the periodogram ordinates computed from the autocovariance function $\{g_k\}$ of the errors $\varepsilon_1, \ldots, \varepsilon_n$. Under the null that $\{\varepsilon_t\}$ is a white noise, the statistic T_n is asymptotically Kolmogorov-Smirnov distributed. See Bartlett (1955) or, more recently, Durlauf (1991) or Anderson (1993). If model (1.1) is correct, it is natural to expect white noise characteristics of the residuals and, therefore, that the asymptotic distribution of $\hat{T}_n$ is to be close to the limit distribution of T_n.

2.1 The work of Durbin (1975) on the asymptotics of $\hat{D}_n$

Introducing the change of variable $\lambda = \pi t$, $\hat{T}_n$ is the sup over [0,1] of the stochastic process $\{\hat{W}_n(t)\colon 0\le t\le 1\}$, where

$$\hat{W}_n(t) = (\sqrt{2}/\pi)\ n^{1/2} \sum_{k=1}^{n-1} \hat{r}_k \frac{\sin(k\pi t)}{k}. \qquad (2.3)$$

Durbin (1975, p.136) proofs that $\{\hat{W}_n(t)\colon 0\le t\le 1\}$ converges weakly to a zero mean gaussian process with covariance function

$$G(t,s) = [\min(t,s) - ts] - (2\pi^2)^{-1} h(\pi t)'[\mathcal{I}(\phi,\theta)]^{-1} h(\pi s), \qquad (2.4)$$

where, for $0\le\lambda\le\pi$, $h(\lambda) = \int_0^\lambda (\partial \log f(t)/\partial\mu)dt$ is a (p+q)x1 vector, $f(\lambda) = (2\pi)^{-1}|\theta(e^{-i\lambda})|^2/|\phi(e^{-i\lambda})|^2$, and $\mathcal{I}(\phi,\theta)$ is the (p+q)x(p+q) information matrix for $\mu = (\phi,\theta)$. The first term in (2.4) is the covariance function of the brownian bridge $\{B(t)\colon 0\le t\le 1\}$ while the second summand can be explained by the distorting effect of parameter estimation on the covariance of the limiting process. As a consequence of (2.4), Durbin (1975, p.138) comments on: *i)* The dependence on unknown parameters of the asymptotic distribution of $\hat{D}_n$; and *ii)* The apparent unfeasibility of constructing an asymptotically correct test based on the maximum deviation of the cumulated periodogram without further modification of the problem.

2.2 An alternative asymptotic theory for $\hat{D}_n$

The results of Durbin (1975) are based on a theorem on the asymptotic validity of inferences built replacing parameters by estimates (Durbin, 1970). An alternative theory can be constructed based on techniques of weak convergence of processes. Methods of this section are, in some way, more illuminating than previous developments since they lead, in section 3, to a modification of the cumulated periodogram goodness-of-fit test that makes it asymptotically valid.

Let $m = m_n$ be a function of the sample size n that goes to infinity at the rate $n^{1/b}$ for some $b > 2$, and decompose

$$\hat{W}_n(t) = \hat{U}_n(t) + \hat{V}_n(t)$$

$$= (\sqrt{2}/\pi)\ n^{1/2} \sum_{k=1}^{m} \hat{r}_k \frac{\sin(k\pi t)}{k} + (\sqrt{2}/\pi)\ n^{1/2} \sum_{k>m}^{n-1} \hat{r}_k \frac{\sin(k\pi t)}{k}. \qquad (2.5)$$

Each of the terms in (2.5) is treated separately. Consider the coefficients $\{a_k\}_{k\geq 0}$ of the expansion

$$\alpha(z) = [\beta(z)]^{-1} = \sum_{k=1}^{\infty} a_k z^k, \tag{2.6}$$

where $\beta(z) = \phi(z)\theta(z) = 1 - \beta_1 z - \dots - \beta_{p+q} z^{p+q}$, put $a_k = 0$ for $k < 0$, and define $h_{jk} = \sum_{r=1}^{p+q} \sum_{s=1}^{p+q} a_{j-r} a_{k-s} \tilde{\Gamma}^{rs}_{p+q}$ for $j,k\geq 1$, where $\tilde{\Gamma}^{rs}_{p+q}$ is the (r,s) element of the inverse of the $(p+q)x(p+q)$ matrix $\tilde{\Gamma}_{p+q} = (\sum_{k=0}^{\infty} a_k a_{k+|r-s|}: 1\leq r,s\leq p+q)$. Under (1.1), using results in Velilla (1994), $\{\hat{U}_n(t): 0\leq t\leq 1\}$ converges weakly to a zero mean continuous gaussian process $\{U(t): 0\leq t\leq 1\}$ with covariance function

$$\gamma(t,s) = [\min(t,s) - ts] - g(t,s), \tag{2.7}$$

$0\leq t,s\leq 1$, where $g(t,s) = (2/\pi^2) \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} h_{jk} \frac{\sin(j\pi t)}{j} \frac{\sin(k\pi s)}{k}$. If model (1.1) has been correctly identified and fitted, the correlations $\hat{r}_k$ of the residuals behave, for large values of the lag k and the sample size n, similarly as the correlations $r_k = g_k/g_0$ of the true white noise errors (Box and Pierce, 1970, p. 1517). It seems then reasonable to approximate $\hat{V}_n(t)$ by $V_n(t) = (\sqrt{2}/\pi)\ n^{1/2} \sum_{k>m}^{n-1} r_k \frac{\sin(k\pi t)}{k}$. From an argument in Grenander and Rosenblatt (1957, p. 189), $\sup_{0\leq t\leq 1} |V_n(t)|$ goes to zero in probability.

The asymptotic distribution of $\hat{D}_n$ is then given by the distribution of the sup of the zero mean continuous gaussian process $\{U(t): 0\leq t\leq 1\}$ with covariance function $\gamma(t,s)$. Using the explicit expression for $\mathcal{I}(\phi,\theta)$ given in Bruce and Martin (1989, p. 399), is not difficult to see that expressions (2.4) and (2.7) coincide.

Example 2.1. The AR(1) model

For the AR(1) model $X_t = \phi X_{t-1} + \varepsilon_t$, where $|\phi| < 1$, $a_k = \phi^k$ and then $h_{jk} = \phi^{j+k-2}(1 - \phi^2)$, $j,k \geq 1$. The function $g(t,s)$ is

$$g(t,s) = (2/\pi^2)\phi^{-2}(1 - \phi^2)c(t,\phi)c(s,\phi),$$

where $c(t,\phi) = \sum_{j=1}^{\infty} \phi^j \frac{\sin(j\pi t)}{j}$. Using a summation formula in Gradshtein and Ryzhik (1994, p.48), $c(t,\phi) = \mathrm{arctg}[\phi\sin(\pi t)/(1-\phi\cos(\pi t))]$. ■

2.3 Asymptotic rejection probabilities for $\hat{D}_n$

From either (2.4) or (2.7), is easily seen that the covariance function $\gamma(t,s)$ is smaller that the covariance of $\{B(t): 0 \le t \le 1\}$ in the Loewner sense, that is $\int_0^1\int_0^1 \gamma(t,s)l(t)l(s)dtds \le \int_0^1\int_0^1 [\min(t,s)-ts]l(t)l(s)dtds$, for any $l(.)$ such that the integrals are defined. From Anderson (1955), the inequality below holds for any $d > 0$,

$$P[\sup_{0\le t\le 1} |B(t)| \ge d] \ge P[\sup_{0\le t\le 1} |U(t)| \ge d]. \tag{2.8}$$

As a conclusion, the Kolmogorov-Smirnov distribution underestimates the statistical significance of an observed value of $\hat{D}_n$. The standard parallel lines superimposed to the cumulated periodogram are at a larger distance from the line $y = x$ than they should be and this decreases the ability of the goodness-of-fit statistic (1.6) to detect lack of fit when it exists. This phenomenon is similar to the one that appears when treating, for diagnostic checking purposes, the residual correlations $\hat{r}_k$ for small lags as the error correlations r_k. The correct confidence bounds for $\hat{r}_k$ are smaller than those for r_k. See, for example, Box and Pierce (1970, p. 1520).

Example 2.2. Empirical quantiles of the asymptotic distribution of $\hat{D}_n$

Consider the AR(1) model $X_t = \phi X_{t-1} + \varepsilon_t$ and put p_d and $p_d(\phi)$ for, respectively, the left and right hand sides of inequality (2.8). Exact computation of the function $p_d(\phi)$ is intractable and resorting to the empirical distribution of $\hat{D}_n$ to get estimates $\hat{p}_d(\phi)$ seems appropriate. For sample sizes n = 100, 200, and for each value of ϕ = .1, .5 and .9, N = 5000 independent series of length n are generated from $X_t = \phi X_{t-1} + \varepsilon_t$, where $\{\varepsilon_t\}$ are i.i.d N(0,1). The values of $\hat{D}_n$ are computed and used to find empirical estimates $\hat{p}_d(\phi)$ = (# of values of $\hat{D}_n \geq d$)/N of $p_d(\phi)$. Figure 1.1 (n = 100) and 1.2 (n = 200) compare the dashed and dotted curves $\hat{p}_d(\phi)$ with the continuous curve p_d obtained from table 1 in Shorack and Wellner (1987, p. 143). Observe the dominance $p_d \geq \hat{p}_d(.9) \geq \hat{p}_d(.5) \geq \hat{p}_d(.1)$.

Figure 1.1 Figure 1.2

Figure 1.3 Figure 1.4

As a complement to figures 1.1 and 1.2, tables 1.1 and 1.2 give the values of the .90, .95 and .99 upper quantiles of the Kolmogorov-Smirnov distribution, first row, and the .90, .95 and .99 empirical upper quantiles of the distribution of $\hat{D}_n$ under the different values of the parameter ϕ. The difference is quite remarkable, casting some doubts about the statistical behaviour of the goodness-of-fit methodology based on (1.6).

Finally, figures 1.3 and 1.4 display the empirical tail probabilities for $\hat{D}_n$ obtained from N = 5000 independent series of lengths, respectively, n = 100 and 200, generated from the AR(2) model $(1-.5B)(1-aB)X_t = \varepsilon_t$ with i.i.d. N(0,1) errors, for values of a = .1, .4 and .7. The dashed and dotted lines are, again, far from the continuous line given by the tail probability function of the Kolmogorov-Smirnov distribution.

Table 1.1 Table 1.2

Example 2.3. Cumulated periodogram

Figure 2 below shows the cumulated periodogram for a simulated series of length n = 200 from the AR(2) model $(1-.5B)(1-aB)X_t = \varepsilon_t$ with a = .2. An AR(1) model $X_t = \phi X_{t-1} + \varepsilon_t$ is postulated and fitted. The value of $\hat{D}_n$ = 1.104 is declared in accordance with an AR(1) explanation using the continuous standard bands with an approximate observed significance level around .175. However, when the dashed correct asymptotic bands from table 1.2, corresponding to $\phi = .5$ (a = .0), are used, the value of $\hat{D}_n$ is declared significant with an approximate p-value of .03.

Figure 2

3. A MODIFIED CUMULATED PERIODOGRAM GOODNESS-OF-FIT TEST STATISTIC

The obvious remedial action for the anomalies detected in the behaviour of $\hat{D}_n$ in section 2 would be to superimpose the correct asymptotic bands to the cumulated periodogram. This is both mathematically intractable and statistically unrealistic because it requires knowledge of the parameters (ϕ,θ). In this section, a modification of the goodness-of-fit test statistic $\hat{D}_n$ is constructed so that the asymptotic distribution is the sup of the brownian bridge.

Take the errorwise version of the process (2.3), namely

$$W_n(t) = (\sqrt{2}/\pi)\, n^{1/2} \sum_{k=1}^{n-1} r_k \frac{\sin(k\pi t)}{k}\,, \tag{3.1}$$

where $\{r_k\}$ is the correlogram of $\varepsilon_1, \ldots, \varepsilon_n$, and consider the representation

$$B(t) = (\sqrt{2}/\pi) \sum_{k=1}^{\infty} u_k \frac{\sin(k\pi t)}{k}\,, \tag{3.2}$$

of the brownian bridge, where $\{u_k\}_{1\le k}$ is a sequence of independent N(0,1) variables. Put $R_m = (r_1, \ldots, r_m)'$, $\hat{R}_m = (\hat{r}_1, \ldots, \hat{r}_m)'$ and assume that n is large. Since for every fixed m, $n^{1/2}R_m$ is approximately $N_m(0,I_m)$ (Anderson, 1942), it is natural to expect closeness of $W_n(t)$ to B(t). In contrast, Box and Pierce (1970) establish, under certain conditions on m, that

$$\hat{R}_m \cong (I_m - H_m)R_m, \tag{3.3}$$

where $H_m = X_m(X'_mX_m)^{-1}X_m$ is the mxm orthogonal projection matrix on the manifold spanned by the columns of a certain mx(p+q) matrix X_m of rank p+q.

The expression for X_m is given in equation (A.1) of the appendix. From (3.3) the asymptotic behaviour of the correlations of the residuals at low lags k differs markedly from the asymptotic behaviour of their error counterparts This explains the convergence of the process $\{\hat{W}_n(t)\colon 0\le t\le 1\}$ to a gaussian process with covariance function (2.7) and, as a consequence, the distinct limit behaviour of the statistics T_n and $\hat{D}_n$.

To overcome this situation, Velilla (1994) proposes to write $I_m - H_m = C_m C_m'$, where $C_m = C_m(\phi,\theta)$ is an mx[m-(p+q)] matrix such that $C_m' C_m = I_{m-(p+q)}$ and $C_m' X_m = 0$. If the autocorrelation sequence $\hat{R}_m = (\hat{r}_1, \ldots, \hat{r}_m)'$ is transformed linearly

$$\hat{S}_m = \hat{C}_m' \hat{R}_m, \tag{3.4}$$

where $\hat{C}_m = C_m(\hat{\phi},\hat{\theta})$, it is reasonable to approximate $n^{1/2}\hat{S}_m \sim N_{m-(p+q)}[0, I_{m-(p+q)}]$. If $\hat{S}_m = (\hat{s}_{1+p+q}, \ldots, \hat{s}_m)'$ and $\hat{s}_{k+p+q} = \hat{r}_{k+p+q}$ for $m-(p+q) < k \le n-(p+q+1)$, it can be then conjectured that the process

$$\hat{Z}_n(t) = (\sqrt{2}/\pi)\, n^{1/2} \sum_{k=1}^{n-(p+q+1)} \hat{s}_{k+p+q} \frac{\sin(k\pi t)}{k}, \tag{3.5}$$

is close to (3.2). By choosing $\{C_m(\phi,\theta)\}$ as in the appendix and assuming that $m = m_n = O(n^{1/b})$, b>2, theorem 3.4 in Velilla (1994) can be used to establish weak convergence of $\{\hat{Z}_n(t)\colon 0\le t\le 1\}$ to $\{B(t)\colon 0\le t\le 1\}$.

Consider now the modified residual periodogram ordinates for $0\le\lambda\le\pi$,

$$\hat{J}_n(\lambda) = [1 + 2 \sum_{k=1}^{n-(p+q+1)} \hat{s}_{k+p+q} \cos(k\lambda)]/2\pi, \tag{3.6}$$

and put $\hat{G}_n(\lambda) = 2\int_0^\lambda \hat{J}_n(t)dt$. The asymptotic distribution of the statistic

$$\hat{\mathcal{J}}_n = (n/2)^{1/2} \sup_{0\le\lambda\le\pi} |\hat{G}_n(\lambda) - F_o(\lambda)| = \sup_{0\le t\le 1} |\hat{Z}_n(t)|, \tag{3.7}$$

is the distribution of $\sup_{0\le t\le 1} |B(t)|$. Replacing the integrals in (3.7) by Riemann sums evaluated at the Fourier frequencies $\lambda_j = 2\pi j/n$, the modified cumulated periodogram goodness-of-fit test statistic

$$\hat{D}_{M,n} = h^{1/2} \max_j |\hat{V}_j - j/h|, \tag{3.8}$$

where $\hat{V}_j = (4\pi/n)\sum_{k=1}^{j}\hat{I}_n(2\pi k/n)$, $j = 1, \ldots, h$, can be properly compared, for n large, with the critical values of the Kolmogorov-Smirnov distribution. The explicit construction of the matrix $C_m(\phi,\theta)$ of (3.4) is given in the appendix. The choice of the lag m is studied in section 4.4.

4. EXAMPLES, SIMULATIONS AND COMPARISONS

This section studies applied aspects of the goodness-of-fit methodology $\hat{D}_{M,n}$. The asymptotic distribution is shown to be accurately reflected by the Kolmogorov-Smirnov distribution in some simple cases. $\hat{D}_{M,n}$ arises from taking the sup of the process $\{\hat{Z}_n(t)\colon 0 \le t \le 1\}$. Other functionals could be considered, for example, the Cramér-von Mises functional

$$\hat{M}_n = \int_0^1 [\hat{Z}_n(t)]^2 dt = (n/\pi^2) \sum_{k=1}^{n-(p+q+1)} \hat{s}^2_{k+p+q}/k^2, \tag{4.1}$$

whose asymptotic distribution is given, by the continuous mapping theorem, by the integral $\int_0^1 [B(t)]^2 dt$ of the brownian bridge. This distribution is tabulated in Shorack and Wellner (1987, p. 147). The structure of (4.1) resembles to the structure of the Ljung-Box statistic $\hat{Q}_n$ of (1.4). The significance levels, power and dependence on the lag m of the three goodness-of-fit statistics $\hat{D}_{M,n}$, $\hat{M}_n$ and $\hat{Q}_n$ are studied and compared. For completeness, the significance levels and power of $\hat{D}_n$ using both the Kolmogorov-Smirnov and the correct asymptotic bands are also analyzed.

4.1 Asymptotic distribution of $\hat{D}_{M,n}$

Put $q_d(\phi) = P[\hat{D}_{M,n} \ge d]$ for the tail probability function of the distribution of $\hat{D}_{M,n}$ under the null that $X_t = \phi X_{t-1} + \varepsilon_t$. For values of $n = 100$ ($m = 10$) and 200 ($m = 15$), $N = 5000$ independent samples are generated from an AR(1) model with values $\phi = .1, .5$ and $.9$ and i.i.d. N(0,1) errors.

Empirical estimates $\hat{q}_d(\phi)$ of $q_d(\phi)$ are computed in the usual way. The matrix C_m is determined using expression (A.3) in the appendix. Figure 3.1 and 3.2 display the curves $\hat{q}_d(\phi)$ and the Kolmogorov-Smirnov curve p_d. Notice the accurate approximation $\hat{q}_d(\phi) \cong p_d$ for large values of d. Figures 3.3 and 3.4 are obtained simulating N = 5000 independent series of sizes n = 100 (m = 10) and n = 200 (m = 15) respectively, from AR(2) models $(1-.5B)(1-aB)X_t = \varepsilon_t$, for values of a = .1, .4 and .7. The statistic $\hat{D}_{M,n}$ is obtained using the expression (A.4) for the matrix C_m. Again, notice the closeness of the empirical tail probability curves to p_d.

Figure 3.1

Figure 3.2

Figure 3.3

Figure 3.4

4.2 Figure 2 (continued)

For the sample of figure 2, figure 4 below shows the plot of the magnitudes $\hat{V}_j$ against j/h superimposed to the plot of the ordinary cumulated periodogram of the residuals. $\hat{D}_{M,n}$ correctly identifies the problem of lack of fit. Observe the slight non monotonic pattern of the plot in the lower left corner. This is because the definition (3.6) does not guarantee the condition $\hat{J}_n(2\pi j/n) \geq 0$ for all j.

Figure 4

4.3 Empirical significance levels and power of the goodness-of-fit criteria

Tables 2.1, 2.2 and 2.3 display the empirical significance levels and power of the criteria $\hat{D}_{n,KS}$, $\hat{D}_{n,C}$, $\hat{D}_{M,n}$, $\hat{M}_n$ and $\hat{Q}_n$ for a nominal significance level .05 for, respectively, sample sizes n = 150, 200, and 400. In all tables, a model of the form $X_t - \phi X_{t-1} = \varepsilon_t$ is fitted, while the true model is of the form $(1-.5B)(1-aB)X_t = \varepsilon_t$ for values of a = .1, .2, .4., .7 and .9. The simulation size is N = 5000 and the errors are i.i.d. N(0,1). $\hat{D}_{n,KS}$

stands for the rejection rule using the .95 quantile of the Kolmogorov–Smirnov distribution and $\hat{D}_{n,C}$ for the rule using the asymptotic .95 quantile obtained from an empirical tabulation of the asymptotic distribution of $\hat{D}_n$ under the model $X_t - .5X_{t-1} = \varepsilon_t$.

Table 2.1

Table 2.2

Table 2.3

Notice the low size and power of $\hat{D}_{n,KS}$. In general, the most powerful criterion is, not unexpectingly, $\hat{D}_{n,C}$, followed, in this order, by criterions $\hat{D}_{M,n}$ and $\hat{M}_n$ that, in turn, are better than $\hat{Q}_n$. The size and power of criteria (3.8) and (4.1) are stable against the choice of the lag m. In contrast, the power of the Ljung–Box statistic declines when m increases. For more complete empirical studies on the size and power of $\hat{Q}_n$, see Davies, Triggs and Newbold (1977), Ljung and Box (1978), and Davies and Newbold (1979).

4.4 Numerical stability of criterions $\hat{D}_{M,n}$ and $\hat{M}_n$ and choice of the lag m

As seen in section 4.3, the size and power of both $\hat{D}_{M,n}$ and $\hat{M}_n$ is not much affected by the choice of the lag m. This empirical finding can be analytically justified exploiting a property of the matrix sequence $\{C_m = C_m(\phi,\theta)\}$. By the causality and invertibility assumptions on the polynomials $\phi(z)$ and $\theta(z)$, the coefficients $\{a_k\}_{k\geq 0}$ are bounded in the form $|a_k| \leq A\,Q^k$ for some constants $A > 0$ and $0 < Q < 1$. Therefore, from expression (A.2) in the appendix, the general term of the sequence $\{C_m\}$ is of the form

$$C_m = \left(\begin{array}{c|c} C_M & 0_{M\times r} \\ \hline 0_{r\times s} & I_{m-M} \end{array}\right),$$

where $r = m - M$ and $s = M - (p+q)$, for m larger than a certain integer value M. The modified correlation sequence (3.4) has the structure

$$\hat{S}_m = \hat{C}'_m \hat{R}_m = \left(\begin{array}{c|c} \hat{C}'_M & 0_{s\times r} \\ \hline 0_{r\times M} & I_{m-M} \end{array}\right) \hat{R}_m = \left|\begin{array}{c} \hat{S}_M \\ \hline \hat{r}_{M+1} \\ \vdots \\ \hat{r}_m \end{array}\right|,$$

for $m > M$ and, as a consequence, the numerical value of the statistics $\hat{D}_{M,n}$ and $\hat{M}_n$ is the same for any choice of the lag beyond the threshold M. This is illustrated in figures 5.1 and figures 5.2 below that display, for values of m from 2 up to 30, the values of, respectively, criterions (3.8) and (4.1) for a sample of size $n = 200$ generated from an AR(2) model $X_t = .9X_{t-1} - .2X_{t-2} + \varepsilon_t$ when the model fitted is $X_t = \phi X_{t-1} + \varepsilon_t$. The continuous lines represent the critical .05 levels of the asymptotic distribution. Notice the stability of the values of both $\hat{D}_{M,n}$ and $\hat{M}_n$. As a comparison, figure 5.3 represents the behaviour of $\hat{Q}_n$ for the same sample. The continuous line is given by the critical .95 χ^2_{m-1} quantiles. For low values of m, $\hat{Q}_n$ rejects the AR(1) null but, as more correlations are added, $\hat{Q}_n$ fails to detect model misspecification.

Figure 5.1 Figure 5.2 Figure 5.3

The guidelines above suggest that choosing the final lag m when using either $\hat{D}_{M,n}$ or $\hat{M}_n$ can be accomplished using a graphical display like figures 5.1 or 5.2. m can be taken as the lag M such that the value of (3.8) or (4.1) is constant for every $m > M$. For example, in the figures above, $M = 11$ for $\hat{D}_{M,n}$ and $M = 7$ for $\hat{M}_n$.

5. FINAL COMMENTS

Testing the adequacy of an ARMA model in the frequency domain has a long tradition in the time series literature. Priestley (1981, sec. 6.2.6) presents an excellent survey of previously proposed spectral based goodness-of-fit tests. Existing methods are generally based on full specification of the spectral distribution function or, in other words, on knowledge of the parameters (ϕ,θ,σ^2).

The latter framework is very restrictive in applications because to check the appropriateness of a model for the given $X_1, \ldots, X_n$, it should be only assumed that the data have been generated by an ARMA(p,q) model with unknown parameters (ϕ,θ,σ^2). Criterions (3.8) and (4.1) overcome this deficiency, since their construction depends on the residuals of the fit of a parametric model. Both are computationally more expensive than the standard Ljung-Box statistic since they require the linear transformation (3.4) that, nevertheless, can be determined using standard software and has an explicit expression for simple models as illustrated in the appendix. In the test statistics $\hat{D}_{M,n}$ and $\hat{M}_n$, m is merely a lag that should be large enough so that the finite sample size distribution is close to its limit that is known to be of the Kolmogorov-Smirnov type. In contrast, the approximating distribution of the Ljung-Box statistic depends on m. As a result, criterions (3.8) and (4.1) are less sensitive to the choice of m and have better power properties.

ACKNOWLEDGEMENTS

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APPENDIX: CONSTRUCTION OF THE MATRIX C_m

Write columnwise $C_m = (\gamma_{1+p+q}, \ldots, \gamma_m)$. Following Velilla (1994), for every integer $m \geq p+q+1$, the $m\times[m-(p+q)]$ matrix C_m must satisfy $C'_m X_m = 0$ and $C'_m C_m = I_{m-(p+q)}$, where $X_m = X_m(\phi,\theta)$ is the $m\times(p+q)$ matrix

$$X_m = \begin{vmatrix} 1 & 0 & \dots & 0 \\ a_1 & 1 & \dots & 0 \\ a_2 & a_1 & \dots & 0 \\ \vdots & \vdots & & 1 \\ a_{m-1} & a_{m-2} & \dots & a_{m-(p+q)} \end{vmatrix}, \tag{A.1}$$

and the coefficients $\{a_k\}_{k\geq 0}$ are as in (2.6). These conditions merely state that the columns of C_m form an orthonormal basis for the orthogonal complement in $\mathbb{R}^m$ of the linear manifold spanned by the columns of X_m. To get weak convergence of $\hat{Z}_n(t)$ to the brownian bridge, a particular basis must be chosen. Specifically, for $1\leq k\leq m-(p+q)$,

$$\gamma_{k+p+q} = \|u_{k+p+q}\|^{-1} \begin{pmatrix} u_{k+p+q} \\ 0_{m-(k+p+q)} \end{pmatrix},$$

where u_{k+p+q} is of $(k+p+q)\times 1$, $0_{m-(k+p+q)}$ is the null vector of $[m-(k+p+q)]\times 1$, and $\|.\|$ is the euclidean length. The $\{u_{k+p+q} : 1\leq k\leq m-(p+q)\}$ are defined as follows:

1. $u_{1+p+q} = (-\beta_{p+q}, -\beta_{p+q-1}, \dots, 1)'$;
2.

$$u_{k+p+q} = \begin{pmatrix} \alpha_{(k-1)+p+q} \\ \hline 1 \end{pmatrix}, \tag{A.2}$$

$2\leq k\leq m-(p+q)$, where $\alpha_{(k-1)+p+q} = -X_{(k-1)+p+q}(X'_{(k-1)+p+q}X_{(k-1)+p+q})^{-1}\xi_{k+p+q} \in \mathbb{R}^{(k-1)+p+q}$ and $\xi_{k+p+q} = (a_{(k-1)+p+q}, \dots, a_k)'$. The explicit expression of C_m for the particular cases of models AR(1) and AR(2) is given below.

AR(1).

For the AR(1) model $X_t - \phi X_{t-1} = \varepsilon_t$ the sequence $\{a_k\}_{k\geq 0}$ is given by $a_k = \phi^k$. Put, for $2\leq k\leq m-1$, $D_{k+1} = \sum_{j=0}^{k-1}\phi^{2j}$ and $A_{k+1} = -\phi^k/D_{k+1}$. The vectors, u_2, ..., u_m are $u_2 = (-\phi,1)'$ and

$$u_{k+1} = \begin{pmatrix} A_{k+1}\begin{pmatrix} 1 \\ \phi \\ \vdots \\ \phi^{k-1} \end{pmatrix} \\ 1 \end{pmatrix}, \qquad (A.3)$$

for $2 \le k \le m-1$. The norms of the u_k are $\|u_2\| = (1 + \phi^2)^{1/2}$ and $\|u_{k+1}\| = (1 - a_k A_{k+1})^{1/2}$ for $2 \le k \le m-1$. ■

AR(2).

For the AR(2) model $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = \varepsilon_t$, the coefficients $\{a_k\}$ are found putting $a_0 = 1$, $a_1 = \phi_1$ and $a_k = \phi_1 a_{k-1} + \phi_2 a_{k-2}$, for $k \ge 2$. Define, for $k \ge 2$, the $k \times 2$ matrices

$$Z_k = \begin{vmatrix} 0 & -1 \\ 1 & -a_1 \\ a_1 & -a_2 \\ \vdots & \vdots \\ a_{k-2} & -a_{k-1} \end{vmatrix},$$

and the quadratic forms $Q_k = (a_k, a_{k-1})Z'_k Z_k (a_k, a_{k-1})' = a_{k-1}^2 + \sum_{j=1}^{k-1} [a_{j-1} a_k - a_j a_{k-1}]^2$. By using an argument of induction on $k \ge 2$, is easy to see:

i) $|X'_k X_k| = \sum_{j=1}^{k-1} Q_j$; *ii)* $(X'_k X_k)^{-1} = Z'_k Z_k / |X'_k X_k|$,

where $Q_1 = 1$, and X_k is as in (A.1) above. Put, for $2 \le k \le m-2$, $D_{k+2} = \sum_{j=1}^{k} Q_j$ and $A_{k+2} = [-a_{k+1}(\sum_{j=0}^{k-1} a_j^2) + a_k(\sum_{j=0}^{k-1} a_j a_{j+1})]/D_{k+2}$, $B_{k+2} = [a_{k+1}(\sum_{j=0}^{k-1} a_j a_{j+1}) - a_k(\sum_{j=0}^{k} a_j^2)]/D_{k+2}$. The vectors $u_3, \dots, u_m$ are $u_3 = (-\phi_2, -\phi_1, 1)'$ and

$$u_{k+2} = \begin{pmatrix} A_{k+2}\begin{pmatrix} 1 \\ a_1 \\ \vdots \\ a_k \end{pmatrix} + B_{k+2}\begin{pmatrix} 0 \\ 1 \\ \vdots \\ a_{k-1} \end{pmatrix} \\ 1 \end{pmatrix}, \qquad (A.4)$$

for $2 \le k \le m-2$. The norms are $\|u_3\| = [1 + \phi_1^2 + \phi_2^2]^{1/2}$ and $\|u_{k+2}\| = (1 - a_{k+1}A_{k+2} - a_k B_{k+2})^{1/2}$ for $2 \le k \le m-2$. ■

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CAPTIONS FOR FIGURES AND TABLES

Figure 1. Tail probability functions of the distribution of $\hat{D}_n$ in models AR(1) and AR(2).

Table 1. Empirical upper quantiles of the distribution of $\hat{D}_n$ under an AR(1) formulation.

Figure 2. Cumulated periodogram of a simulated sample of size n = 200 with standard and correct superimposed bands.

Figure 3. Tail probability functions of the distribution of $\hat{D}_{M,n}$ in models AR(1) and AR(2).

Figure 4. Modified cumulated periodogram of the sample of figure 2.

Table 2. Empirical significance levels and power of criteria $\hat{D}_{n,,KS}$, $D_{n,C}$, $\hat{D}_{M,n}$, $\hat{M}_n$ and $\hat{Q}_n$.

Figure 5. Dependence on the lag m of the goodness-of-fit statistics $\hat{D}_{M,n}$, $\hat{M}_n$ and $\hat{Q}_n$ in a simulated sample of size n = 200.

Figure 1.1

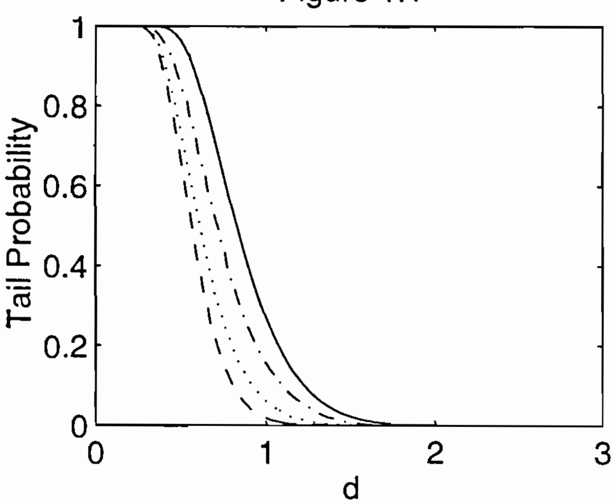


Figure 1.2

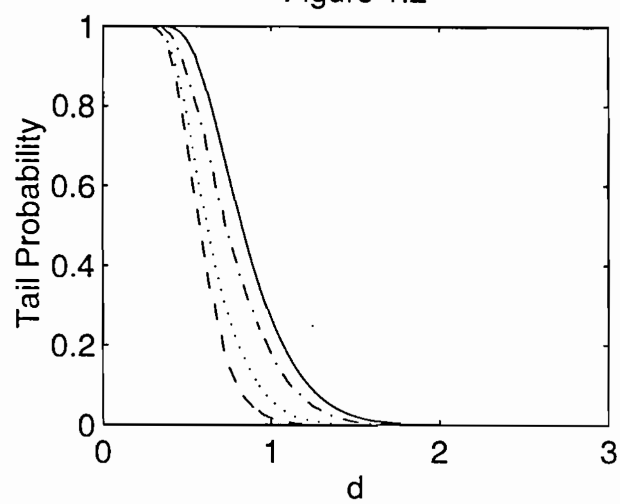


Figure 1.3

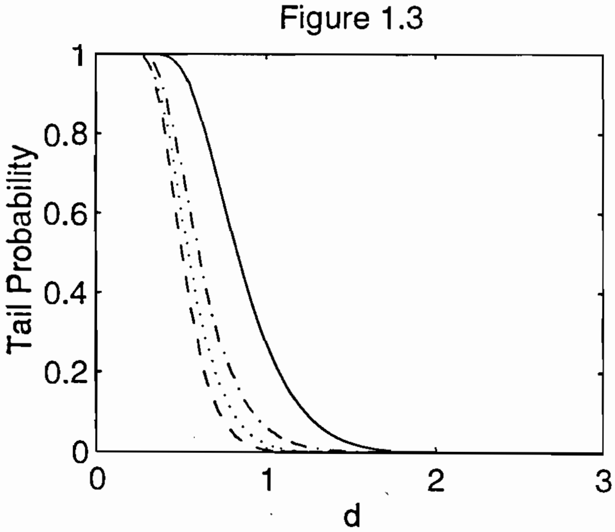
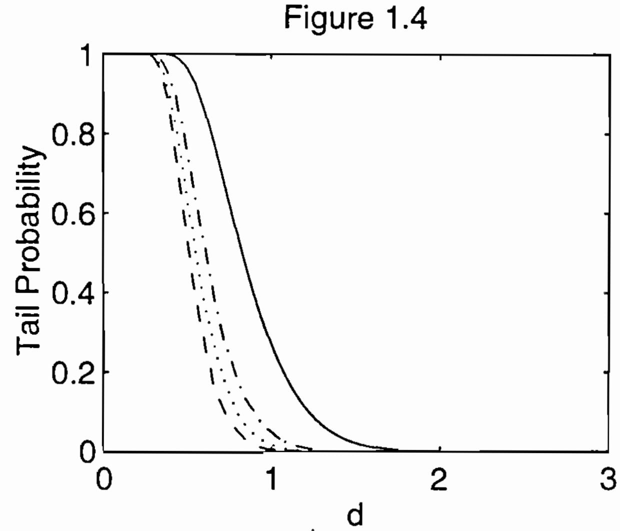


Figure 1.4



	n = 100		
	.90	.95	.99
KS	1.224	1.358	1.628
ϕ = .9	1.097	1.228	1.465
ϕ = .5	.920	1.037	1.253
ϕ = .1	.805	.892	1.075

Table 1.1

	n = 200		
	.90	.95	.99
KS	1.224	1.358	1.628
ϕ = .9	1.114	1.243	1.487
ϕ = .5	.925	1.028	1.250
ϕ = .1	.797	.885	1.056

Table 1.2

Figure 2

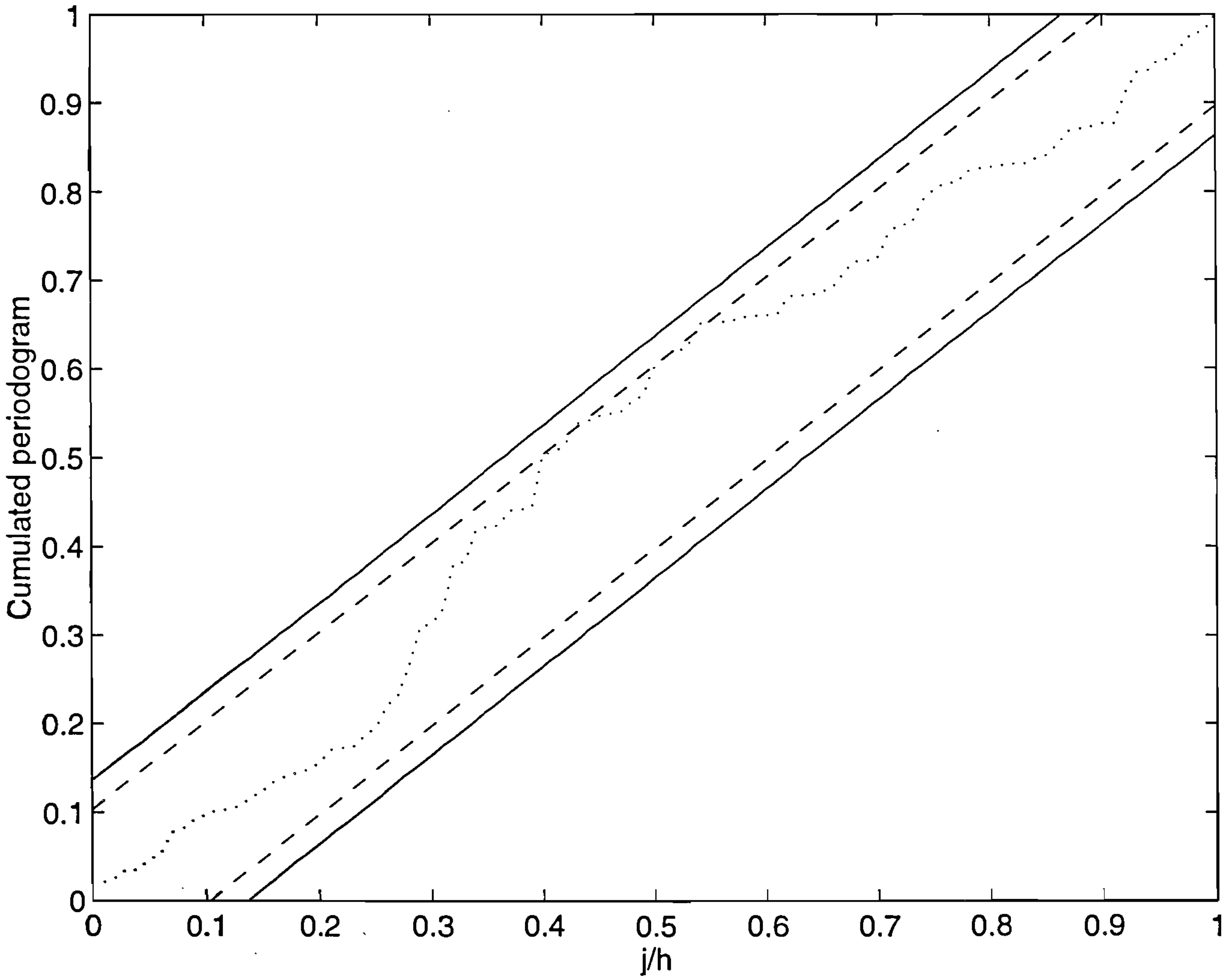


Figure 3.1

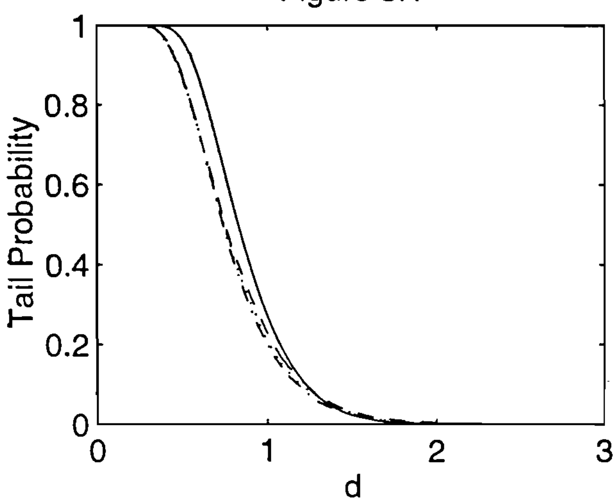


Figure 3.2

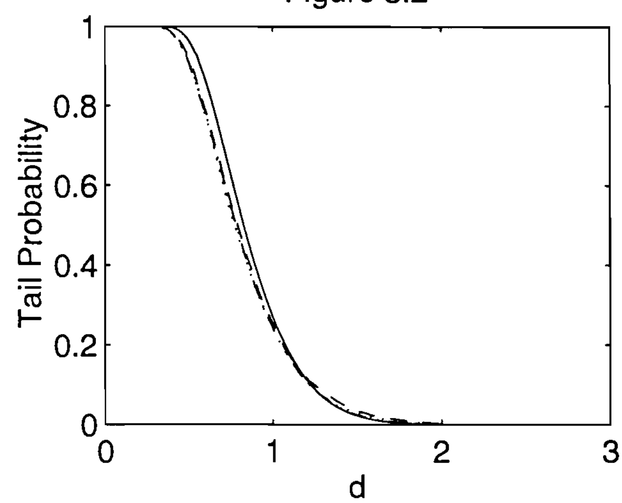


Figure 3.3

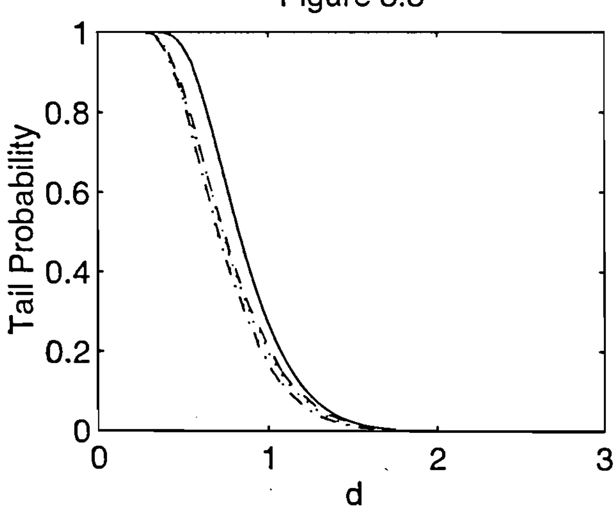


Figure 3.4

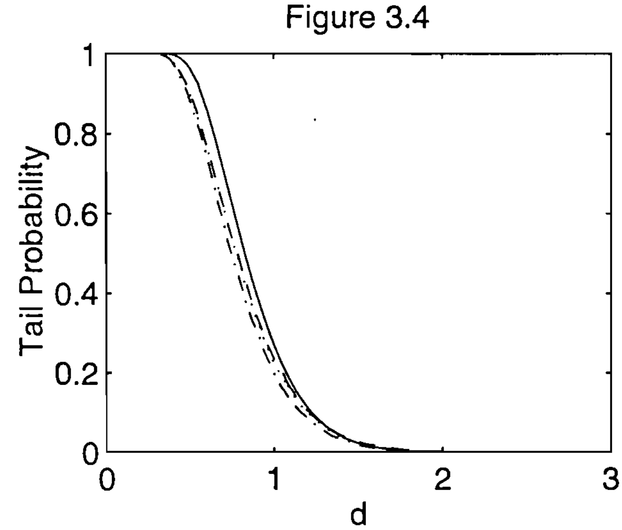
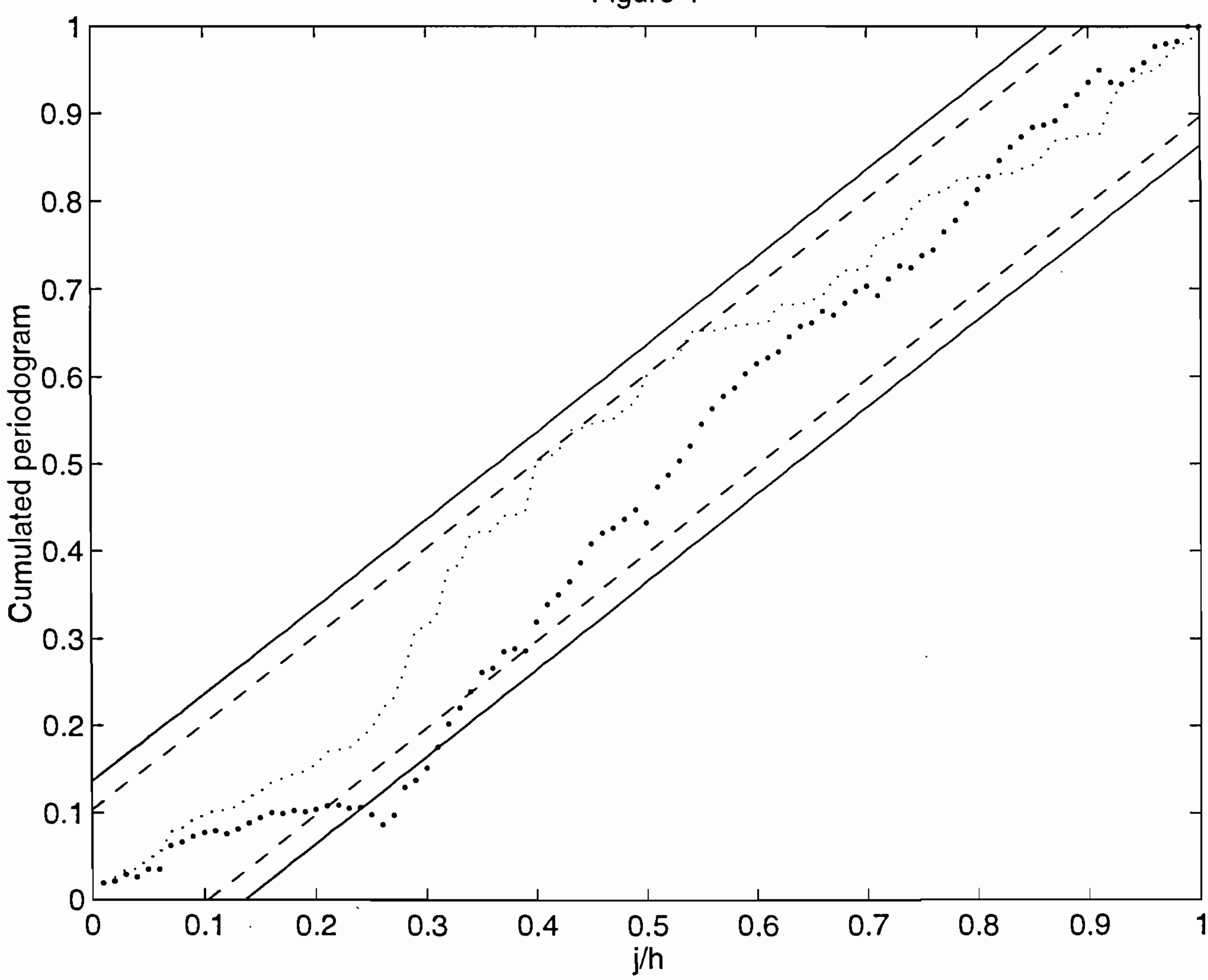


Figure 4



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Cumulated periodogram
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
j/h

n = 150

a	$\hat{D}_{n,KS}$	$\hat{D}_{n,C}$	$\hat{D}_{M,n}$	$\hat{M}_n$	$\hat{Q}_n$
			m = 8		
.0	.0044	.0490	.0590	.0500	.0462
.1	.0156	.1244	.1242	.0720	.0732
.2	.0706	.2834	.2132	.1466	.1178
.4	.4276	.7583	.5032	.4402	.3824
.7	.9629	.9956	.8433	.8271	.8995
.9	.9934	.9992	.8216	.8244	.9762
			m = 10		
.0	.0028	.0442	.0554	.0380	.0462
.1	.0170	.1156	.1130	.0646	.0684
.2	.0672	.2920	.2184	.1564	.1124
.4	.4252	.7557	.4940	.4374	.3494
.7	.9587	.9936	.8319	.8291	.8643
.9	.9918	.9978	.8150	.8260	.9628
			m = 12		
.0	.0034	.0510	.0554	.0436	.0528
.1	.0152	.1252	.1172	.0704	.0666
.2	.0668	.2904	.2116	.1516	.1098
.4	.4392	.7637	.5086	.4558	.3376
.7	.9575	.9958	.8235	.8227	.8371
.9	.9914	.9992	.8178	.8274	.9508
			m = 18		
.0	.0024	.0472	.0496	.0374	.0620
.1	.0118	.1206	.1124	.0666	.0640
.2	.0666	.2858	.2118	.1464	.1050
.4	.4278	.7531	.5028	.4466	.2914
.7	.9623	.9952	.8339	.8321	.7991
.9	.9924	.9990	.8090	.8260	.9244
			m = 25		
.0	.0028	.0446	.0544	.0398	.0662
.1	.0166	.1258	.1208	.0706	.0774
.2	.0682	.2970	.2260	.1590	.1084
.4	.4224	.7615	.5040	.4482	.2706
.7	.9585	.9950	.8245	.8191	.7367
.9	.9914	.9992	.8058	.8352	.8931

Table 2.1

n = 200

a	$\hat{D}_{n,KS}$	$\hat{D}_{n,C}$	$\hat{D}_{M,n}$	$\hat{M}_n$	$\hat{Q}_n$
		m = 10			
.0	.0052	.0540	.0606	.0448	.0496
.1	.0174	.1424	.1310	.0844	.0646
.2	.0946	.3674	.2742	.2114	.1390
.4	.5710	.8583	.6407	.5838	.4750
.7	.9915	.9994	.9407	.9367	.9485
.9	.9992	1.0000	.9538	.9550	.9930
		m = 12			
.0	.0050	.0498	.0538	.0422	.0550
.1	.0122	.1430	.1258	.0774	.0616
.2	.0948	.3586	.2682	.2042	.1302
.4	.5834	.8649	.6369	.5922	.4462
.7	.9925	.9998	.9493	.9429	.9461
.9	.9984	.9998	.9480	.9546	.9906
		m = 15			
.0	.0046	.0518	.0616	.0460	.0518
.1	.0226	.1426	.1308	.0830	.0780
.2	.0956	.3560	.2670	.2022	.1200
.4	.5800	.8697	.6377	.5898	.4074
.7	.9889	.9990	.9445	.9427	.9243
.9	.9986	1.0000	.9480	.9586	.9840
		m = 25			
.0	.0042	.0504	.0542	.0388	.0552
.1	.0192	.1336	.1398	.0828	.0800
.2	.0964	.3654	.2814	.2018	.1170
.4	.5778	.8627	.6417	.5916	.3522
.7	.9889	.9992	.9421	.9373	.8729
.9	.9986	1.0000	.9482	.9576	.9654
		m = 30			
.0	.0034	.0524	.0646	.0414	.0632
.1	.0188	.1470	.1388	.0876	.0756
.2	.0972	.3652	.2664	.2068	.1246
.4	.5736	.8587	.6393	.5924	.3194
.7	.9927	.9994	.9435	.9457	.8495
.9	.9988	.9998	.9402	.9520	.9548

Table 2.2

Figure 5.1

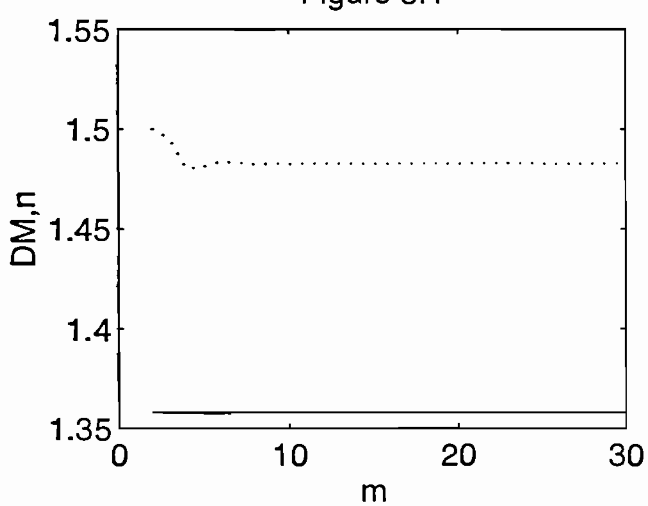


Figure 5.2

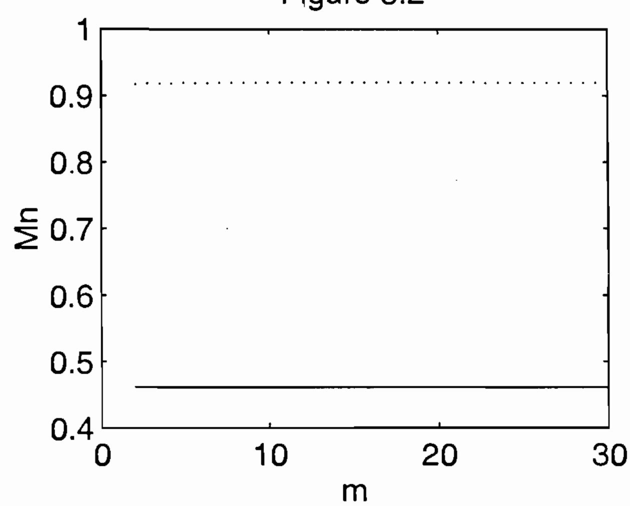
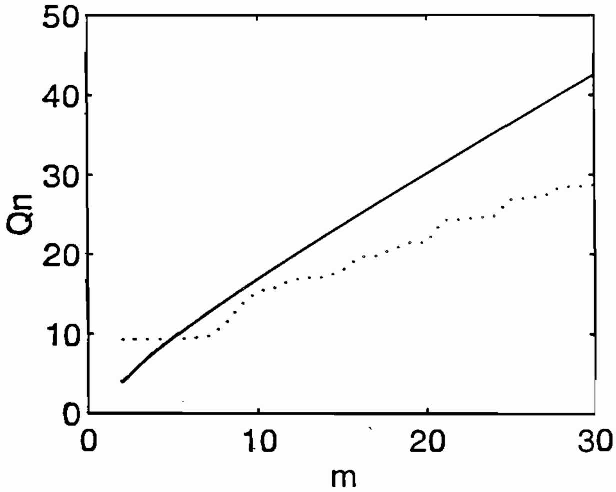


Figure 5.3



n = 400

a	$\hat{D}_{n,KS}$	$\hat{D}_{n,C}$	$\hat{D}_{M,n}$	$\hat{M}_n$	$\hat{Q}_n$
			m = 10		
.0	.0008	.0447	.0590	.0510	.0550
.1	.0122	.0960	.1926	.1352	.0916
.2	.1262	.3922	.4542	.3956	.2294
.4	.8296	.9665	.9211	.9087	.8212
.7	1.0000	1.0000	1.0000	1.0000	1.0000
.9	1.0000	1.0000	1.0000	1.0000	1.0000
			m = 15		
.0	.0006	.0531	.0546	.0444	.0470
.1	.0120	.0978	.1918	.1352	.0804
.2	.1256	.3996	.4566	.3858	.2008
.4	.8248	.9627	.9145	.9045	.7428
.7	1.0000	1.0000	1.0000	.9996	.9995
.9	1.0000	1.0000	1.0000	1.0000	1.0000
			m = 20		
.0	.0012	.0479	.0568	.0460	.0514
.1	.0124	.1038	.2012	.1392	.0786
.2	.1234	.3930	.4406	.3886	.1722
.4	.8460	.9681	.9301	.9103	.6892
.7	1.0000	1.0000	1.0000	1.0000	.9991
.9	1.0000	1.0000	1.0000	1.0000	1.0000
			m = 30		
.0	.0006	.0501	.0574	.0440	.0556
.1	.0104	.1018	.1960	.1328	.0824
.2	.1194	.4022	.4608	.3906	.1634
.4	.8272	.9619	.9237	.9103	.6052
.7	1.0000	1.0000	1.0000	1.0000	.9969
.9	1.0000	1.0000	1.0000	1.0000	1.0000
			m = 40		
.0	.0012	.0493	.0562	.0434	.0588
.1	.0124	.1052	.1938	.1348	.0804
.2	.1298	.4070	.4626	.4008	.1554
.4	.8228	.9617	.9121	.9093	.5462
.7	1.0000	1.0000	1.0000	1.0000	.9917
.9	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2.3