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COOPERATIVE PRODUCTION AND EFFICIENCY*

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Abstract

We characterize the sharing rule for which a contribution mechanism achieves efficiency in a cooperative production setting when agents are heterogeneous. The sharing rule bears no resemblance to those considered by the previous literature. We also show for a large class of sharing rules that if Nash equilibrium yields efficient allocations, the production function displays constant returns to scale, a case in which cooperation in production is useless.

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1. Introduction

The Cooperative Production problem arises when n agents use a commonly owned technology to transform inputs into outputs. Output is distributed by means of a *sharing rule,* a function yielding consumption of each agent as a function of inputs. In "classical" economies there are allocations that are efficient and belong to any given continuous sharing rule (Corchón and Puy (2002), Proposition 1). Also, there is a mechanism that implements these allocations in Nash equilibrium (Corchón and Puy (2002), Proposition 2, Nandeibam (2003)). The implementing mechanism, however, is complicated so in this note we examine the performance of a natural mechanism in which each agent decides her own input contribution and receives the consumption dictated by the sharing rule. Holmstrom (1982) and Fabella (1988) showed in two special cases that such a mechanism does not yield efficient allocations as Nash Equilibria.[1] Sen (1966) showed that a particular mix of the egalitarian and the proportional sharing rules achieves efficiency when all agents are identical. However, when agents are heterogeneous, Browning (1983) showed that the natural mechanism described above achieves efficiency only when the production function fulfills a separability property. A result from Gradstein (1995) can be generalized to our framework to show that the natural mechanism achieves efficiency only when the production function is a polynomial of, at most, degree $n-1$.

In this paper we delve into the kind of sharing rules for which the natural mechanism achieves efficiency when the domain of admissible preferences is large enough. We first characterize the sharing rule for which the natural mechanism yields efficient allocations (Proposition 1 and Remark 1). We call this rule the *Incremental Sharing Rule*. We show that the incremental sharing rule yields non negative returns in two cases: when the production function is a polynomial with all coefficients but one are

[1] Both assume utility functions quasi-linear in consumption. Holmstrom considers only sharing rules which depend on aggregate output and Fabella considers only the proportional sharing rule.

negative (Proposition 2) or when all coefficients but two are positive (Proposition 3). Unfortunately, we have been unable to guarantee non negativity in general. Define the incremental consumption of an agent as the difference between the consumption when this agent works and the consumption when he does not work. We show that the incremental consumption yielded by the incremental sharing rule is bounded by the incremental consumption yielded by the proportional and the egalitarian sharing rules (Proposition 4). Finally, we show for a class of sharing rules, which to the best of our knowledge, includes all of those used by the literature, that if Nash equilibrium yields efficient allocations, the production function displays constant returns to scale, a case in which cooperation in production is useless (Proposition 5). Thus, implementation of sharing rules in this class requires a different mechanism from the one considered here, possibly, a complex one. This implies that Sen´s result is an artifact of his assumption that agents are identical.

2. The Model and the Results

We have n agents that supply labor denoted by l_i, $i \in N = \{1, 2, ...n\}$. Let $l \equiv (l_1, l_2, ..., l_n)$, $l_{-i} \equiv (l_1, ..., l_{i-1}, l_{i+1}, ..., l_n)$, $\ell \equiv \sum_{j=1}^{n} l_j$, $\ell_{-i} \equiv \sum_{j \neq i} l_j$, $\ell_{-ik} = \sum_{j \neq i, j \neq k} l_j$, and so on. There is a maximum quantity of labor that any agent can supply, $\bar{l}$.

Agents share a technology that is able to generate a consumption good whose production function is written as $X(\ell)$. It is assumed to be concave, increasing and differentiable in $[0, n\bar{l}]$ with $X(0) = 0$. The production function displays Constant Returns to Scale if $X(\ell) = a\ell$, $a > 0$.

Let x_i be the consumption of i and $x \equiv (x_1, x_2, ..., x_n)$. The pair (x, l) is an *allocation.* An allocation (x, l) is feasible if $\sum_{i=1}^{n} x_i = X(\ell)$ and $0 \leq l_i \leq \bar{l}$, $i \in N$. The set of feasible allocations is denoted by A.

Each agent, say i, has preferences over consumption and labor representable by a strictly concave and differentiable utility function $U_i = U_i(x_i, l_i)$ which is strictly increasing (resp. decreasing) in the first (second) argument.

Efficient allocations are found by

$$\max \sum_{i=1}^{n} \alpha_i U_i(x_i, l_i) \text{ with } (x, l) \in A \tag{2.1}$$

for given $(\alpha_1, \alpha_2, ..., \alpha_n)$ with $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. This is the maximization of a continuous function over a compact set and, hence, it has a solution by Weierestrass theorem. The program is concave and thus first order conditions gives the maximum. Assuming interiority, we have that

$$\frac{\partial U_i}{\partial x_i} \frac{dX}{d\ell} + \frac{\partial U_i}{\partial l_i} = 0, \ i \in N. \tag{2.2}$$

A *Sharing Rule* specifies the consumption allocated to each agent as a function of labor inputs. Formally, a sharing rule, $x(\cdot)$, is a collection of functions $(x_1(\cdot), x_2(\cdot), ..., x_n(\cdot))$ with $x_i : \Re^n_+ \to \Re_+$, $i \in N$, such that $\sum_{j=1}^{n} x_j(l) = X(\ell)$, $\forall l \in [0, \bar{l}]^n$. Two well-known examples of sharing rules are:

$$x_i^P(l) = \frac{l_i}{\sum_{j=1}^{n} l_j} X(\ell), \text{ for all } i \in N \tag{Proportional}$$

$$x_i^E(l) = \frac{1}{n} X(\ell), \text{ for all } i \in N \tag{Equal Sharing}$$

Consider now the non-cooperative part of the problem. If labor contributions are voluntary the strategy space for each agent is $[0, \bar{l}]$. Let $x(l) = (x_1(l), x_2(l), ..., x_n(l))$. Thus, the payoff functions are $U_i(x_i(l), l_i)$, $i \in N$. A Nash equilibrium of this mechanism is a vector of strategies (l^*) such that

$$U_i(x_i(l^*), l_i^*) \geq U_i(x_i(l_1^*, .., l_i, .., l_n^*), l_i) \text{ for all } l_i \in [0, ,\bar{l}], \text{ for each agent } i.$$

We will speak of a Nash equilibrium associated to a sharing rule, since the latter enters into the payoff functions. The first order conditions of an interior Nash equilibrium are

$$\frac{\partial U_i}{\partial x_i} \frac{\partial x_i}{\partial l_i} + \frac{\partial U_i}{\partial l_i} = 0. \tag{2.3}$$

We will consider that the sharing rule and the technology are fixed but preferences are not. An economy, denoted by $U \equiv (U_i(\ ), U_i(\ ), ..., U_i(\ ))$, is a list of utility functions. The set of efficient allocations in U is denoted by $\varphi^E(U)$. The input vector (l) is efficient in U if $(x(l), l) \in \varphi^E(U)$. Let $\mathcal{E}$ be the set of all admissible economies. Define,

$$R = \{(l \mid \exists U \in \mathcal{E}, (x(l), l) \in \varphi^E(U)\}.$$

In words, R is the set of input allocations that are efficient for some economy. Define,

$$R_i(l_{-i}) = \{l_i \mid \exists U \in \mathcal{E}, (x(l_i, l_{-i}), (l_i, l_{-i}) \in \varphi^E(U)\}.$$

In words, $R_i(l_{-i})$ is the set of input contributions for i, l_i, such that (l_i, l_{-i}) is an efficient input allocation for some economy.

We assume that the set of admissible economies is large in the following sense:

$$\mathcal{E} \text{ is such that } R_i(l_{-i}) \text{ is the interval } (0, \bar{l}),\ \forall i \in N. \tag{LD}$$

It is easy to generate a space of economies for which (LD) holds. For instance let $U_i = x_i - \beta_i l_i^2/2$. In order to have allocation $(x(\hat{l}), \hat{l})$ as efficient, choose β's such that

$$\frac{dX(\hat{\ell})}{d\ell} = \beta_i \hat{l}_i. \tag{2.4}$$

Since the second order conditions of (2.1) hold, the allocation above is efficient.

We adapt a result from Gradstein (1995) in the framework of the Cournot oligopoly model (a special case of the model considered in this paper) that we use as a Lemma.

Lemma 1. *If Nash equilibrium yields efficient and interior allocations in any $U \in E$, the production function is a polynomial of, at most, degree $(n-1)$.*

Proof. Take any $U \in \mathcal{E}$ and consider a Pareto efficient allocation $(x(l^*), l^*)$ such that l^* is a Nash equilibrium. Thus, from (2.2) and (2.3),

$$\frac{\partial x_i(l^*)}{\partial l_i} = \frac{dX(\sum_{j=1}^n l_j^*)}{d\ell},\ \forall i \in N. \tag{2.5}$$

The above equation holds in the interval $R_i(l^*_{-i})$. Integrating on $(0, l_i)$ we get

$$x_i(l_i, l^*_{-i}) = X(l_i + \sum_{j \neq i} l^*_j) - Q_i, \ \forall l_i \in R_i(l^*_{-i}), \ \forall i \in N, \tag{2.6}$$

where Q_i depends on l^*_{-i}. Since the above equation holds for all $l_j \in R_j(l_{-j})$, $\forall j \neq i$,

$$x_i(l) \equiv X(\sum_{j=1}^{n} l_j) - Q_i(l_{-i}), \ \forall (l_i, l_{-i}) \in R, \ \forall i \in N. \tag{2.7}$$

Adding over i and considering feasibility we obtain

$$(n-1)X(\sum_{j=1}^{n} l_j) \equiv \sum_{j=1}^{n} Q_j(l_{-j}), \ \forall l \in R. \tag{2.8}$$

(see Browning (1983)). Consider now all the possible vectors with one component equal to zero. For each of these vectors we apply equation (2.8) and we subtract the resulting equations from equation (2.8). We do the same for all possible vectors with two components equal to zero and we add the equations to the result of the previous step. We proceed in this way subtracting from the previous step the equation resulting from considering all possible vectors with an odd number of components equal to zero and adding the equations resulting from considering all possible vectors with an even number of components equal to zero. As a result of these operations we get the following functional equation:

$$X(\ell) - \sum_{k=1}^{n} X(\ell_{-k}) + \sum_{(k,t)/k<t} X(\ell_{-kt}) + \ldots\ldots + (-1)^{n-1} \sum_{j=1}^{n} X(l_j) = 0. \tag{2.9}$$

The solution of (2.9) is a polynomial of, at most, degree $(n-1)$ (Aczel (1966) pp. 129-130). ■

An implication of the previous lemma is that when $n = 2$ and the production function is strictly concave, Nash equilibrium cannot be efficient in all economies in $\mathcal{E}$. We now characterize the sharing rules whose Nash equilibria are efficient.

Proposition 1. *If all Nash equilibria associated with an anonymous sharing rule are efficient, the sharing rule has the form:*

$$x_i^I(l) = X(\ell) - (n-1)X(\ell_{-i}) + \frac{n-1}{2}\sum_{k\neq i} X(\ell_{-ik}) + ... + (-1)^{n-1}\sum_{j\neq i} X(l_j). \quad (2.10)$$

Proof. Given l, for each agent i, we can find $Q_i(l_{-i})$ applying equation (2.8) successively to $X(\ell_{-i})$, $X(\ell_{-ik})$ for all possible k different from i, $X(\ell_{-ikh})$ for all possible k and h, $k < h$, different from i, and so on up to $X(l_j)$ for all possible j different from i. Given the equations obtained in this way, we apply the following operation:

$$(n-1)X(\ell_{-i}) - \frac{n-1}{2}\sum_{k\neq i} X(\ell_{-ik}) - .... - (-1)^{n-1}\frac{n-1}{n-1}\sum_{j\neq i} X(l_j). \quad (2.11)$$

By anonymity we know that for any vector such that $l_{-i} = l_{-j}$, $Q_i(l_{-i}) = Q_j(l_{-j})$. Thus, applying anonymity to the result of the above operation we get that:

$$Q_i(l_{-i}) = (n-1)X(\ell_{-i}) - \frac{n-1}{2}\sum_{k\neq i} X(\ell_{-ik}) + ... - (-1)^{n-1}\sum_{j\neq i} X(l_j), \quad (2.12)$$

as we wanted to prove. ■

When $n = 3$ and $n = 4$ the sharing rule (2.10) looks as follows:

$$x_i^I(l) = X(\ell) - 2X(\ell_{-i}) + \sum_{j\neq i} X(l_j), \text{ and} \quad (2.13)$$

$$x_i^I(l) = X(\ell) - 3X(\ell_{-i}) + \frac{3}{2}\sum_{k\neq i} X(\ell_{-ik}) - \sum_{j\neq i} X(l_j). \quad (2.14)$$

The sharing rule (2.10) awards each agent the whole output minus a measure of the contributions of others. We will call it the *Incremental Sharing Rule* because it comes from equation (2.7) which implies that

$$x_i^I(l) - x_i^I(l_{-i}) = X(\ell) - X(\ell_{-i}). \quad (2.15)$$

Thus, the incremental sharing rule, despite the complex analytical form is really simple. It demands the equalization between private gain in consumption of i and public gain in aggregate output for each variation of the labor supplied by i.[2]

The existence of a Nash equilibrium associated with the incremental sharing rule is proved by noting that, under the conditions stated below, the second derivative of i´s payoff function with respect to i´s labor is negative. Thus payoffs are concave and a standard fixed point argument shows the existence of a Nash equilibrium. Formally,

Remark 1. *If the production function is a polynomial of, at most, degree $n-1$ and $\frac{\partial^2 U_i}{\partial x_i l_i} \leq 0$ for all i there is a Nash equilibrium associated to the sharing rule (2.10).*[3]

Unfortunately we have not been able to prove that, in general, the incremental sharing rule yields non negative returns. Clearly, it yields non negative returns when $n = 2$, since in this case the production is lineal. When $n > 2$ we only have a partial result stated in the next propositions.

Proposition 2. *Let $n \geq 3$, and let $X(\ell) = a_{n-1}\ell^{n-1} + a_{n-2}\ell^{n-2} + ... + a_2\ell^2 + a_1\ell$ an increasing and concave polynomial in $[0, n\bar{l}]$ with $a_t \leq 0$ for all $t \in \{2, ..., n-1\}$. Then, $x_i^I(l) \geq 0$ for all i.*

Proof. Since $x_i^I(l)$ is increasing in l_i (recall that $x_i^I(l) = X(\ell) - Q_i(l_{-i})$), to prove the proposition it is enough to show that $x_i^I(l_{-i}) \geq 0$. Without lost of generality let us consider $i = n$. From (2.10) we know that

$$x_n^I(l_{-n}) = -(n-2)X(\ell_{-n}) + \frac{n-1}{2}\sum_{k=1}^{n-1} X(\ell_{-nk}) + ... + (-1)^{n-1}\sum_{j=1}^{n-1} X(l_j). \qquad (2.16)$$

[2] In a model of cost-sharing Moulin (2006) characterizes the (polynomial) cost function whose Nash equilibria are efficient when utility functions are quasi-linear. Despite the similarity in the techniques used by Moulin and us, none of the results imply the other since the inverse of a polynomial is not a polynomial. Moreover cost-sharing and surplus sharing problems are not equivalent (see Leroux (2005)).

[3] Existence results in the literature assume quasi-linear utility in labor (Holmstrom (1982), Fabella (1988) and Moulin (2006)). In this case $\frac{\partial^2 U_i}{\partial^2 x_i l_i} = 0$.

Let $X_t(\ell) = a_t \ell^t$, $1 \leq t \leq n-1$, and let us show that for all $t \in \{1, .., n-1\}$,

$$-(n-2)X_t(\ell_{-n}) + \frac{n-1}{2} \sum_{k=1}^{n-1} X_t(\ell_{-nk}) + ... + (-1)^{n-1} \sum_{j=1}^{n-1} X_t(l_j) \geq 0. \tag{2.17}$$

Notice first that, for a given set of m components, $1 \leq m \leq n-1$, without loss of generality, let us call them $l_1, ..., l_m$,

$$X_t(l_1 + .. + l_m) = a_t \sum_{t_1, .., t_m} \frac{t!}{t_1! ... t_m!} l_1^{t_1} l_2^{t_2} ... l_m^{t_m}, \tag{2.18}$$

where the sum is taken over all non negative integers $t_1, .., t_m$ such that $t_1 + ... + t_m = t$. Thus, expression (2.17) can be rewritten as an expression with terms of the form $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ with $1 \leq m \leq t$, $t_h > 0$ for all $h \in \{1, ..., m\}$ and $t_1 + ... + t_m = t$. Let us see that all the coefficients of such terms are positive. Fix m, and $t_1, ..., t_m$ all positive and such that $t_1 + ... + t_m = t$. For each of the terms involving the sum of k components ($k \geq m$) in expression (2.17), $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ appears as many times as the number of combinations of $(n-m-1)$ elements taken $(k-m)$ at a time. Thus, the coefficient of $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ is:

$$a_t \frac{t!}{t_1! ... t_m!} \left[-(n-2) + \sum_{k=m}^{n-2} (-1)^{n-k} \frac{(n-1)}{(n-k)} \binom{n-m-1}{k-m} \right]. \tag{2.19}$$

We prove in the Appendix (Lemma 2) that

$$-(n-2) + \sum_{k=m}^{n-2} (-1)^{n-k} \frac{(n-1)}{n-k} \binom{n-m-1}{k-m} = \frac{-m+1}{n-m}. \tag{2.20}$$

Since $a_t \leq 0$ for all $t \in \{2, ..., n-1\}$ the coefficient of $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ is non negative. Therefore expression (2.17) is non negative for all $t \in \{2, .., n-1\}$. For $t = 1$, a_1 is positive and sufficiently large to guarantee that the polynomial is increasing, but in this case we only have terms of the form t_j, and all the coefficients of these terms are zero (since $m = 1$). Thus, expression (2.17) is non negative for all t as we wanted to prove. ■

Proposition 3. *Let $n \geq 4$, and let $X(\ell) = a_{n-1}\ell^{n-1} + a_{n-2}\ell^{n-2} + ... + a_2\ell^2 + a_1\ell$ an increasing and concave polynomial in $[0, n\bar{l}]$ with $a_t \geq 0$ for all $t \in \{3, ..., n-1\}$. Then, $x_i^I(l) \geq 0$ for all i.*

Proof. Notice first that the proof of Proposition 2 can be reproduced here up to (2.20). From expressions (2.19), and (2.20) we know that for a given t, $1 \leq t \leq n-1$, for a fixed m, $1 \leq m \leq t$, and for $t_1, ..., t_m$ all positive and such that $t_1 + ... + t_m = t$, the coefficient of terms of the form $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ is given by

$$a_t \frac{t!}{t_1!...t_m!}(\frac{-m+1}{n-m}). \tag{2.21}$$

For $m = 1$, the coefficient is zero. So we just fix attention to $m \geq 2$. For $t = 2$, we only have terms of the form $l_j l_k$ with coefficient $-\frac{2}{(n-2)}a_2$. Thus, expression (2.17) is non positive for all $t \in \{3, .., n-1\}$ since $a_t \geq 0$, it is positive for $t = 2$ since $a_2 < 0$, and it is cero for $t = 1$. However, let us see that (2.16) is positive. By concavity of the polynomial,

$$2a_2 \leq -\sum_{t=3}^{n-1} t(t-1)a_t\ell^{t-2} \text{ for all } \ell \in [0, n\bar{l}]. \tag{2.22}$$

In particular, (2.22) holds for $\ell = \frac{n}{n-1}(\sum_{j=1}^{n-1} l_j)$. Thus, for all possible combinations of $l_j l_k$.

$$-\frac{2}{(n-2)}a_2 l_j l_k \geq \frac{1}{(n-2)}\sum_{t=3}^{n-1} t(t-1)a_t(\frac{n}{n-1})^{t-2} l_j l_k (\sum_{j=1}^{n-1} l_j)^{t-2}, \tag{2.23}$$

For a given t, $3 \leq t \leq n-1$, for a fixed m, $2 \leq m \leq t$, and for $t_1, ..., t_m$ all positive and such that $t_1 + ... + t_m = t$, the term $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ appears in all the inequalities in (2.23) involving all possible order pairs $l_j l_k$ among $(l_1, ..., l_m)$. Thus, the coefficient of $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ that is obtained from those inequalities is:

$$\sum_{T_{jk}^m} \frac{1}{(n-2)}(\frac{n}{n-1})^{t-2} t(t-1)a_t \frac{(t-2)!}{t_1!..(t_j-1)!..(t_k-1)!...t_m!}, \tag{2.24}$$

where the sum is taken over all possible pairs of indexes in the set $T^m_{jk} = \{(j,k)/j < k$, and $j,k \in \{1,...,m\}\}$. Notice that expression (2.24) can be rewritten as:

$$\sum_{T^m_{jk}} \frac{1}{(n-2)}(\frac{n}{n-1})^{t-2} a_t \frac{t! t_j t_k}{t_1!..t_j!..t_k!...t_m!}. \tag{2.25}$$

Since $t_j t_k \geq 1$, and the cardinality of the set T^m_{jk} is equal to the combinations of m elements taken two at a time, expression (2.25) is bigger than

$$\frac{1}{(n-2)}(\frac{n}{n-1})^{t-2} a_t \frac{t!}{t_1!..t_j!..t_k!...t_m!} \frac{m(m-1)}{2}. \tag{2.26}$$

Thus, combining (2.21) and (2.26), we get that the coefficient of $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ is

$$a_t \frac{t!}{t_1!...t_m!} \left[\frac{-m+1}{n-m} + \frac{1}{(n-2)}(\frac{n}{n-1})^{t-2} \frac{m(m-1)}{2} \right]. \tag{2.27}$$

We show in the Appendix (Lemma 3) that (2.27) is positive for all m, $2 \leq m \leq t$, $3 \leq t \leq n-1$, which implies that (2.16) is positive as we wanted to prove. ■

Next we focus on the properties of the incremental sharing rule.

The consumption yielded by Sen´ sharing rule is between those yielded by the proportional and the egalitarian sharing rules. This property does not hold for the incremental sharing rule, as shown by the following example:

Example 1. *Let* $n = 3$. *Thus* $x_1^I(l_1, l_2, l_3) = X(l_1 + l_2 + l_3) - 2X(l_2 + l_3) + X(l_2) + X(l_3)$. *When* $2l_1 = l_2 + l_3$ *the consumption for agent* 1 *yielded by the proportional and the egalitarian sharing rule coincide. Thus, if the consumption for agent* 1 *yielded by the incremental rule were in between those yielded by the proportional and the egalitarian, the three must coincide. Thus,*

$$x_1^I(\frac{l_2+l_3}{2}, l_2, l_3) = X(3\frac{l_2+l_3}{2}) - 2X(l_2+l_3) + X(l_2) + X(l_3) = \frac{X(3\frac{l_2+l_3}{2})}{3}$$

which implies

$$\frac{2}{3} X(3\frac{l_2+l_3}{2}) = 2X(l_2+l_3) - X(l_2) - X(l_3)$$

Taking $l_2 = 0$, *the previous equation yields*

$$\frac{2}{3}X(3\frac{l_3}{2}) = 2X(l_3) - X(l_3) = X(l_3)$$

which only is true under constant returns to scale.

However a related property holds for this sharing rule. The increase in consumption resulting from a increase in i´s labor in the incremental sharing rule (2.10) is between those yielded by the proportional and the egalitarian sharing rules.

Proposition 4. $x_i^E(l) - x_i^E(l_{-i}) \leq x_i^I(l) - x_i^I(l_{-i}) \leq x_i^P(l) - x_i^P(l_{-i})$.

Proof. For the sharing rule (2.10), it holds that

$$x_i^I(l) - x_i^I(l_{-i}) = X(\ell) - X(\ell_{-i}).$$

For the egalitarian sharing rule,

$$x_i^E(l) - x_i^E(l_{-i}) = \frac{1}{n}(X(\ell) - X(\ell_{-i})).$$

Thus,

$$x_i^E(l) - x_i^E(l_{-i}) \leq x_i^I(l) - x_i^I(l_{-i}).$$

For the relation of sharing rule (2.10) with the Proportional, notice that since $X()$ is concave with $X(0) = 0$, $X(\ell)/\ell$ is decreasing. Thus,

$$\frac{X(\ell_{-i})}{\ell_{-i}} \geq \frac{X(\ell)}{\ell}, \ \Rightarrow X(\ell_{-i}) \geq X(\ell) - \frac{l_i}{\ell}X(\ell).$$

Thus,

$$x_i^P(l) - x_i^P(l_{-i}) = \frac{l_i}{\ell}X(\ell) \geq X(\ell) - X(\ell_{-i}) = x_i^I(l) - x_i^I(l_{-i}),$$

as we wanted to prove. ■

Given the novelty of the form of (2.10), we investigate the implications of postulating sharing rules like those considered by the literature (see, e.g. Moulin (1987), Pfingsten

(1991) and Roemer and Silvestre (1993)). The next proposition characterizes the technology for which Nash equilibrium yields efficient allocations for a class of sharing rules that, to the best of our knowledge, contains all sharing rules proposed by the literature.

Proposition 5. *Assume that the sharing rule can be written as $x_i = x_i(l_i, \sum_{j=1}^n l_j)$, or it is such that $x_i(0, l_{-i}) = 0$. If Nash equilibrium yields efficient and interior allocations in any $U \in \mathcal{E}$, the production function displays constant returns to scale.*

Proof. Case 1. Let us consider first the case $x_i = x_i(l_i, \sum_{j=1}^n l_j)$. Since the production function depends on the sum of inputs, $Q_i(l_{-i}) = Q_i(\sum_{j \neq i} l_j)$. For any possible vector l such that $\sum_{j=1}^n l_j \leq \bar{l}$, consider another vector such that all components are zero except one (let us say i) and this component is the sum of all components in l. Then, equation (2.8) implies that:

$$(n-1)X(\ell) = \sum_{j \neq i} Q_j(\ell). \tag{2.28}$$

Repeating the argument but considering that the non-zero component is k,

$$(n-1)X(\ell) = \sum_{j \neq k} Q_j(\ell). \tag{2.29}$$

Subtracting both equations we get that

$$Q_i(\ell) = Q_k(\ell). \tag{2.30}$$

Since the above equation is true for any i and k, from (2.28),

$$(n-1)X(\ell) = (n-1)Q_k(\ell), \tag{2.31}$$

which implies $Q_k(\ell) = X(\ell)$ for all k and ℓ. Given l such that $\sum_{j=1}^n l_j \leq \bar{l}$, consider another vector with the first component equal to l_1, the second component equal to $\sum_{j \neq 1} l_j$ and any other component equal to zero. Equation (2.8) now reads

$$(n-1)X(\sum_{j=1}^n l_j) = X(\sum_{j \neq 1} l_j) + X(l_1) + (n-2)X(\sum_{j=1}^n l_j), \tag{2.32}$$

which implies that

$$X(\sum_{j=1}^{n} l_j) = X(\sum_{j\neq 1} l_j) + X(l_1). \tag{2.33}$$

Repeating the argument to $X(\sum_{j\neq 1} l_j)$ and so on, we get that

$$X(\sum_{j=1}^{n} l_j) = \sum_{j=1}^{n} X(l_j), \text{ for all } l \text{ such that} \sum_{j=1}^{n} l_j \leq \bar{l}. \tag{2.34}$$

This is a Cauchy equation whose solutions are linear (Aczel (1966), chapter 2). Thus, the production function displays constant returns to scale for all l such that $\sum_{j=1}^{n} l_j \leq \bar{l}$. By Lemma 1 the production function is a polynomial of degree $n-1$. Combining both results, the production function displays constant returns to scale in the whole domain.

Case 2. If the sharing rule is such that $x_i(0, l_{-i}) = 0$, we have that $x_i(l_{-i}) = 0 = X(\ell_{-i}) - Q_i(l_{-i})$. Thus, $Q_i(l_{-i}) = X(\ell_{-i})$ for all i and (2.8) reads:

$$(n-1)X(\sum_{j=1}^{n} l_j) \equiv \sum_{j=1}^{n} X(\ell_{-j}). \tag{2.35}$$

Let us see that equation (2.35) implies that

$$X(\sum_{j=1}^{n} l_j) = \sum_{j=1}^{n} X(l_j), \tag{2.36}$$

which is a Cauchy's equation whose solutions are linear (see Aczel 1966, chapter 2 for a discussion). We prove the above relation by induction on the number of zero components in a vector l. Let us consider a vector l such that all components but two are zero. Thus,

$$\begin{aligned} (n-1)X(l_i + l_j) &= X(l_i) + X(l_j) + (n-2)X(l_i + l_j), \text{ thus} \\ X(l_i + l_j) &= X(l_i) + X(l_j). \end{aligned}$$

Suppose that the relation is true for all vectors l such that all components but one are different from zero. Then, applying the induction hypothesis to equation (2.35),

$$(n-1)X(\sum_{j=1}^{n} l_j) = \sum_{j=1}^{n} X(l_{-j}) = (n-1)\sum_{j=1}^{n} X(l_j),$$

as we wanted to prove. ■

Proposition 5 implies that the result obtained by Sen is an artifact of his assumption that all agents are identical. In his case our assumption LD fails because $R_i(l_{-i})$ is just a point or the empty set.

3. Appendix

Lemma 2. *Let*

$$S_n = -(n-2) + \sum_{k=m}^{n-2} (-1)^{n-k} \frac{(n-1)}{n-k} \binom{n-m-1}{k-m}.$$

Then, $S_n = \frac{-m+1}{n-m}$.

Proof. Notice first that

$$\begin{aligned} \frac{1}{n-k}\binom{n-m-1}{k-m} &= \frac{1}{n-m}\binom{n-m}{n-k}, \text{ and} \\ \sum_{k=m}^{n-2} (-1)^{n-k} \binom{n-m}{n-k} &= \sum_{k=2}^{n-m} (-1)^k \binom{n-m}{k}. \end{aligned}$$

Thus,

$$S_n = -(n-2) + \frac{(n-1)}{(n-m)} \sum_{k=2}^{n-m} (-1)^k \binom{n-m}{k}.$$

By the Newton's binomial we know that

$$0 = (1 + (-1))^{n-m} = \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k}.$$

Thus,

$$S_n = -(n-2) + \frac{(n-1)}{(n-m)}(-1 + n - m) = \frac{-m+1}{n-m},$$

as we wanted to prove. ■

Lemma 3. *For all* t, $3 \leq t \leq n-1$, *and for all* m, $2 \leq m \leq n-1$,

$$\frac{-m+1}{n-m} + \frac{1}{(n-2)}\left(\frac{n}{n-1}\right)^{t-2}\frac{m(m-1)}{2} \geq 0. \tag{3.1}$$

Proof. Notice first that in order to prove the statement it is enough to show that for all $m, 2 \leq m \leq t$,

$$-1 + \frac{1}{(n-2)}\left(\frac{n}{n-1}\right)^{t-2}\frac{m(n-m)}{2} \geq 0. \tag{3.2}$$

Clearly, for $m = 2$, the statement is true. Notice that $m(n-m)$ as a function of m extended to the real numbers is concave in m,thus, the minimum of this function is reached in $m = 2$ or $m = t$. Notice that $m(n-m)$ gets the same value for $m = 2$ and for $m = n-2$. Therefore, if $t \leq n-2$, for all $2 \leq m \leq t$, $m(n-m) \geq 2(n-2)$. Thus, the expression in (3.2) is positive. If $t = n-1$, the minimum is obtained in $m = t$. Let us show that also in this case the statement of the Lemma holds, that is:

$$\begin{aligned} -1 + \frac{1}{(n-2)}\left(\frac{n}{n-1}\right)^{n-3}\frac{(n-1)}{2} &\geq 0, \text{ or equivalently,} && (3.3)\\ \frac{(n-1)}{(n-2)}\left(\frac{n}{n-1}\right)^{n-3} &\geq 2. && (3.4)\end{aligned}$$

Let $a_n = \frac{(n-1)}{(n-2)}\left(\frac{n}{n-1}\right)^{n-3}$, $b_n = \frac{(n-1)}{(n-2)}\left(\frac{n-1}{n}\right)^2$ and $c_n = \left(\frac{n}{n-1}\right)^{n-1}$. Notice that $a_n = b_n c_n$. The sequence c_n is a well studied sequence which is increasing and converges to the number e, and it is easy to prove that the sequence b_n is increasing in n for all $n \geq 4$. Thus, a_n is increasing in n and since $a_4 = 2$, $a_n \geq 2$ for all $n \geq 4$. ■

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