FIRST PASSAGE OF A MARKOV ADDITIVE PROCESS AND GENERALIZED JORDAN CHAINS

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Abstract

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Keywords: Lévy processes; Fluctuation theory; Markov Additive Processes

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Abstract

In this paper we consider the first passage process of a spectrally negative Markov additive process (MAP). The law of this process is uniquely characterized by a certain matrix function, which plays a crucial role in fluctuation theory. We show how to identify this matrix using the theory of Jordan chains associated with analytic matrix functions. This result provides us with a technique, which can be used to derive various further identities.

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1. Introduction

Continuous-time Markov additive processes (MAPs) with one-sided jumps have proven to be an important modelling tool in various application areas, such as communications networking [21, Ch. 6-7] and finance [3, 14]. Over the past decades a vast body of literature has been developed; see for instance [1, Ch. XI] for a collection of

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results. A MAP can be thought of as a Lévy process whose Laplace exponent depends on the state of a (finite-state) Markovian background process (with additional jumps at transition epochs of this background process). It is a non-trivial generalization of the standard Lévy process, with many analogous properties and characteristics, as well as new mathematical objects associated to it, posing new challenges. Any Lévy process is characterized by a Laplace exponent $\psi(\alpha)$; its counterpart for MAPs is the matrix exponent $F(\alpha)$, which is essentially a multi-dimensional analogue of $\psi(\alpha)$.

In this paper we consider the *first passage process* τ_x defined as the first time the process exceeds level x. We concentrate on the case of a *spectrally negative* MAP (that is, all jumps are negative), so that the first passage process is a MAP itself. Knowledge of the matrix exponent of this process, which we in the sequel denote by the matrix function $\Lambda(q)$, is of crucial interest when addressing related fluctuation theory issues. Indeed it can be considered as the multi-dimensional generalization of $-\Phi(q)$, where $\Phi(q)$ is the (one-dimensional) right-inverse of $\psi(\alpha)$, as given in [16, Eqn. (3.15)]. Our main result concerns the identification of the matrix function $\Lambda(q)$ in terms of the matrix exponent $F(\alpha)$ of the original MAP. We provide the Jordan normal form of $\Lambda(q)$ relying on the theory of Jordan chains associated with analytic matrix functions.

The problem of identification of $\Lambda(q)$ received a lot of attention in the literature. It is known that $\Lambda(q)$ is a unique (admissible) solution of a certain matrix integral equation. This result in different degrees of generality appears in [2, 8, 18, 19, 20, 22]. Alternatively, one can use an iterative method to compute $\Lambda(q)$, see for example [6]. Some spectral considerations (under the assumption that $\Lambda(q)$ has distinct eigenvalues) can be found in [2, 19]. It is plausible that an iterative method is preferable if one aims to compute $\Lambda(q)$ numerically. Our result, however, provides a better understanding of how $\Lambda(q)$ is related to $F(\alpha)$, and can be used to prove various further identities. As an example we provide a simple proof of the fact that $\Lambda(q)$ is the unique solution of the above mentioned integral equation.

It has been realized before that the zeros of $\det(F(\alpha))$ with positive real parts and the corresponding null spaces of $F(\alpha)$ play an important role in many problems concerning fluctuations of MAPs, see for example [4, 5, 12]. The problem is that one has to assume that $\det(F(\alpha))$ has a sufficient number of distinct zeros. The number of zeros was determined in [12], and in [13] it was shown that if a MAP is time-reversible

then the zeros are semi-simple, that is, they can be treated as distinct. In general though, this is not the case. This paper provides a final answer to the above problem through the use of generalized Jordan chains associated to $F(\alpha)$. A number of examples is given in the extended version of this paper [7].

This paper is organized as follows. Section 2 reviews some main results from analytic matrix function theory, while in Section 3 we identify the matrix exponent $\Lambda(q)$ by relating the Jordan pairs of the matrix functions $F(\alpha) - q\mathbb{I}$ and $\alpha\mathbb{I} + \Lambda(q)$ for a fixed $q \geq 0$. This result, which is Thm. 1 and which can be considered as the main contribution of our work, is explicit in the sense that it is given in terms of computable quantities associated with $F(\alpha)$. Finally, in Section 4 we discuss applicability of our results.

The remainder of this section is devoted to the definitions of the quantities of interest, with a focus on spectrally negative MAPs and their first passage process. Throughout this work we use bold symbols to denote column vectors unless otherwise specified. In particular, $\mathbf{1}$ and $\mathbf{0}$ are the vectors of 1's and 0's respectively, and $\boldsymbol{e}_i$ is a vector with ith element being 1 and all others being 0.

1.1. Spectrally negative MAP

A MAP is a bivariate Markov process $(X(t), J(t))$ defined as follows. Let $J(\cdot)$ be an irreducible continuous-time Markov chain with finite state space $E = \{1, \dots, N\}$, transition rate matrix $Q = (q_{ij})$ and a (unique) stationary distribution $\boldsymbol{\pi}$. For each state i of $J(\cdot)$ let $X_i(\cdot)$ be a Lévy process with Laplace exponent $\psi_i(\alpha) = \log(\mathbb{E}e^{\alpha X_i(1)})$. Letting T_n and T_{n+1} be two successive transition epochs of $J(\cdot)$, and given that $J(\cdot)$ jumps from state i to state j at T_n, we define the additive process $X(\cdot)$ in the time interval $[T_n, T_{n+1})$ through

$$X(t) = X(T_n-) + U_{ij}^n + [X_j(t) - X_j(T_n)], \tag{1}$$

where (U_{ij}^n) is a sequence of independent and identically distributed random variables with moment generating function

$$\tilde{G}_{ij}(\alpha) = \mathbb{E}e^{\alpha U_{ij}^1}, \quad \text{where} \quad U_{ii}^1 \equiv 0, \tag{2}$$

describing the jumps at transition epochs. To make the MAP spectrally negative, it is required that $U_{ij}^1 \leq 0$ (for all $i, j \in \{1, \dots, N\}$) and that $X_i(\cdot)$ is allowed to have

only negative jumps (for all $i \in \{1, \dots, N\}$). As a consequence, the moment generation functions $\tilde{G}_{ij}(\alpha)$ are well defined for $\alpha \geq 0$.

A Lévy process is called a *downward subordinator* if it has non-increasing paths a.s. We denote the subset of indices of E corresponding to such processes by $E_{\downarrow}$. Let also $E_+ = E \backslash E_{\downarrow}$, $N_{\downarrow} = |E_{\downarrow}|$ and $N_+ = |E_+|$. It is convenient to assume that $E_+ = \{1, \dots, N_+\}$, which we do throughout this work. We use $\boldsymbol{v}_+$ and $\boldsymbol{v}_{\downarrow}$ to denote the restrictions of a vector $\boldsymbol{v}$ to the indices from E_+ and $E_{\downarrow}$ respectively. Finally, in order to exclude trivialities it is assumed that $N_+ > 0$.

Define the matrix $F(\alpha)$ through

$$F(\alpha) = Q \circ \tilde{G}(\alpha) + \text{diag}[\psi_1(\alpha), \dots, \psi_N(\alpha)], \tag{3}$$

where $\tilde{G}(\alpha) = (\tilde{G}_{ij}(\alpha))$; for matrices A and B of the same dimensions we define $A \circ B = (a_{ij} b_{ij})$. One can see that in the absence of positive jumps $F(\alpha)$ is analytic on $\mathbb{C}^{\text{Re} > 0} = \{\alpha \in \mathbb{C} : \text{Re}(\alpha) > 0\}$. Moreover, it is known that

$$\mathbb{E}_i[e^{\alpha X(t)}; J(t) = j] := \mathbb{E}_i[e^{\alpha X(t)} \mathbf{1}_{\{J(t)=j\}}] = (e^{F(\alpha)t})_{ij}, \tag{4}$$

cf. [1, Prop. XI.2.2], where $\mathbb{E}_i(\cdot)$ denotes expectation given that $J(0) = i$. We also write $\mathbb{E}[e^{\alpha X(t)}; J(t)]$ to denote the matrix with ij-th element given in (4). Hence $F(\alpha)$ can be seen as the multi-dimensional analog of a Laplace exponent, defining the law of the MAP. In the following we call $F(\alpha)$ the *matrix exponent* of the MAP $(X(t), J(t))$.

An important quantity associated to a MAP is the *asymptotic drift*:

$$\kappa = \lim_{t \to \infty} \frac{1}{t} X(t) = \sum_i \pi_i \left(\psi_i'(0) + \sum_{j \neq i} q_{ij} \tilde{G}_{ij}'(0) \right), \tag{5}$$

which does not depend on the initial state i of $J(t)$ [1, Cor. XI.2.7]. Finally for $q \geq 0$ we define $F^q(\alpha) = F(\alpha) - q\mathbb{I}$, with $\mathbb{I}$ being the identity matrix, which can be seen as the matrix exponent of the MAP 'killed' at random time e_q:

$$\mathbb{E}[e^{\alpha X(t)}; t < e_q, J(t)] = e^{(F(\alpha) - q\mathbb{I})t}, \tag{6}$$

where e_q is an exponential random variable of rate q independent of everything else and $e_0 \equiv \infty$ by convention.

1.2. First passage process

Define the first passage time over level $x > 0$ for the (possibly killed) process $X(t)$ as

$$\tau_x = \inf\{t \geq 0 : X(t) > x\}. \tag{7}$$

It is known that on $\{J(\tau_x) = i\}$ the process $(X(t + \tau_x) - X(\tau_x), J(t + \tau_x)), t \geq 0$ is independent from $(X(t), J(t)), t \in [0, \tau_x]$ and has the same law as the original process under $\mathbb{P}_i$. Therefore, in the absence of positive jumps the time-changed process $J(\tau_x)$ is a time-homogeneous Markov process and hence is a Markov chain. Letting $\{\partial\}$ be an absorbing state corresponding to $J(\infty)$, we note that $J(\tau_x)$ lives on $E_+ \cup \{\partial\}$, because $X(t)$ can not hit new maximum when $J(t)$ is in a state corresponding to a downward subordinator; see also [17]. Let $\Lambda(q)$ be the $N_+ \times N_+$ dimensional transition rate matrix of $J(\tau_x)$ restricted to E_+, that is

$$\mathbb{P}(J(\tau_x) = j, \tau_x < e_q \mid J(\tau_0) = i) = (e^{\Lambda(q)x})_{ij}, \text{ where } i, j \in E_+. \tag{8}$$

This, in fact, shows that the first passage process $(\tau_x, J(\tau_x)), x \geq 0$ is a MAP itself, and $\Lambda(-q)$ is its matrix exponent: $\mathbb{E}_{J(\tau_0)}[e^{-q\tau_x}; J(\tau_x)] = e^{\Lambda(q)x}$.

Another matrix of interest is $N \times N_+$ matrix $\Pi(q)$ defined by

$$\Pi(q)_{ij} = \mathbb{P}_i(J(\tau_0) = j, \tau_0 < e_q), \text{ where } i \in E \text{ and } j \in E_+. \tag{9}$$

This matrix specifies initial distributions of the time-changed Markov chain $J(\tau_x)$, so that $\mathbb{E}[e^{-q\tau_x}; J(\tau_x)] = \Pi(q)e^{\Lambda(q)x}$. Note also that $\Pi(q)$ restricted to the rows in E_+ is the identity matrix, because $\tau_0 = 0$ a.s. when $J(0) \in E_+$ [16, Thm. 6.5]. We note that the case of $q = 0$ is a special case corresponding to no killing. In order to simplify notation we often write Λ and Π instead of $\Lambda(0)$ and $\Pi(0)$.

It is noted that if $q > 0$ or $q = 0, \kappa < 0$ then $\Lambda(q)$ is a transient transition rate matrix: $\Lambda(q)\mathbf{1}_+ \leq \mathbf{0}_+$, with at least one strict inequality. If, however, $\kappa \geq 0$, then Λ is a recurrent transition rate matrix: $\Lambda\mathbf{1}_+ = \mathbf{0}_+$; also $\Pi\mathbf{1}_+ = \mathbf{1}$. These statements follow trivially from [1, Prop. XI.2.10]. Finally, note that Λ is an irreducible matrix, because so is Q. Hence if Λ is recurrent then by Perron-Frobenius theory [1, Thm. I.6.5] the eigenvalue 0 is simple, because it is the eigenvalue with maximal real part.

It is instructive to consider the 'degenerate' MAP, i.e., the one with dimension $N = 1$. Such a MAP is just a Lévy process, and $\Lambda(q) = -\Phi(q)$, where $\Phi(q)$ is the

right-inverse of $\psi(\alpha), \alpha \geq 0$. Note also that Λ being recurrent (and hence singular) corresponds to $\Phi(0) = 0$.

2. Preliminaries

In this section we review some basic facts from analytic matrix function theory. Let $A(z)$ be an analytic matrix function ($n \times n$ dimensional), defined on some domain $D \subset \mathbb{C}$, where it is assumed that $\det(A(z))$ is not identically zero on this domain. For any $\lambda \in D$ we can write

$$A(z) = \sum_{i=0}^{\infty} \frac{1}{i!} A^{(i)}(\lambda)(z - \lambda)^i, \tag{10}$$

where $A^{(i)}(\lambda)$ denotes the i-th derivative of $A(z)$ at λ. We say that λ is an *eigenvalue* of $A(z)$ if $\det(A(\lambda)) = 0$.

Definition 1. We say that vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1} \in \mathbb{C}^n$ with $\boldsymbol{v}_0 \neq \boldsymbol{0}$ form a *Jordan chain* of $A(z)$ corresponding to the eigenvalue λ if

$$\sum_{i=0}^{j} \frac{1}{i!} A^{(i)}(\lambda) \boldsymbol{v}_{j-i} = \boldsymbol{0}, \qquad j = 0, \ldots, r-1. \tag{11}$$

Note that this definition is a generalization of the well-known notion of a Jordan chain for a square matrix A. In this classical case $A(z) = z\mathbb{I} - A$, and (11) reduces to

$$A\boldsymbol{v}_0 = \lambda \boldsymbol{v}_0, \quad A\boldsymbol{v}_1 = \lambda \boldsymbol{v}_1 + \boldsymbol{v}_0, \quad \ldots, \quad A\boldsymbol{v}_{r-1} = \lambda \boldsymbol{v}_{r-1} + \boldsymbol{v}_{r-2}. \tag{12}$$

The following result is well known [11] and is an immediate consequence of (12).

Proposition 1. *Let $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1}$ be a Jordan chain of $A(z)$ corresponding to the eigenvalue λ, and let $C(z)$ be $m \times n$ dimensional matrix. If $B(z) = C(z)A(z)$ is $r - 1$ times differentiable at λ, then*

$$\sum_{i=0}^{j} \frac{1}{i!} B^{(i)}(\lambda) \boldsymbol{v}_{j-i} = \boldsymbol{0}, \qquad j = 0, \ldots, r-1. \tag{13}$$

Note that if $B(z)$ is a square matrix then $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1}$ is a Jordan chain of $B(z)$ corresponding to the eigenvalue λ. It is, however, not required that $C(z)$ and $B(z)$ be square matrices.

Let m be the multiplicity of λ as a zero of $\det(A(z))$ and p be the dimension of the null space of $A(\lambda) = A_0$. It is known, see e.g. [11], that there exists a *canonical system of Jordan chains* corresponding to λ

$$\boldsymbol{v}_0^{(k)}, \boldsymbol{v}_1^{(k)}, \ldots, \boldsymbol{v}_{r_k-1}^{(k)}, \qquad k = 1, \ldots, p, \tag{14}$$

such that the vectors $\boldsymbol{v}_0^{(1)}, \ldots, \boldsymbol{v}_0^{(p)}$ form the basis of the null space of A_0 and $\sum_{i=1}^p r_i = m$. We write such a canonical system of Jordan chains in matrix form:

$$V = [\boldsymbol{v}_0^{(1)}, \boldsymbol{v}_1^{(1)}, \ldots, \boldsymbol{v}_{r_1-1}^{(1)}, \ldots, \boldsymbol{v}_0^{(p)}, \boldsymbol{v}_1^{(p)}, \ldots, \boldsymbol{v}_{r_p-1}^{(p)}], \quad \Gamma = \mathrm{diag}[\Gamma^{(1)}, \ldots, \Gamma^{(p)}], \tag{15}$$

where $\Gamma^{(i)}$ is the Jordan block of size $r_i \times r_i$ with eigenvalue λ, i.e. a square matrix having zeros everywhere except along the diagonal, whose elements are equal to λ, and the superdiagonal, whose elements are equal to 1.

Definition 2. A pair of matrices (V, Γ) given by (15) is called a *Jordan pair* of $A(z)$ corresponding to the eigenvalue λ.

We note that, unlike in the classical case, the vectors forming a Jordan chain are *not* necessarily linearly independent; furthermore a Jordan chain may contain a null vector.

We conclude this section with a result on entire functions of matrices defined through

$$f(M) = \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(0) M^i, \tag{16}$$

for an entire function $f : \mathbb{C} \to \mathbb{C}$ and a square matrix M. The next lemma will be important for applications.

Lemma 1. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let Γ be a Jordan block of size k with λ on the diagonal, then for an arbitrary set of vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{k-1}$ the $(j+1)$-st column of the matrix $[\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{k-1}] f(\Gamma)$ equals*

$$\sum_{i=0}^{j} \frac{1}{i!} f^{(i)}(\lambda) \boldsymbol{v}_{j-i}, \tag{17}$$

where $j = 0, \ldots, k-1$.

Proof. Immediate from [9, Thm. 6.6].

3. Jordan normal form of $\Lambda(q)$

In this section we consider a spectrally negative MAP $(X(t), J(t))$ with matrix exponent $F(\alpha)$ and asymptotic drift κ. Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of $F^q(\alpha)$, to be understood as the zeros of $\det(F^q(\alpha))$, for a given $q \geq 0$, in its region of analyticity $\mathbb{C}^{\mathrm{Re}>0}$. Let (V_i, Γ_i) be a Jordan pair corresponding to the eigenvalue λ_i. Define the matrices V and Γ in the following way:

$$
\begin{aligned}
V &= [V_1, \ldots, V_k] \\
\Gamma &= \mathrm{diag}[\Gamma_1, \ldots, \Gamma_k] \qquad \text{if } q > 0 \text{ or } q = 0, \kappa < 0; \\[2mm]
V &= [\mathbf{1}, V_1, \ldots, V_k] \\
\Gamma &= \mathrm{diag}[0, \Gamma_1, \ldots, \Gamma_k] \qquad \text{if } q = 0, \kappa \geq 0.
\end{aligned}
\tag{18}
$$

and let the matrices V_+ and $V_\downarrow$ be the restrictions of the matrix V to the rows corresponding to E_+ and $E_\downarrow$ respectively.

Theorem 1. *It holds that Γ and V_+ are $N_+ \times N_+$-dimensional matrices, V_+ is invertible, and*

$$
\Lambda(q) = -V_+ \Gamma V_+^{-1}, \qquad \Pi(q) = V V_+^{-1}.
\tag{19}
$$

We start by establishing a lemma, which can be considered as a weak analog of Thm. 1.

Lemma 2. *If $\boldsymbol{v}^0, \ldots, \boldsymbol{v}^{r-1}$ is a Jordan chain of $F^q(\alpha)$ corresponding to the eigenvalue $\lambda \in \mathbb{C}^{\mathrm{Re}>0}$ then $\boldsymbol{v}_+^0, \ldots, \boldsymbol{v}_+^{r-1}$ is a Jordan chain of $\alpha \mathbb{I} + \Lambda(q)$ corresponding to the eigenvalue $\alpha = \lambda$ and $\Pi(q)\boldsymbol{v}_+^i = \boldsymbol{v}^i$ for $i = 0, \ldots, r-1$.*

Proof. It is known from [4, Thm. 2.1] that for $\alpha \in \mathbb{C}^{\mathrm{Re}>0}$

$$
M_\alpha(t) = \left[\int_0^t e^{\alpha X(s)} \boldsymbol{e}'_{J(s)} \mathrm{d}s \right] \cdot F(\alpha) + \boldsymbol{e}'_k - e^{\alpha X(t)} \boldsymbol{e}'_{J(t)},
$$

is a row vector valued zero mean martingale under the probability measure $\mathbb{P}_k$, where ''' denotes the transposition operation. Apply the optional sampling theorem to $M_\alpha(\cdot)$ with the finite stopping time $t \wedge \tau_x \wedge e_q$ and note that

$$
\mathbb{E}_k \left[e^{\alpha X(e_q)} 1_{\{t \wedge \tau_x > e_q\}} \boldsymbol{e}'_{J(e_q)} \right] = q \mathbb{E}_k \left[\int_0^{t \wedge \tau_x \wedge e_q} e^{\alpha X(s)} \boldsymbol{e}'_{J(s)} \mathrm{d}s \right],
$$

to obtain

$$
C(\alpha) F^q(\alpha) = B(\alpha),
\tag{20}
$$

where

$$C(\alpha) = \mathbb{E}_k\left[\int_0^{t\wedge\tau_x\wedge e_q} e^{\alpha X(s)}\boldsymbol{e}'_{J(s)}\mathrm{d}s\right], \quad B(\alpha) = \mathbb{E}_k\left[e^{\alpha X(t\wedge\tau_x)}1_{\{t\wedge\tau_x<e_q\}}\boldsymbol{e}'_{J(t\wedge\tau_x)}\right] - \boldsymbol{e}'_k.$$

Noting that $X(\cdot) \le x$ on $[0, \tau_x]$ and using usual dominated convergence argument we conclude that $B(\alpha)$ is infinitely differentiable in $\alpha \in \mathbb{C}^{\mathrm{Re}>0}$. Apply Prop. 1 to (20) to see that for all $j = 0, \ldots, r-1$ the following holds true:

$$\sum_{i=0}^j \frac{1}{i!}\mathbb{E}_k\left[X^i(t\wedge\tau_x)e^{\lambda X(t\wedge\tau_x)}1_{\{t\wedge\tau_x<e_q\}}\boldsymbol{e}'_{J(t\wedge\tau_x)}\right]\boldsymbol{v}^{j-i} - \boldsymbol{e}'_k\boldsymbol{v}^j = 0.$$

Letting $t \to \infty$ we obtain

$$\sum_{i=0}^j \frac{1}{i!}x^i e^{\lambda x}\mathbb{P}_k(J(\tau_x), \tau_x < e_q)\boldsymbol{v}^{j-i} - \boldsymbol{e}'_k\boldsymbol{v}^j = 0, \tag{21}$$

where $\mathbb{P}_k(J(\tau_x), \tau_x < e_q)$ denotes a row vector with ℓ-th element given by $\mathbb{P}_k(J(\tau_x) = \ell, \tau_x < e_q)$. Note that the case when $q = 0$ and $\mathbb{P}_k(\tau_x = \infty) > 0$ should be treated with care. In this case $\kappa < 0$ and thus $\lim_{t\to\infty} X(t) = -\infty$ a.s. [1, Prop. XI.2.10], so the above limit is still valid.

Considering (21) for all $k \in E$ and choosing $x = 0$ we indeed obtain $\Pi(q)\boldsymbol{v}_+^j = \boldsymbol{v}^j$. If, however, we pick $k \in E_+$, then

$$\sum_{i=0}^j \frac{1}{i!}x^i e^{(\lambda\mathbb{I}+\Lambda(q))x}\boldsymbol{v}_+^{j-i} - \boldsymbol{v}_+^j = \boldsymbol{0}_+. \tag{22}$$

Take the right derivative in x at 0 of both sides to see that

$$(\lambda\mathbb{I} + \Lambda(q))\boldsymbol{v}_+^j + \boldsymbol{v}_+^{j-1} = \boldsymbol{0}_+, \tag{23}$$

which shows that $\boldsymbol{v}_+^0, \ldots, \boldsymbol{v}_+^{r-1}$ is a Jordan chain of $\alpha\mathbb{I} + \Lambda(q)$ corresponding to the eigenvalue λ.

We are now ready to give a proof of our main result, Thm. 1.

Proof of Thm. 1. Lemma 2 states that $\boldsymbol{v}_+^0, \ldots, \boldsymbol{v}_+^{r-1}$ is a classical Jordan chain of the matrix $-\Lambda(q)$. Recall that if $q = 0, \kappa \ge 0$ then $\Lambda(q)\boldsymbol{1}_+ = \boldsymbol{0}_+$ and $\Pi(q)\boldsymbol{1}_+ = \boldsymbol{1}$. Therefore the columns of V_+ are linearly independent [10, Prop. 1.3.4] and

$$-\Lambda(q)V_+ = V_+\Gamma, \qquad \Pi(q)V_+ = V. \tag{24}$$

Consider the case when $q > 0$. Now [12, Thm. 1] states that $\det(F^q(\alpha))$ has N_+ zeros (counting multiplicities) in $\mathbb{C}^{\mathrm{Re}>0}$; see also [12, Rem. 1.2], so the matrices V_+ and Γ are of size $N_+ \times N_+$ by construction (18). Note there is one-to-one correspondence between the zeros of $\det(F^q(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$ and the eigenvalues of $-\Lambda(q)$ when $q > 0$.

Assume now that $q = 0$. We only need to show that $\det(F(\alpha))$ has $N_+ - 1_{\{\kappa \geq 0\}}$ zeros (counting multiplicities) in $\mathbb{C}^{\mathrm{Re}>0}$. Pick a sequence of q_n converging to 0 and consider a sequence of matrix exponents $F^{q_n}(\alpha) = F(\alpha) - q_n \mathbb{I}$ and transition rate matrices $\Lambda(q_n)$. From (8) it follows that $e^{\Lambda(q_n)} \to e^{\Lambda}$, hence the eigenvalues of $\Lambda(q_n)$ converge to the eigenvalues of Λ (preserving multiplicities) as $n \to \infty$. Moreover, all the eigenvalues of Λ have negative real part except a simple one at 0 if $\kappa \geq 0$. The above mentioned one-to-one correspondence and the convergence statement of [12, Thm. 9] conclude the proof.

The above proof strengthens [12, Thm. 2]; we remove the assumption that κ is non-zero and finite.

Corollary 1. *It holds that* $\det(F(\alpha))$ *has* $N_+ - 1_{\{\kappa \geq 0\}}$ *zeros (counting multiplicities) in* $\mathbb{C}^{\mathrm{Re}>0}$.

4. Applications

A number of applications of our result is discussed in detail in the extended version of this paper [7]. These applications include finding the stationary distributions of a one-sided MAP reflected at 0, and a Markov-modulated Brownian motion (MMBM) reflected to stay in the strip $[0, B]$. Moreover, we solve two-sided exit problem for a MMBM. It is noted that MMBM is a MAP with continuous paths and hence our result can be applied to both $X(t)$ and $-X(t)$, which hints why the two-sided problems become tractable.

The approach to the above problems consists of the following steps:

- use a martingale argument to arrive at an initial equation involving the unknown quantities and $F(\alpha)$,

- use the properties of Jordan chains such as Prop. 1 and Lemma 1 to rewrite the initial equation in terms of (V, Γ),

- use the special structure of (V, Γ), such as invertibility of V, to simplify the equation,

- eliminate the Jordan pair by introducing Λ and Π using Thm. 1 to recover the probabilistic interpretation of the involved matrices and claim uniqueness of the solution.

It is noted that this approach can be seen as an extension of the ideas known as 'martingale calculations for MAPs' [1, Ch. XI, 4a] to its final and general form. It is important that no assumptions about the number and simplicity of the eigenvalues are needed. For some problems certain eigenvalues are inherently non-simple. For example, the special but important case of a MMBM with zero drift immediately leads to a non-simple eigenvalue 0. In this case an additional equation associated to the null Jordan chain is required to obtain the solution, see also [22]. In our framework, this equation comes out in a natural way.

4.1. Matrix integral equation

In this section we demonstrate how our result can be used to show in a simple way that a pair $(\Pi(q), \Lambda(q))$ is a unique solution of a certain matrix integral equation. This equation under different assumptions appears in [2, 8, 18, 19, 20, 22].

Define two sets of matrices: let $\mathcal{M}$ be a set of all $N_+ \times N_+$ irreducible transition rate matrices, and $\mathcal{P}$ be a set of $N \times N_+$ matrices P satisfying $P_+ = \mathbb{I}$. Furthermore, partition $\mathcal{M}$ into two disjoint sets $\mathcal{M}_0$ and $\mathcal{M}_1$ of transient and recurrent matrices respectively. Clearly, $\Lambda(q) \in \mathcal{M}_i, i = 1_{\{q=0, \kappa \geq 0\}}$ and $\Pi(q) \in \mathcal{P}$. We use the following notation for arbitrary matrices $P \in \mathcal{P}$ and $-M \in \mathcal{M}$

$$
\begin{aligned}
F^q(P, M) &= \Delta_{\boldsymbol{a}} \, P M + \frac{1}{2} \Delta_{\boldsymbol{\sigma}}^2 \, P M^2 + \int_{-\infty}^0 \Delta_{\boldsymbol{\nu}}(\mathrm{d}x) P \left(e^{Mx} - \mathbb{I} - Mx 1_{\{x > -1\}} \right) \\
&\quad + \int_{-\infty}^0 Q \circ G(\mathrm{d}x) \, P \, e^{Mx} - qP,
\end{aligned}
\tag{25}
$$

where $(a_i, \sigma_i, \nu_i(\mathrm{d}x))$ are the Lévy triplets corresponding to the Lévy processes $X_i(\cdot)$, that is $\psi_i(\alpha) = a_i \alpha + \sigma_i^2 / 2\alpha^2 + \int_{-\infty}^0 (e^{\alpha x} - 1 - \alpha x 1_{\{x > -1\}}) \nu_i(\mathrm{d}x)$, $G_{ij}(\mathrm{d}x)$ is the distribution of U_{ij}. It will be clear from the following that the integrals converge for the above choice of P and M.

Theorem 2. $(\Pi(q), \Lambda(q))$ *is the unique pair* $(P, M) \in \mathcal{P} \times \mathcal{M}_i$ *with* $i = 1_{\{q=0, \kappa \geq 0\}}$ *which satisfies* $F^q(P, -M) = \mathbb{O}_+$.

Proof. In the proof we drop the superscript q to simplify notation. Let $-M = V_+ \Gamma V_+^{-1}$ be a Jordan decomposition of the matrix $-M$. One can extend V_+ to a $N \times N_+$ matrix V through $V = P V_+$, because $P_+ = \mathbb{I}$. Let also $\boldsymbol{v}_+^0, \ldots, \boldsymbol{v}_+^{r-1}$ be the columns of V corresponding to some Jordan block of size r and eigenvalue λ. Observe that $\lambda \in \mathbb{C}^{\mathrm{Re}>0}$ or $\lambda = 0$ in which case it must be simple, because $M \in \mathcal{M}$. Note that $g(-M) = V_+ g(\Gamma) V_+^{-1}$ for an entire function $g : \mathbb{C} \to \mathbb{C}$, and use Lemma 1 to see that the column of $F(P, -M) V_+$ corresponding to $\boldsymbol{v}_+^j$, $j = 0, \ldots, r-1$, equals

$$\sum_{i=0}^{j} \frac{1}{i!} F^{(i)}(\lambda) P \boldsymbol{v}_+^{j-i} = \sum_{i=0}^{j} \frac{1}{i!} F^{(i)}(\lambda) \boldsymbol{v}^{j-i}. \tag{26}$$

We also used the fact that differentiation of $F(\alpha)$ at λ, $\mathrm{Re}(\lambda) > 0$, can be done under the integral signs and no differentiation is needed for a simple eigenvalue $\lambda = 0$ if such exists.

If $M = \Lambda$ and $P = \Pi$, then according to Thm. 1 the matrices V and Γ can be chosen as in (18). Hence (26) becomes $\boldsymbol{0}_+$, because $\boldsymbol{v}^0, \ldots, \boldsymbol{v}^{r-1}$ is a Jordan chain of $F(\alpha)$, see (11). But V_+ is an invertible matrix and so $F(\Pi, -\Lambda) = \mathbb{O}_+$.

Suppose now that $F(P, -M) = \mathbb{O}_+$ with $M \in \mathcal{M}_i$ and $P \in \mathcal{P}$. Then the vectors $\boldsymbol{v}^0, \ldots, \boldsymbol{v}^{r-1}$ form a Jordan chain of $F(\alpha)$ corresponding to an eigenvalue $\lambda \in \mathbb{C}^{\mathrm{Re}>0}$ or $\lambda = 0$. If $q = 0, \kappa \geq 0$ and $\lambda = 0$, which is a simple eigenvalue of M, then $F(0)\boldsymbol{v}^0 = Q\boldsymbol{v}^0 = \boldsymbol{0}$ implies $\boldsymbol{v}^0 = c\boldsymbol{1}$, where $c \neq 0$ is a constant. Combining this observation and Lemma 2 we obtain $\Lambda V_+ = -V_+ \Gamma$ and $\Pi V_+ = V$, and hence $M = \Lambda, P = \Pi$.

From the above theorem we immediately get the following corollaries.

Corollary 2. *If* $N = N_+$, *then* $M = \Lambda(q)$ *is the unique solution of* $F^q(\mathbb{I}, -M) = \mathbb{O}$, *where* $M \in \mathcal{M}_i$, $i = 1_{\{q=0, \kappa \geq 0\}}$.

For the case of a MMBM, i.e. a continuous MAP, we obtain a generalization of the result in [22] and [2].

Corollary 3. *If* $(X(t), J(t))$ *is a MMBM then* $(\Pi(q), \Lambda(q))$ *is the unique pair* $(P, M) \in \mathcal{P} \times \mathcal{M}_i$ *with* $i = 1_{\{q=0, \kappa \geq 0\}}$ *which satisfies*

$$\frac{1}{2} \Delta_{\boldsymbol{\sigma}}^2 \, P \, M^2 - \Delta_{\boldsymbol{a}} \, P \, M + (Q - q\mathbb{I})P = \mathbb{O}_+. \tag{27}$$

Acknowledgements

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