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Departamento de Economía de la Empresa
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 916249607

Stability of the optimal reinsurance with respect to the risk measure*

Alejandro Balbás [1], Beatriz Balbás [2] and Antonio Heras [3]

Abstract

The optimal reinsurance problem is a classic topic in Actuarial Mathematics. Recent approaches consider a coherent or expectation bounded risk measure and minimize the global risk of the ceding company under adequate constraints. However, there is no consensus about the risk measure that the insurer must use, since every risk measure presents advantages and shortcomings when compared with others.

This paper deals with a discrete probability space and analyzes the stability of the optimal reinsurance with respect to the risk measure that the insurer uses. We will demonstrate that there is a "stable optimal retention" that will show no sensitivity, insofar as it will solve the optimal reinsurance problem for many risk measures, thus providing a very robust reinsurance plan. This stable optimal retention is a stop-loss contract, and it is easy to compute in practice. A fast algorithm will be given and a numerical example presented.

Keywords: Optimal reinsurance, Risk measure, Sensitivity, Stable optimal retention, Stop-loss reinsurance.
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[1] University Carlos III of Madrid, Department of Business Administration. alejandro.balbas@uc3m.es

[2] Rey Juan Carlos University, Department of Financial Economics and Accounting. beatriz.balbas@urjc.es

[3] Complutense University of Madrid, Department of Actuarial and Financial Economics. aheras@ccee.ucm.es

1. Introduction

General risk measures are becoming more and more important in insurance and finance. The paper by Artzner *et al.* (1999) on coherent measures of risk launched this topic, and, since then, many authors have further extended the discussion. So, among others, Goovaerts *et al.* (2004) have introduced the consistent risk measures, also studied in Burgert and Rüschendorf (2006), Frittelli and Scandolo (2005) have analyzed risk measures for stochastic processes, and Rockafellar *et al.* (2006) have defined the deviations and the expectation bounded risk measures.

Many classical actuarial and financial problems have been revisited using risk measures beyond the variance. For example, Laeven and Goovaerts (2004) and Dhaene *et al.* (2008) analyze the capital allocation problem, Nakano (2004) and Balbás *et al.* (2010) draw on risk measures when pricing in incomplete markets, Mansini *et al.* (2007) and Schied (2007) deal with portfolio choice and optimal investment, and Annaert *et al.* (2009) check the efficiency of the classical portfolio insurance problem if the risk level is given by the Value at Risk (VaR) or the Conditional Value at Risk ($CVaR$).

The optimal reinsurance problem is a classical issue in Actuarial Science. Usually, authors consider the primary (or ceding) company viewpoint. A common approach attempts to minimize some measure of the first insurer risk after reinsurance. Seminal papers by Borch (1960) and Arrow (1963) used the variance as a risk measure and proved that the stop-loss reinsurance minimizes the retained risk if premiums are calculated following the Expected Value Premium Principle..

The subsequent research followed the ideas outlined in the articles above, trying to take into account more general risk measures and premium principles, which may give optimal contracts other than stop-loss. In recent years some interesting articles devoted to this subject have appeared. For example, Gajec and Zagrodny (2004) consider more general symmetric and even asymmetric risk functions such as the absolute deviation and the truncated variance of the retained loss, under the standard deviation premium principle. Young (1999) maximizes the expected utility of the final wealth under the distortion premium principle. Kaluszka (2005) studies reinsurance contracts with many convex premium principles (exponential, semi-deviation and semi-variance, Dutch, distortion, etc.). Other well known financial risk measures such as the VaR or the tail value at risk ($TVaR$) are also being considered. For example, Kaluszka (2005) uses the $TVaR$ as a premium principle and Cai and Tan (2007) calculate the optimal retention for a stop-loss reinsurance by considering the VaR and the conditional tail expectation risk measures (CTE), under the expected value premium principle.

The most recent papers have finally incorporated coherent and/or expectation bounded risk measures in the objective function to be minimized by the ceding company. Along with the paper of Cai and Tan (2007) above, other interesting examples are Cai *et al.* (2008), Balbás *et al.* (2009) or Bernard and Tian (2009). The differences among their approaches are caused by the insurer behavior. Very complete information may be found in the survey of Centeno and Simoes (2009).

Despite the interest of the problem, as far as we know there are no analyses focusing on the stability of the optimal reinsurance. This should be an important topic since the optimality of many reinsurance plans will critically depend on the risk measure and the pricing principle. There is no consensus about the risk measure that the insurer must use, since every risk measure presents advantages and shortcomings when compared with others.

This paper considers that the reinsurer's premium principle is given by a convex function and deals with the optimal reinsurance problem if risk is measured by coherent and expectation bounded risk measures.[1] The focus is on the stability of the optimal retention plan with respect to the chosen risk measure.

The paper's outline is as follows. Section 2 will present the basic conditions and properties of the risk measure ρ to be used. Section 3 provides our general optimal reinsurance problem. We will present the problem in a discrete probability space. Actually, this simplifies the mathematical exposition, and every probability space admits a discrete approximation which achieves as much accuracy as needed. Many actuarial and financial analyses are done by means of discrete probability spaces (see Benati, 2003, Konno *et al.*, 2005, Mansini *et al.*, 2007, or Miller and Ruszczynski, 2008, among many others), since this is not a restriction in practice. The proposed optimal reinsurance problem seems to be quite flexible and general, since it allows us to incorporate many particular situations such as budget constraints, the maximization of the insurer expected wealth, etc. The most important results in Section 3 are Theorem 1 and Corollary 3, since they characterize the optimal retention by means of Karush-Kuhn-Tucker (KKT) like conditions and permit us to introduce the "stable optimal retention", which will solve the problem for all of the risk measures the KKT multipliers of which satisfy adequate properties. Therefore, the stable optimal retention may be understood as a robust optimal reinsurance plan.

Section 4 is devoted computing in practice the stable optimal retention. Here we will assume that the reinsurer uses a linear value principle, containing the expected value premium principle as a particular case. Of course it is not necessary, since practical optimality conditions have been given in a much more general framework, but the specific solution of the optimization problem depends on the premium principle we take, and considering more than one would significantly enlarge the paper. As already indicated, previous literature measuring the insurer risk by a general risk

[1] Insurance premiums are usually given by convex functions. See, for instance, Deprez and Gerber (1985).

measure is still limited, so it seems to be natural and of interest to analyze concrete problems by taking the most used premium principle.

The most important result of this section is Theorem 7, because it gives explicit expressions for the stable optimal retention and the $KKT-$multipliers of the problem. According to Theorem 7, the stable optimal retention is a stop-loss reinsurance.

Theorem 7 is used in Section 5 so as to introduce an algorithm that gives the stable optimal retention in numerical applications. An illustrative numerical example is also provided, which clarifies how to use the algorithm in practice and shows the robustness of the given reinsurance, in the sense that most of the usual risk measures lead to this solution.

The last section of the paper points out the most important conclusions.

2. Preliminaries and notations

As usual, consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of "states of the world" Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. As said above, we will be dealing with a discrete framework, so Ω will be composed of a finite number of elements,

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n\}.$$

We will consider the probability of every single event

$$p_i = \mathbb{P}(\omega_i) > 0$$

$i = 1, 2, ..., n$.

Denote by $\mathbb{E}(y)$ the mathematical expectation of every random variable y, and denote by L^2 the Hilbert space of $\mathbb{R}-$valued random variables y on Ω endowed with the norm

$$\|y\|_2 = \left(\mathbb{E}\left(|y|^2\right)\right)^{1/2}$$

for every $y \in L^2$.[2]

Let $[0, T]$ be a time interval. From an intuitive point of view, one can interpret that every $y \in L^2$ may represent the wealth at T of an arbitrary insurer. Let

$$\rho : L^2 \longrightarrow \mathbb{R}$$

be the general risk function that a insurer uses in order to control the risk level of his final wealth at T. Denote by

$$\Delta_\rho = \left\{z \in L^2; -\mathbb{E}(yz) \leq \rho(y), \ \forall y \in L^2\right\}. \quad (1)$$

[2] Actually, Ω being discrete the dimension of L^2 is finite and equals $n \in \mathbb{N}$. Thus, $L^2 = L^p$ for every $p \in [1, \infty]$ and the norm $\|.\|_2$ above is equivalent to the norm $\|.\|_p$. Though we have chosen $p = 2$, every $p \in [1, \infty]$ may play the same role.

We will assume that Δ_ρ is convex and compact, and

$$\rho(y) = Max \ \{-\mathbb{E}(yz) : z \in \Delta_\rho\} \tag{2}$$

holds for every $y \in L^2$. Furthermore, we will also suppose that the constant random variable $z = 1$ is in Δ_ρ and

$$\Delta_\rho \subset \left\{z \in L^2; \mathbb{E}(z) = 1\right\}. \tag{3}$$

Summarizing, we have:

Assumption 1. The set Δ_ρ given by (1) is convex and compact, (2) holds for every $y \in L^2$, $z = 1$ is in Δ_ρ, and (3) holds. □

The assumption above is closely related to the Representation Theorem of Risk Measures stated in Rockafellar *et al.* (2006). Following their ideas, it is easy to prove that the fulfillment of Assumption 1 holds if and only if ρ satisfies:

a)

$$\rho(y + k) = \rho(y) - k \tag{4}$$

for every $y \in L^2$ and $k \in \mathbb{R}$.

b)

$$\rho(\alpha y) = \alpha \rho(y) \tag{5}$$

for every $y \in L^2$ and $\alpha > 0$.

c)

$$\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2) \tag{6}$$

for every $y_1, y_2 \in L^2$.

d)

$$\rho(y) \geq -\mathbb{E}(y) \tag{7}$$

for every $y \in L^2$.[3]

It is easy to see that if ρ satisfies Properties *a*), *b*), *c*) and *d*) then it is also coherent in the sense of Artzner *et al.* (1999) if and only if

$$\Delta_\rho \subset L^2_+ = \left\{z \in L^2; \mathbb{P}(z \geq 0) = 1\right\}. \tag{8}$$

Particular interesting examples are the Conditional Value at Risk ($CVaR$) of Rockafellar *et al.* (2006), the Weighted Conditional Value at Risk ($WCVaR$) of Cherny (2006), the Dual Power Transform (DPT) of Wang (2000) and the Wang Measure (Wang, 2000), among many others. Furthermore, following the original idea

[3] Actually, the properties above are almost similar to those used by Rockafellar *et al.* (2006) in order to introduce their Expectation Bounded Risk Measures.

of Rockafellar *et al.* (2006) to identify their Expectation Bounded Risk Measures and their Deviation Measures, it is easy to see that

$$\rho(y) = \sigma(y) - \mathbb{E}(y) \tag{9}$$

satisfies *a*), *b*), *c*) and *d*) if $\sigma : L^2 \longrightarrow \mathbb{R}$ is a deviation, that is, if σ is satisfies *b*), *c*),

e)

$$\sigma(y+k) = \sigma(y)$$

for every $y \in L^2$ and $k \in \mathbb{R}$, and

f)

$$\sigma(y) \geq 0$$

for every $y \in L^2$.

Among many others, a particular example is the classical $p-$deviation for every $p \in [1, \infty)$, given by

$$\sigma_p(y) = \left[\mathbb{E}\left(\left|\mathbb{E}(y) - y\right|^p\right)\right]^{1/p},$$

or the downside $p-$semi-deviation, given by

$$\sigma_p^-(y) = \left[\mathbb{E}\left(\left|Max\ \left\{\mathbb{E}(y) - y, 0\right\}\right|^p\right)\right]^{1/p}.$$

The classical Separation Theorems allow us to prove that there is a one to one identification $\rho \longleftrightarrow \Delta_\rho$ between the risk measures satisfying Assumption 1 that are coherent and the set of convex and compact subsets of L^2 such that $z = 1$ is in Δ_ρ, and (3) and (8) hold. Furthermore, (2) shows that this identification is increasing, *i.e.*, $\rho_1(y) \leq \rho_2(y)$ holds for every $y \in L^2$ if and only if $\Delta_{\rho_1} \subset \Delta_{\rho_2}$ holds. Accordingly, the maximum coherent risk measure satisfying Assumption 1 is that Γ associated with the set

$$\Delta_\Gamma = \left\{z \in L_+^2 : \mathbb{E}\ (z) = 1\right\}. \tag{10}$$

It is easy to see that the risk measure Γ is

$$\Gamma(y) = -Min\ \left\{y(\omega_i) :\ i = 1, 2, ..., n\right\} \tag{11}$$

for every $y \in L^2$. Similarly, $y \to -\mathbb{E}\ (y)$ is the minimum risk measure satisfying the conditions above, since $\Delta_{\mathbb{E}} = \{1\}$. Thus

$$\Gamma(y) \geq \rho(y) \geq -\mathbb{E}(y) \tag{12}$$

holds for every $y \in L^2$ and every coherent ρ satisfying Assumption 1.

Finally, once again the Separation Theorems allow us to prove that every convex combination

$$\rho = \sum_{i=1}^{m} w_i \rho_i$$

of risk measures satisfying (4), (5), (6), (7) and (8) also satisfies (4), (5), (6), (7) and (8), and

$$\Delta_\rho = \sum_{i=1}^{m} w_i \Delta_{\rho_i} \tag{13}$$

holds.

3. Optimal reinsurance: General problem and optimality conditions

Consider that the insurance company receives the fixed amount S_0 (premium) and will have to pay the random variable $y_0 \in L^2_+$ within a given period $[0, T]$ (claims). Without loss of generality we will assume that $\mathbb{P}(y_0 > 0) = 1$, since the absence of claims is an unrealistic situation in practice.

Suppose that a reinsurance contract is signed in such a way that the company will only pay $y \in L^2$, whereas the reinsurer will pay $y_0 - y$. If the reinsurer premium principle is given by the convex (and therefore continuous) and increasing function,

$$\pi : L^2 \longrightarrow \mathbb{R}$$

such that $\pi(0) = 0$, and $S_1 > 0$ is the highest amount that the insurer would like to pay for the contract, then the insurance company will choose y (optimal retention) so as to solve the bi-criteria optimization problem

$$\begin{cases} Min\ \rho_0 (S_0 - y - \pi(y_0 - y)) \\ Max\ \mathbb{E}(S_0 - y - \pi(y_0 - y)) \\ \pi(y_0 - y) \le S_1 \\ 0 \le y \le y_0 \end{cases} \tag{14}$$

ρ_0 being a coherent risk measure that satisfies Assumption 1. Conditions $\pi(0) = 0$ and $S_1 > 0$ imply that $y = y_0$ satisfies the constraint, so (14) is never unfeasible. Notice that, if desired, constraint $\pi(y_0 - y) \le S_1$ may be removed without modifying (14), since π is increasing and therefore it is sufficient to choose $S_1 > \pi(y_0)$.

The multiobjective optimization problem (14) is convex, so it may be solved by scalarization methods. Thus, take w_0 and w_1 non negative and such that $w_0 + w_1 = 1$, let $\rho = w_0 \rho_0 - w_1 \mathbb{E}$, and solve

$$\begin{cases} Min\ \rho (S_0 - y - \pi(y_0 - y)) \\ \pi(y_0 - y) \le S_1 \\ 0 \le y \le y_0 \end{cases} . \tag{15}$$

Bearing in mind the ideas of the previous section, ρ satisfies Assumption 1 and is coherent, since it is a convex combination of ρ_0 and $-\mathbb{E}$.

It is worth remarking that the first (second) objective of (14) may be removed and the problem still fits in (15), because one can take $w_0 = 0$ and $w_1 = 1$ ($w_0 = 1$ and $w_1 = 0$).

Next we will give necessary and sufficient Karush-Kuhn-Tucker optimality conditions. We will not prove this result since it is parallel to one of Balbás *et al.* (2009).

Theorem 1. *Problem (15) is bounded and solvable. Moreover, the existence of $(\tau^*, z^*) \in \mathbb{R} \times L^2$ satisfying the following Karush-Kuhn-Tucker conditions is necessary and sufficient to guarantee the optimality of $y^* \in L^2$.*

$$\begin{cases} \mathbb{E}\left(y^* z\right) \leq \mathbb{E}\left(y^* z^*\right), & \forall z \in \Delta_\rho \\ \tau^* \left(\pi\left(y_0 - y^*\right) - S_1\right) = 0 & \\ \pi\left(y_0 - y^*\right) - S_1 \leq 0 & \\ \mathbb{E}\left(y^* z^*\right) + \left(1 + \tau^*\right) \pi\left(y_0 - y^*\right) \leq \mathbb{E}\left(y z^*\right) + \left(1 + \tau^*\right) \pi\left(y_0 - y\right), & \forall 0 \leq y \leq y_0 \\ \tau^* \in \mathbb{R},\ \tau^* \geq 0,\ 0 \leq y^* \leq y_0,\ z^* \in \Delta_\rho & \end{cases} \tag{16}$$

(τ^*, z^*) will be called $KKT-$multiplier of (15). □

A first important consequence is that one can give conditions ensuring that the solution of (15) remains the same if ρ is replaced by a lower one.[4] Hence we can give the first result guaranteeing the stability of the optimal insurance (retention) with respect to the risk measure.

Corollary 2. *Suppose that $y^* \in L^2$ solves (15) and (τ^*, z^*) is a $KKT-$multiplier. Take the coherent risk measure $\tilde{\rho}$ satisfying Assumption 1 and such that $\tilde{\rho} \leq \rho$. If $z^* \in \Delta_{\tilde{\rho}}$ and $\tilde{\rho}$ replaces ρ then $y^* \in L^2$ still solves (15) and (τ^*, z^*) is still a $KKT-$multiplier.*

Proof. On the one hand, y^* and (τ^*, z^*) satisfy (16). On the other hand, according to that properties given in the previous section, $\Delta_{\tilde{\rho}} \subset \Delta_\rho$ because $\tilde{\rho} \leq \rho$. Thus, $z^* \in \Delta_{\tilde{\rho}}$ implies that (16) still holds if $\Delta_{\tilde{\rho}}$ replaces Δ_ρ. □

Corollary 3. *Suppose that $y^*_\Gamma \in L^2$ solves (15) and $(\tau^*_\Gamma, z^*_\Gamma)$ is a $KKT-$multiplier for the risk measure Γ of (11). Then y^*_Γ still solves (15) and $(\tau^*_\Gamma, z^*_\Gamma)$ is still a $KKT-$multiplier for every ρ such that $z^*_\Gamma \in \Delta_\rho$.*

[4] With the notations of (14) , notice that ρ decreases if so does ρ_0, *i.e.*,

$$\rho_0 \geq \tilde{\rho}_0 \Rightarrow w_0 \rho_0 - w_1 \mathbb{E} \geq w_0 \tilde{\rho}_0 - w_1 \mathbb{E}.$$

Proof. It trivially follows from the previous corollary and (12). □

Remark 1. *With the notations of Corollary* 3*, if* $z^*_\Gamma \notin \Delta_\rho$ *one still can look for a risk measure* $\tilde{\rho} \geq \rho$ *quite similar to* ρ *and such that* $z^*_\Gamma \in \Delta_{\tilde{\rho}}$*, and therefore* y^*_Γ *still solves (15) and* $(\tau^*_\Gamma, z^*_\Gamma)$ *is still a* $KKT-$*multiplier if one considers* $\tilde{\rho}$*. Indeed, it is sufficient to take the convex and compact set*

$$\Delta_{\tilde{\rho}} = Co\left(\Delta_\rho \cup \{z^*_\Gamma\}\right),$$ [5]

obviously associated with the risk measure

$$\tilde{\rho}(y) = Max \ \left\{\rho(y), -\mathbb{E}\left(y z^*_\Gamma\right)\right\} \tag{17}$$

for every $y \in L^2$*. For this reason hereafter the solution* $y^*_\Gamma \in L^2$ *of (15) for the risk measure* Γ *of (11) will be called "stable optimal retention".* □

Remark 2. *If the ceding company is also interested in maximizing the expected wealth and deals with problem (14), then* Γ *may be replaced by* $w_0\Gamma - w_1\mathbb{E}$ *(with* $w_i \geq 0$*,* $i = 0, 1$*, and* $w_0 + w_1 = 1$*). Indeed, in such a case, (10) and (13) show that*

$$\Delta_{w_0\Gamma - w_1\mathbb{E}} = \left\{z \in L^2;\ \mathbb{E}(z) = 1 \text{ and } z \geq w_1\right\}. \tag{18}$$

Obviously, Corollary 2 *proves that if* $y^*_{w_0\Gamma} \in L^2$ *solves (14) and* $\left(\tau^*_{w_0\Gamma}, z^*_{w_0\Gamma}\right)$ *is a* $KKT-$*multiplier for the risk measure* $w_0\Gamma - w_1\mathbb{E}$ *above, then* $y^*_{w_0\Gamma}$ *still solves (14) and* $\left(\tau^*_{w_0\Gamma}, z^*_{w_0\Gamma}\right)$ *is still a* $KKT-$*multiplier for every* ρ *such that* $z^*_{w_0\Gamma} \in \Delta_{w_0\rho - w_1\mathbb{E}}$*. Furthermore, a new comment similar to Remark* 1 *applies.* □

4. Characterizing and computing the stable optimal retention

Let us give properties making it easier to verify the fulfillment of the inequalities of (16). To this purpose, and taking into account Corollary 3, Remark 2 and the first condition in (16), let us give an instrumental lemma.

Lemma 4. *Suppose that* $0 \leq y^* \leq y_0$ *and* $z^* \in \Delta_{w_0\Gamma - w_1\mathbb{E}}$ *(see (18)).* $\mathbb{E}(y^* z) \leq \mathbb{E}(y^* z^*)$ *holds for every* $z \in \Delta_{w_0\Gamma - w_1\mathbb{E}}$ *if and only if*

$$y^*(\omega_j) = Max \ \left\{y^*(\omega_i) :\ i = 1, 2, .., n\right\}$$

holds for every $j = 1, 2, ..., n$ *such that* $z^*(\omega_j) > w_1$*.*

[5] As usual, $Co(A)$ denotes the convex hull of every set $A \subset L^2$.

Proof. The inequality above holds if and only if z^* solves the linear optimization problem

$$\begin{cases} Max \ \sum_{i=1}^{n} y^*(\omega_i) z(\omega_i) p_i \\ \sum_{i=1}^{n} z(\omega_i) p_i = 1 \\ w_1 \leq z(\omega_i), \ i = 1, 2, ..., n \end{cases}$$

According to the classical Karush-Kuhn-Tucker conditions, this is equivalent to the existence of $\mu_0, \mu_1, ..., \mu_n \in \mathbb{R}$ such that

$$\begin{cases} -y^*(\omega_i) p_i + \mu_0 p_i - \mu_i = 0, & i = 1, 2, ..., n \\ \sum_{i=1}^{n} z^*(\omega_i) p_i = 1 & \\ (z^*(\omega_i) - w_1) \mu_i = 0, & i = 1, 2, ..., n \\ \mu_i \geq 0, & i = 1, 2, ..., n \\ z^*(\omega_i) \geq w_1, & i = 1, 2, ..., n \end{cases}.$$

Hence, the result trivially follows if one takes

$$\mu_0 = Max \ \{y^*(\omega_i) : \ i = 1, 2, .., n\}$$

and

$$\mu_i = (\mu_0 - y^*(\omega_i)) p_i,$$

$i = 1, 2, ..., n.$ □

Despite the level of generality of the previous analyses, the solutions of (16) will depend on the specific assumptions one imposes. Henceforth we will assume that the reinsurer uses a linear premium principle. Actually, as indicated in the introduction, previous literature considering a general risk measure is scant, so it seems to be natural and of interest to analyze concrete problems by taking the most used premium principle, which is the expected value premium principle, *i.e.*, there exists $k > 1$ such that

$$\pi(y) = k\mathbb{E}(y) \tag{19}$$

for every $y \in L^2$. We will impose something strictly weaker, such as the existence of $z_\pi \in L^2$ such that

$$\mathbb{P}(z_\pi > 0) = 1, \tag{20}$$

$$\mathbb{E}(z_\pi) > 1 \tag{21}$$

and

$$\pi(y) = \mathbb{E}(y z_\pi) \tag{22}$$

for every $y \in L^2$.[6]

[6]Notice that (19) is a particular case of (22) that arises if z_π remains constant and equals k.

Assumption 2. Henceforth we will assume the existence of $z_\pi \in L^2$ such that (20), (21) and (22) hold. $\square$

Nevertheless, it is worth pointing out that the previous developments are more general, and therefore they also apply to alternative premium principles.

From Assumption 2 the necessary and sufficient optimality conditions (16) become

$$\left\{ \begin{array}{ll} \mathbb{E}\left(y^* z\right) \leq \mathbb{E}\left(y^* z^*\right), & \forall z \in \Delta_\rho \\ \tau^* \left(\mathbb{E}\left(\left(y_0 - y^*\right) z_\pi\right) - S_1\right) = 0 & \\ \mathbb{E}\left(\left(y_0 - y^*\right) z_\pi\right) - S_1 \leq 0 & \\ \mathbb{E}\left(y^* \left(z^* - \left(1+\tau^*\right) z_\pi\right)\right) \leq \mathbb{E}\left(y \left(z^* - \left(1+\tau^*\right) z_\pi\right)\right), & \forall 0 \leq y \leq y_0 \\ \tau^* \in \mathbb{R},\ \tau^* \geq 0,\ 0 \leq y^* \leq y_0,\ z^* \in \Delta_\rho & \end{array} \right. \tag{23}$$

Next let us present two simple lemmas. The first one simplifies the fourth condition of (23).

Lemma 5. *Let $z^* \in L^2$, $y^* \in L^2$ with $0 \leq y^* \leq y_0$, and $\tau^* \in \mathbb{R}$. Then,*

$$\mathbb{E}\left(y^* \left(z^* - \left(1+\tau^*\right) z_\pi\right)\right) \leq \mathbb{E}\left(y \left(z^* - \left(1+\tau^*\right) z_\pi\right)\right)$$

holds for every $y \in L^2$ with $0 \leq y \leq y_0$ if and only if there exists a measurable partition

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$$

such that

$$\left\{ \begin{array}{lll} z^*(\omega) > \left(1+\tau^*\right) z_\pi, & y^*(\omega) = 0, & if\ \omega \in \Omega_1 \\ z^*(\omega) = \left(1+\tau^*\right) z_\pi, & & if\ \omega \in \Omega_2 \\ z^*(\omega) < \left(1+\tau^*\right) z_\pi, & y^*(\omega) = y_0(\omega), & if\ \omega \in \Omega_3 \end{array} \right. \tag{24}$$

Proof. It is obvious if we realize that the solution of

$$\left\{ \begin{array}{l} Min\ \mathbb{E}\left(y \left(z^* - \left(1+\tau^*\right) z_\pi\right)\right) \\ 0 \leq y \leq y_0 \end{array} \right.$$

must be as large as possible (*i.e.*, must equal y_0) whenever $z^* - \left(1+\tau^*\right) z_\pi < 0$ and as small as possible (*i.e.*, zero) if $z^* - \left(1+\tau^*\right) z_\pi > 0$, whereas its value is not relevant at all if $z^* - \left(1+\tau^*\right) z_\pi = 0$. $\square$

Lemma 6. *$y^* = 0$ does not solve (15).*

Proof. If $y^* = 0$ solved (15) then (24) would lead to $z^* \geq (1+\tau^*)\, z_\pi$. Bearing in mind (3) and (21), and taking expectations, one has the contradiction $1 \geq (1+\tau^*)\, \mathbb{E}\,(z_\pi) > 1$. □

As already said the stop–loss reinsurance is often obtained as the optimal retention (Balbás *et al.*, 2009). Recall that $y \in L^2$ and lying between 0 and y_0 is said to be a stop-loss reinsurance if there exists $\alpha \geq 0$ such that

$$y = \begin{cases} y_0, & y_0 \leq \alpha \\ \alpha, & y_0 > \alpha \end{cases}. \tag{25}$$

Hereafter the random variable of (25) will be denoted by y_0^α.

Corollary 3 and Remark 2 show the importance of solving (15) when $\rho = w_0\Gamma - w_1\mathbb{E}$, since the solution will provide a very stable optimal reinsurance contract.

Theorem 7. *Consider Problem (15) with the risk measure $w_0\Gamma - w_1\mathbb{E}$. Suppose that $\mathbb{P}\,(z_\pi > w_1) = 1$.*[7]

a) There exists $\alpha^* > 0$ such that $y_0^{\alpha^*}$ solves (15).

b) Suppose that $y_0^{\alpha^*}$ solves (15), $\mathbb{P}\,(y_0 = \alpha^*) = 0$ and $\left(\tau^*_{w_0\Gamma}, z^*_{w_0\Gamma}\right)$ is a KKT-multiplier of (15). Then

$$z^*_{w_0\Gamma} = \begin{cases} w_1, & if \quad y_0 < \alpha^* \\ \left(1+\tau^*_{w_0\Gamma}\right) z_\pi, & if \quad y_0 > \alpha^* \end{cases}. \tag{26}$$

c) Suppose that $y_0^{\alpha^*}$ solves (15), there is a unique $\omega_{i_0} \in \Omega$ with $y_0\,(\omega_{i_0}) = \alpha^*$ and $\left(\tau^*_{w_0\Gamma}, z^*_{w_0\Gamma}\right)$ is a KKT-multiplier of (15). Then

$$z^*_{w_0\Gamma}\,(\omega) = \begin{cases} w_1, & y_0 < \alpha^* \\ \dfrac{1 - \sum\limits_{y_0(\omega)>\alpha^*} \left(1+\tau^*_{w_0\Gamma}\right) z_\pi(\omega) - w_1 \sum\limits_{y_0(\omega)<\alpha^*} \left(1+\tau^*_{w_0\Gamma}\right)}{p_{i_0}}, & \omega = \omega_{i_0} \\ \left(1+\tau^*_{w_0\Gamma}\right) z_\pi\,(\omega)\,, & y_0 > \alpha^* \end{cases}, \tag{27}$$

and

$$\frac{1 - \sum\limits_{y_0(\omega)>\alpha^*} \left(1+\tau^*_{w_0\Gamma}\right) z_\pi\,(\omega_{i_0}) - w_1 \sum\limits_{y_0(\omega)<\alpha^*} \left(1+\tau^*_{w_0\Gamma}\right)}{p_{i_0}} \leq \left(1+\tau^*_{w_0\Gamma}\right) z_\pi\,(\omega_{i_0}) \tag{28}$$

[7] (20) implies the fulfillment of this property whenever $w_0 = 1$. Since $w_0 \leq 1$, the property holds if the reinsurer draws on the Expected Value Premium Principle, since then $z_\pi = k > 1$.

hold.

d) Suppose that $y_0^{\alpha^*}$ *solves (15) and* $\left(\tau^*_{w_0\Gamma}, z^*_{w_0\Gamma}\right)$ *is a* KKT*-multiplier of (15). Suppose that* ρ *is coherent and satisfies Assumption* 1. *If* $z^*_{w_0\Gamma - w_1 \mathbb{E}} \in \Delta_\rho$ *then* $y_0^{\alpha^*}$ *solves (15) for* $w_0\Gamma - w_1\mathbb{E}$.

Proof. a) Take the solution y^* of (15) whose existence is guaranteed by Theorem 1, and define

$$\alpha^* = Max\ \left\{y^*\left(\omega_i\right):\ i = 1, 2, .., n\right\}.$$

Lemma 6 implies that $\alpha^* > 0$. Let us see that $y^* = y_0^{\alpha^*}$. Indeed, y^* being (15)-feasible we have that $y^* \leq y_0$, so $\alpha^* \leq y_0\left(\omega\right)$ whenever $y^*\left(\omega\right) = \alpha^*$. Besides, if $y^*\left(\omega\right) < \alpha^*$ and $\left(\tau^*_{w_0\Gamma}, z^*_{w_0\Gamma}\right)$ is a KKT-multiplier (its existence follows from Theorem 1), then the first condition in (23) and Lemma 4 lead to $z^*_{w_0\Gamma}\left(\omega\right) = w_1$. Hence, the fourth condition in (23), Expression (24) and $z_\pi > w_1$ lead to $y^*\left(\omega\right) = y_0\left(\omega\right)$, and therefore $y^* = y_0^{\alpha^*}$.

b) As in the proof of a), $z^*_{w_0\Gamma}\left(\omega\right) = w_1$ whenever $y_0^{\alpha^*} < \alpha^*$. Suppose that $y_0^{\alpha^*}\left(\omega\right) = \alpha^*$. Then, consider the partition of Lemma 5 and obviously $\omega \in \Omega_2$ or $\omega \in \Omega_3$, since $y_0^{\alpha^*}\left(\omega\right) \neq 0$. But $\omega \in \Omega_3$ would imply $y_0\left(\omega\right) = \alpha^*$, which cannot hold.

c) As in the proof of b), $z^*_{w_0\Gamma}\left(\omega\right) = w_1$ whenever $y_0^{\alpha^*} < \alpha^*$. Suppose that $y_0^{\alpha^*}\left(\omega\right) = \alpha^*$. Then, consider the partition of Lemma 5 and obviously $\omega \in \Omega_2$ or $\omega \in \Omega_3$, since $y_0^{\alpha^*}\left(\omega\right) \neq 0$. But $\omega \in \Omega_3$ implies that $y_0\left(\omega\right) = \alpha^*$, and therefore $\omega = \omega_{i_0}$. Thus, taking into account (3), we have (27). Finally, (28) comes from (24), because $\mathbb{P}\left(y_0^{\alpha^*} = 0\right) = 0$ implies that $z^*_{w_0\Gamma} \leq \left(1 + \tau^*_{w_0\Gamma}\right) z_\pi$.

d) It trivially follows from Corollary 3 and Remark 2. □

Remark 3. *According to the previous theorem the "stable optimal retention" of Remark* 1 *is a stop-loss reinsurance* $y_0^{\alpha^*}$. *Theorem 7 also provides the multiplier* $z^*_{w_0\Gamma - w_1\mathbb{E}}$ *(see (26) or (27)), so the condition* $z^*_{w_0\Gamma} \in \Delta_\rho$ *is very easy to verify in practical examples. Actually, we will see in the next section that the assumptions of Statements 7b and 7c are always fulfilled in practice.* □

Remark 4. *Rockafellar et al. (2006) introduced the risk measure* $CVaR_{\mu_0}$, $\mu_0 \in (0, 1)$ *being the level of confidence.* $CVaR_{\mu_0}$ *is becoming very important and popular among practitioners and researchers for its interesting properties. Indeed, it is coherent and expectation bounded (Rockafellar et al., 2006), and compatible with the second order stochastic dominance and the classical utility functions (Ogryczak and Ruszczynski, 2002).*[8] *Rockafellar et al. (2006) proved that*

$$\Delta_{CVaR_{\mu_0}} = \left\{z \in L^2;\ 0 \leq z \leq \frac{1}{1-\mu_0},\ \mathbb{E}\left(z\right) = 1\right\}. \tag{29}$$

[8] Recall that the standard deviation is not compatible with the second order stochastic dominance if asymmetries are involved (Ogryczak and Ruszczynski, 1999), and the stop-loss reinsurance obviously generates asymmetric results.

Consider $w_0 = 1$ *(the expected wealth is not optimized by the ceding company). Thus, if* $\rho = CVaR_{\mu_0}$ *in Problem (15), then* $y_0^{\alpha^*}$ *will solve the problem (i.e., (29) will contain the random variable* z_Γ^**) as long as*

$$\frac{1}{1-\mu_0} \geq z_\Gamma^*, \tag{30}$$

which clearly holds for μ_0 *close enough to* 100%. *Analogously, if the insurance company deals with problem (14) and* $\rho_0 = CVaR_{\mu_0}$*, then the solution* $y_0^{\alpha^*}$ *of (14) for* $w_0\Gamma - w_1\mathbb{E}$ *will be still the solution for the* $w_0 CVaR_{\mu_0} - w_1\mathbb{E}$ *as long as*

$$\frac{w_0}{1-\mu_0} + w_1 \geq z_{w_1\Gamma}^*,$$

which is also obvious for $w_0 > 0$ *and* μ_0 *large enough. An illustrative numerical example will be given in Section* 5. □

5. Algorithm and numerical experiment

Next let us point out that the conditions of Theorem 7 always hold in practice, and the stable optimal retention $y_0^{\alpha^*}$ and the $KKT-$multiplier $(\tau_\Gamma^*, z_\Gamma^*)$ may be easily calculated by drawing on an appropriate algorithm. First of all we will introduce the algorithm and then we will present a numerical example. In order to simplify the exposition, in this section we will assume that $w_1 = 0$ (the expected wealth is not maximized, and only the risk level is minimized), though the extension for $w_1 > 0$ is straightforward.

Notice that, according to Theorem 7, $y_0^{\alpha^*}$ and $(\tau_\Gamma^*, z_\Gamma^*)$ will be known once we compute α^* and τ_Γ^*, *i.e.*, we only have to estimate two real numbers.

Without loss of generality we can suppose that

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n\} \subset \mathbb{R},$$

$$0 < \omega_1 < \omega_2 < ... < \omega_n,$$

and

$$\mathbb{P}(y_0 = \omega_i) = p_i,$$

$i = 1, 2, ..., n$.

Define

$$\alpha_{Max} = \omega_n.$$

Obviously, $y_0^{\alpha_{Max}} = y_0$ is (15)-feasible because $S_1 > 0$ and $\pi(0) = 0$. Due to (20), the premium principle of (22) generates a strictly increasing function π.[9] Consequently,

[9] *i.e.*, $\pi(y_1) < \pi(y_2)$ whenever $y_1 \leq y_2$ and $y_1 \neq y_2$.

$\pi(y_0 - y_0^\alpha)$ strictly decreases as α grows. Consider a first case ($Case\_1$) such that $\pi(y_0) \le S_1$, which implies that y_0^α is (15)-feasible for every $\alpha \ge 0$ and therefore we will consider

$$\alpha_{Min} = 0.$$ [10]

If $\pi(y_0) > S_1$ then the continuity of $\alpha \to \pi(y_0 - y_0^\alpha)$ implies the existence of a unique $\alpha_{Min} \in (0, \alpha_{Max})$ such that

$$\pi\left(y_0 - y_0^{\alpha_{Min}}\right) = S_1.$$

Let us distinguish two situations. $Case\_2$ arises if $\alpha_{Min} \notin \Omega$, in which case we will chose i_0 as the smallest subscript such that

$$\pi(y_0 - \omega_{i_0}) < S_1.$$

$Case\_3$ holds if $\alpha_{Min} = \omega_{i_0-1} \in \Omega$ for some i_0.

Obviously, for the three cases y_0^α is (15)-feasible if and only if

$$\alpha_{Min} \le \alpha \le \alpha_{Max}.$$

Algorithm 1. Suppose that $Case\_1$ holds. Lemma 6 implies that $y_0^{\alpha_{Min}}$ does not solve (15), so the stable optimal retention $y_0^{\alpha^*}$ satisfies $\pi\left(y_0 - y_0^{\alpha^*}\right) < S_1$ and the second condition in (16) leads to $\tau_\Gamma^* = 0$. Hence, we only have to estimate α^*.

$Step - 1$. Define

$$\alpha_1 = \frac{\omega_1}{2}, \alpha_2 = \omega_1, \alpha_3 = \frac{\omega_1 + \omega_2}{2}, \alpha_4 = \omega_2, ...\alpha_{2n-1} = \frac{\omega_{n-1} + \omega_n}{2}, \alpha_{2n} = \omega_n.$$

$Step - 2$. For $j = 1$ to n check whether $y_0^{\alpha_{2j-1}}$ and

$$z_{2j-1}^* = \begin{cases} 0, & if \quad \omega < \alpha_{2j-1} \\ z_\pi, & if \quad \omega \ge \alpha_{2j-1} \end{cases}$$

satisfy (23) and (24). If these conditions are satisfied for some $y_0^{\alpha_{2j-1}}$ then we will have the stable optimal retention and the KKT-multiplier.. Notice that two different values of j cannot satisfy (23) and (24), since (20) implies that $\mathbb{E}\left(z_{2j-1}^*\right)$ strictly decreases with j and therefore (3) cannot hold two times. Furthermore, if these conditions hold for some j then every $\alpha^* \in (\alpha_{2j-2}, \alpha_{2j})$ will generate a stop-loss stable optimal retention $y_0^{\alpha^*}$, since the same KKT-multipliers z_{2j-1}^* and $\tau_{2j-1}^* = 0$ will still apply.

[10] Actually, Constraint $\pi(y_0 - y) \le S_1$ is redundant in this case, and may be removed in (15).

$Step-3$. For $j=1$ to n check whether $y_0^{\alpha_{2j}}$ and

$$z_{2j}^{*}(\omega)=\begin{cases}0, & \omega<\alpha_{2j}\\ \frac{1-\sum_{\omega>\alpha_{2j}} z_{\pi}(\omega)}{p_j}, & \omega=\alpha_{2j}\\ z_{\pi}(\omega), & \omega>\alpha_{2j}\end{cases} \tag{31}$$

satisfy (23) and (24). Every time these conditions are satisfied we will have a solution of (15) for $\rho=\Gamma$. Notice that (24) will imply

$$\frac{1-\sum_{\omega>\alpha_{2j}} z_{\pi}(\alpha_{2j})}{p_j}\leq z_{\pi}(\alpha_{2j}) \tag{32}$$

(see(28)). □

Algorithm 2. Suppose that $Case\_2$ holds. Then proceed as in Algorithm 1 with a minor modification in $Step-1$. Now we must define

$$\alpha_{2i_0-1}=\frac{\alpha_{Min}+\omega_{i_0}}{2},\alpha_{2i_0}=\omega_{i_0},\alpha_{2i_0+1}=\frac{\omega_{i_0}+\omega_{i_0+1}}{2},....\alpha_{2n-1}=\frac{\omega_{n-1}+\omega_n}{2},\alpha_{2n}=\omega_n.$$

Obviously, $Step-2$ and $Step-3$ will start with $j=i_0$ rather than $j=1$.

$Step-4$ Let us finally check the optimality of $y_0^{\alpha_{Min}}$. In this case $\tau_{\Gamma}^{*}>0$ may hold and we are in the conditions of Theorem 7b. We must verify whether $y_0^{\alpha_{Min}}$, τ_{Γ}^{*} and

$$z_{\Gamma}^{*}=\begin{cases}0, & if\ \ y_0<\alpha_{Min}\\ (1+\tau_{\Gamma}^{*})\,z_{\pi}, & if\ \ y_0>\alpha_{Min}\end{cases}$$

satisfy (23) and (24) for some $\tau_{\Gamma}^{*}\geq 0$. Actually the only condition one must check is (3), *i.e.*,

$$(1+\tau_{\Gamma}^{*})\sum_{\omega_i>\alpha_{Min}} z_{\pi}(\omega_i)=1,$$

so the optimality of $y_0^{\alpha_{Min}}$ holds if and only if

$$\tau_{\Gamma}^{*}=\frac{1}{\sum_{\omega_i>\alpha_{Min}} z_{\pi}(\omega_i)}-1\geq 0. \tag{33}$$

Thus, $Step-4$ reduces to the verification of (33). □

Algorithm 3. Suppose that *Case_3* holds. Then proceed as in Algorithm 2 with a minor modification in $Step-1$. Now we must modify α_{2i_0-1} according to

$$\alpha_{2i_0-1}=\frac{\omega_{i_0-1}+\omega_{i_0}}{2},\alpha_{2i_0}=\omega_{i_0},\alpha_{2i_0+1}=\frac{\omega_{i_0}+\omega_{i_0+1}}{2},....\alpha_{2n-1}=\frac{\omega_{n-1}+\omega_n}{2},\alpha_{2n}=\omega_n.$$

Once again, $Step-2$ and $Step-3$ will start with $j=i_0$.

$Step-4$. We still have to check the optimality of $y_0^{\alpha_{Min}}=y_0^{\omega_{i_0-1}}$. This retention level is optimal if and only if we can find $\tau_\Gamma^*\geq 0$ such that $y_0^{\omega_{i_0-1}}$, τ_Γ^* and

$$z_\Gamma^*\left(\omega\right)=\begin{cases}0, & \omega<\omega_{i_0-1}\\ \frac{1-\sum_{\omega>\omega_{i_0-1}}\left(1+\tau_\Gamma^*\right)z_\pi(\omega)}{p_{i_0-1}}, & \omega=\omega_{i_0-1}\\ \left(1+\tau_\Gamma^*\right)z_\pi\left(\omega\right), & \omega>\omega_{i_0-1}\end{cases}$$

satisfy (23) and (24). The existence of τ_Γ^* is easy to verify, because, bearing in mind the findings of Sections 3 and 4, one only needs to check the conditions

$$0\leq 1-\sum_{\omega>\omega_{i_0-1}}\left(1+\tau_\Gamma^*\right)z_\pi\left(\omega\right), \tag{34}$$

$$\frac{1-\sum_{\omega>\omega_{i_0-1}}\left(1+\tau_\Gamma^*\right)z_\pi\left(\omega_{i_0-1}\right)}{p_{i_0-1}}\leq\left(1+\tau_\Gamma^*\right)z_\pi\left(\omega_{i_0-1}\right) \tag{35}$$

and

$$1-\sum_{\omega>\omega_{i_0-1}}\left(1+\tau_\Gamma^*\right)z_\pi\left(\omega\right)+\left(1+\tau_\Gamma^*\right)\sum_{\omega>\omega_{i_0-1}}z_\pi\left(\omega\right)\mathbb{P}\left(\omega\right)=1. \tag{36}$$

Equality (36) yields τ_Γ^*, and then the inequalities (34) and (35) are equivalent to

$$\tau_\Gamma^*\leq\frac{1}{\sum_{\omega>\omega_{i_0-1}}z_\pi\left(\omega\right)}-1 \tag{37}$$

and

$$\tau_\Gamma^*\geq\frac{1}{z_\pi\left(\omega_{i_0-1}\right)\left(p_{i_0-1}+1\right)}-1 \tag{38}$$

respectively. Thus, $Step-4$ reduces to the computation of τ_Γ^* by means of (36) and then the verification of the inequalities (37) and (38). □

Remark 5. *Notice that the existence of solution of (15) and the findings of Sections 3 and 4 show that at least one of the three algorithms must generate a stable optimal retention.* □

Next let us present a simple numerical example. Our only objective is to illustrate the algorithm above.

Example 1. Suppose that y_0 can reach the values 100, 200, 300, 400 and 500 with a similar probability 0.2. Suppose that the reinsurer uses the Expected Value Premium Principle with a price 80% higher than the expected claims, *i.e.*,

$$\pi(y) = 1.8\mathbb{E}(y).$$

Suppose finally that the ceding company does not impose any budget constraint, *i.e.*, we are in *Case_1* above. With the notations of Algorithm 1, define

$$\alpha_1 = 50, \alpha_2 = 100, \alpha_3 = 150, \alpha_4 = 200, ..., \alpha_9 = 450, \alpha_{10} = 500.$$

In $Step - 2$ we have to check the optimality of five stop-loss contracts. The first one is y_0^{50}. Consider

$$z_1^* = \begin{cases} 0, & if \quad \omega < 50 \\ 1.8, & if \quad \omega \geq 50 \end{cases}.$$

Obviously, z_1^* remains constant and equals 1.8, so it is not in the set Δ_Γ of (10). Then, y_0^{50} is not a stable optimal retention. If one repeats the analysis with the four remaining "candidates" then similar results apply, so $Step - 2$ does not generate any stable optimal retention.

In $Step - 3$ we have to check the optimality of the remaining five stop-loss contracts. The first one is y_0^{100}, and (31) gives

$$z_2^* = \begin{cases} 0, & if \quad \omega < 100 \\ -2.2, & if \quad \omega = 100 \\ 1.8, & if \quad \omega > 100 \end{cases}$$

which do not belong to Δ_Γ. Repeat the exercise with the remaining values of α, and for $\alpha = 200$ we get

$$z_4^* = \begin{cases} 0, & if \quad \omega < 200 \\ -0.4, & if \quad \omega = 200 \\ 1.8, & if \quad \omega > 200 \end{cases}$$

which implies that y_0^{200} is not a stable optimal retention either. Analogously, for $\alpha = 300$ we get

$$z_6^* = \begin{cases} 0, & if \quad \omega < 300 \\ 1.4, & if \quad \omega = 300 \\ 1.8, & if \quad \omega > 300 \end{cases}$$

and y_0^{300} is the "stable optimal retention" we are looking for. It is easy to check that y_0^{400} and y_0^{500} are not stable optimal retentions. In fact, for y_0^{400} one obtains

$$z_8^* = \begin{cases} 0, & if \quad \omega < 400 \\ 3.2, & if \quad \omega = 400 \\ 1.8, & if \quad \omega > 400 \end{cases}$$

and this multiplier is not feasible because (32) does not hold. An analogous caveat arises for y_0^{500}.

Reinsurance y_0^{300} will be the optimal retention for many risk measures. For instance, if one considers $\rho = CVaR_{\mu_0}$, according to (30) y_0^{300} solves the problem if

$$\frac{1}{1-\mu_0} \geq 1.8$$

which holds for $\mu_0 \geq 0.45$ (or $\mu_0 \geq 45\%$), and, in particular, for the usual values of this parameter in the industry, which are higher than 90%. Finally, it is worthwhile to point out that the role of the $CVaR_{\mu_0}$ may be also played by many other important risk measures in Actuarial Sciences, such as, $WCVaR$, DPT, Wang, etc. □

6. Conclusions

The optimal reinsurance problem is a classic topic in Actuarial Theory and has been studied under different assumptions and by using different criteria to compute the insurer risk. Since coherent and expectation bounded risk measures are becoming very important in Finance and Insurance, recent approaches deal with them and the optimal reinsurance problem. However, there is no consensus about the risk measure that one must use, since every risk measure presents advantages and shortcomings when compared with others.

This article analyzes the stability of the optimal reinsurance with respect to the risk measure that the insurer uses. It has been pointed out that there is a "stable optimal retention" that will show no sensitivity, insofar as it will solve the optimal reinsurance problem for many risk measures, providing a very robust reinsurance plan. For the expected value premium principle this stable optimal retention is a stop-loss contract, and it is easy to compute in practice. An algorithm has been given and a numerical example presented. The approach is general enough. Actually, if desired, the analysis permits us to incorporate both budget constraints and the simultaneous maximization of the ceding company expected wealth.

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The usual caveat applies.

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