FOURIER-PADÉ APPROXIMANTS FOR NIKISHIN SYSTEMS

GUILLERMO LÓPEZ LAGOMASINO AND JUDIT MÍNGUEZ CENICEROS

Abstract. We study type I Fourier-Padé approximation for certain systems of functions formed by the Cauchy transform of finite Borel measures supported on bounded intervals of the real line. This construction is similar to type I Hermite-Padé approximation. Instead of power series expansions of the functions in the system, we take their development in a series of orthogonal polynomials. We give the exact rate of convergence of the corresponding approximants. The answer is expressed in terms of the extremal solution of an associated vector valued equilibrium problems for the logarithmic potential.

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1. Introduction

We study type I Fourier-Padé approximation for vector valued analytic functions formed by Nikishin systems. Fourier-Padé approximation of general analytic functions was first considered by S.P. Suetin. In [9] and [10], he obtained convergence for row sequences of Fourier-Padé approximants extending to this setting the classical Montessus de Ballore Theorem. Diagonal sequences of Fourier-Padé approximants of Cauchy transforms of measures supported on the real line were studied by A. A. Gonchar, E. A. Rakhmanov, and S. P. Suetin. In [4] they describe the rate of convergence of such approximants in terms of the equilibrium measure of an associated potential theoretic problem.

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M. Bello, G. López and J. Mínguez studied in [1] type II Fourier-Padé approximation for Angelesco systems of functions. Angelesco systems are formed by the Cauchy transform of a finite set of Borel measures supported on non-intersecting intervals of the real line. They obtained the rate of convergence of the error of approximation by means of linear and non-linear Fourier-Padé approximants. Here, the solution depends on a vector valued equilibrium problem for the logarithmic potential.

Let $S = (s_1, \ldots, s_m)$ be a system of finite Borel measures with constant sign, and bounded support consisting of infinitely many points contained in the real line. Let $\widehat{S} = (\widehat{s}_1, \ldots, \widehat{s}_m)$ be the corresponding system of Markov functions; that is,

$$\widehat{s}_k(z) = \int \frac{ds_k(x)}{z-x}, \quad k = 1, \ldots, m. \tag{1}$$

When the support of these measures lie in non-intersecting intervals one obtains an Angelesco system.

Another special system of Markov functions was introduced by E. M. Nikishin in [5]. They constitute an important model class of functions in the theory of multiple orthogonal polynomials and simultaneous rational approximations since many classical results of these theories have found their corresponding analogues; thus, have attracted increasing attention in recent decades. Let us define them.

Let σ_1, σ_2 be two measures with constant sign supported on $\mathbb{R}$ and let Δ_1, Δ_2 denote the smallest intervals containing their supports, $\mathrm{supp}(\sigma_1)$ and $\mathrm{supp}(\sigma_2)$, respectively. We write $\mathrm{Co}(\mathrm{supp}(\sigma_j)) = \Delta_j$. Assume that $\Delta_1 \cap \Delta_2 = \emptyset$ and define

$$\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\sigma_2(t)}{x-t} d\sigma_1(x) = \widehat{\sigma}_2(x) d\sigma_1(x).$$

Therefore, $\langle \sigma_1, \sigma_2 \rangle$ is a measure with constant sign and support equal to that of σ_1.

For a system of intervals $\Delta_1, \ldots, \Delta_m$ contained in $\mathbb{R}$ satisfying $\Delta_j \cap \Delta_{j+1} = \emptyset$, $j = 1, \ldots, m-1$, and finite Borel measures $\sigma_1, \ldots, \sigma_m$ with constant sign in $\mathrm{Co}(\mathrm{supp}(\sigma_j)) = \Delta_j$ and such that, each one has infinitely many points in its support, we define recursively

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_j \rangle = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_j \rangle \rangle, \quad j = 2, \ldots, m.$$

We say that $S = (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$, where

$$s_1 = \langle \sigma_1 \rangle = \sigma_1, \quad s_2 = \langle \sigma_1, \sigma_2 \rangle, \ldots, \quad s_m = \langle \sigma_1, \ldots, \sigma_m \rangle$$

is the Nikishin system of measures generated by $(\sigma_1, \ldots, \sigma_m)$. In the sequel, the system $(\sigma_1, \ldots, \sigma_m)$ is such that $\Delta_j \cap \Delta_{j+1} = \emptyset, j = 1, \ldots, m-1$.

Notice that all the measures in a Nikishin system have the same support, namely $\operatorname{supp}(\sigma_1)$. Take an arbitrary Nikishin system of measures $S = (s_1, \ldots, s_m)$, and let $\widehat{S} = (\widehat{s}_1, \ldots, \widehat{s}_m)$ be the corresponding Nikishin system of Markov functions.

Let σ_0 be a finite Borel measure with constant sign, and bounded support consisting of infinitely many points contained in an interval Δ_0, such that

$$\Delta_0 \cap \Delta_1 = \emptyset.$$

Consider the sequence $\{\ell_j\}$, $j \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, of orthonormal polynomials with respect to σ_0 with positive leading coefficient.

For $\mathbf{n} = (n_0, n_1, \ldots, n_m) \in \mathbb{Z}_+^{m+1}$ we denote $|\mathbf{n}| = n_0 + n_1 + \cdots + n_m$. Let $A_{\mathbf{n},0}, A_{\mathbf{n},1}, \ldots, A_{\mathbf{n},m}$ be polynomials such that:

i) $\deg(A_{\mathbf{n},j}) \leq n_j - 1$, $j = 0, \ldots, m$, not all identically equal to zero.

ii) For $k = 0, \ldots, |\mathbf{n}| - 2$

$$\int \left(A_{\mathbf{n},0}(x) + \sum_{j=1}^{m} A_{\mathbf{n},j}(x)\widehat{s}_j(x) \right) \ell_k(x) d\sigma_0(x) = 0. \tag{2}$$

Finding $A_{\mathbf{n},0}, \ldots, A_{\mathbf{n},m}$ reduces to solving a homogeneous linear system of $|\mathbf{n}| - 1$ equations on $|\mathbf{n}|$ unknowns, so a non-trivial solution is guaranteed. The solution may not be unique. We call $(A_{\mathbf{n},0}, A_{\mathbf{n},1}, \ldots, A_{\mathbf{n},m})$ a type I Fourier-Padé approximant of $(\widehat{s}_1, \ldots, \widehat{s}_m)$ with respect to the multi-index $\mathbf{n}$.

Uniqueness is a desirable condition. A multi-index $\mathbf{n} = (n_0, n_1, \ldots, n_m) \in \mathbb{Z}_+^{m+1}$ is said to be normal if every solution to i)-ii) satisfies $\deg A_{\mathbf{n},j} = n_j - 1, j = 0, \ldots, m$. If an index is normal it is easy to verify that these polynomials are uniquely determined (except for a common factor). Set

$$\mathbb{Z}_+^{m+1}(\bullet) = \{\mathbf{n} \in \mathbb{Z}_+^{m+1} : n_0 \geq n_1 \geq n_2 \geq \cdots \geq n_m\}$$

In Proposition 2.1, we prove that all multi-indices in $\mathbb{Z}_+^{m+1}(\bullet)$ are normal. We normalize $(A_{\mathbf{n},0}, A_{\mathbf{n},1}, \ldots, A_{\mathbf{n},m})$ so that $A_{\mathbf{n},m}$ is monic.

Theorem 1 gives the rate of convergence of the $|\mathbf{n}|$-th root of the linear forms

$$L_{\mathbf{n},0}(z) = A_{\mathbf{n},0}(z) + \sum_{j=1}^{m} A_{\mathbf{n},j}(z)\widehat{s}_j(z).$$

under mild conditions on the sequence of multi-indices assuming that the measures $\sigma_j, j = 0, \ldots, m$, belong to the class **Reg** of regular measures. For different equivalent forms of defining regular measures see sections 3.1 to 3.3 in [8]. In particular, $\sigma_0 \in \mathbf{Reg}$ if and only if

$$\lim_n |\ell_n(z)|^{1/n} = \exp\{g_{\Omega_0}(z;\infty)\}\,,$$

uniformly on compact subsets of the complement of the smallest interval containing $\mathrm{supp}(\sigma_0)$, where $g_{\Omega_0}(\cdot;\infty)$ denotes the Green's function for the region $\Omega_0 = \mathbb{C} \setminus \mathrm{supp}(\sigma_0)$ with singularity at ∞. Analogously, one defines regularity for the other measures $\sigma_1, \ldots, \sigma_m$. In the sequel, we write $(\sigma_0; \sigma_1, \ldots, \sigma_m) \in \mathbf{Reg}$ to mean that $\sigma_k \in \mathbf{Reg}, k = 0, \ldots, m$. The system $(\sigma_1, \ldots, \sigma_m)$ will be used to construct the Nikishin system of functions whereas σ_0 will determine the system of orthogonal polynomials with respect to which the Fourier expansions is taken. Before stating Theorem 1, we need to introduce some notation and results from potential theory.

Let F_k, $k = 0, 1, \ldots, N$, be (not necessarily distinct) closed bounded intervals of the real line and $\mathcal{C} = (c_{j,k})$ a real, positive definite, symmetric matrix of order $N+1$. $\mathcal{C}$ will be called the interaction matrix. By $\mathcal{M}(F_k)$ we denote the class of all finite, positive, Borel measures with compact support consisting of an infinite set of points contained in F_k and $\mathcal{M}_1(F_k)$ is the subclass of probability measures of $\mathcal{M}(F_k)$. Set

$$\mathcal{M}_1 = \mathcal{M}_1(F_0) \times \cdots \times \mathcal{M}_1(F_N)\,.$$

Given a vector measure $\mu = (\mu_0, \mu_1, \ldots, \mu_N) \in \mathcal{M}_1$ and $j = 0, 1, \ldots, N$, we define the combined potential

$$W_j^{\mu}(x) = \sum_{k=0}^{N} c_{j,k} V^{\mu_k}(x)\,, \qquad x \in \Delta_j\,, \tag{3}$$

where

$$V^{\mu_k}(x) = \int \log \frac{1}{|x-t|}\, d\mu_k(t)\,,$$

denotes the standard logarithmic potential of μ_k. We denote

$$\omega_j^{\mu} = \inf\{W_j^{\mu}(x) : x \in F_j\}\,, \quad j = 0, 1, \ldots, N\,.$$

In Chapter 5 of [7] the authors prove (we state the result in a form convenient for our purpose).

Lemma 1. *Let $\mathcal{C}$ be a real, positive definite, symmetric matrix of order $N+1$. If there exists $\overline{\mu} = (\overline{\mu}_0, \overline{\mu}_1, \ldots, \overline{\mu}_N) \in \mathcal{M}_1$ such that for each $j = 0, 1, \ldots, N$*

$$W_j^{\overline{\mu}}(x) = \omega_j^{\overline{\mu}}\,, \qquad x \in supp(\overline{\mu}_j)\,,$$

then $\overline{\mu}$ is unique. Moreover, if $c_{j,k} \geq 0$ when $F_j \cap F_k \neq \emptyset$ then $\overline{\mu}$ exists.

An explanation of why this lemma follows from the results in [7] is contained in section 4 of [1]. The vector measure $\overline{\mu} \in \mathcal{M}_1$ is called the equilibrium solution for the vector potential problem determined by the interaction matrix $\mathcal{C}$ on the system of intervals $F_j\,, j = 0, 1, \ldots, N$.

Let $\Lambda = \Lambda(p_0, p_1, \ldots, p_m) \subset \mathbb{Z}_+^{m+1}(\bullet)$ be an infinite sequence of distinct multi-indices such that

$$\lim_{\mathbf{n} \in \Lambda} \frac{n_j}{|\mathbf{n}|} = p_j \in (0,1)\,, \qquad j = 0, \ldots, m\,.$$

Obviously, $p_0 \geq p_1 \geq \cdots \geq p_m$ and $\sum_{j=0}^m p_j = 1$.

Set

$$P_j = \sum_{k=j}^m p_k.$$

Let us define the interaction matrix $\mathcal{C}$ which is relevant for the rest of the paper. Take

$$\mathcal{C} = \begin{pmatrix} 1 & -\frac{P_1}{2} & 0 & \cdots & 0 & 0 \\ -\frac{P_1}{2} & P_1^2 & -\frac{P_1P_2}{2} & \cdots & 0 & 0 \\ 0 & -\frac{P_1P_2}{2} & P_2^2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & P_{m-1}^2 & -\frac{P_{m-1}P_m}{2} \\ 0 & 0 & 0 & \cdots & -\frac{P_{m-1}P_m}{2} & P_m^2 \end{pmatrix} \tag{4}$$

This matrix satisfies all the assumptions of Lemma 1 on the system of intervals $F_j = \Delta_j, j = 0, \ldots, m$, including $c_{j,k} \geq 0$ when $F_j \cap F_k \neq \emptyset$ and it is positive definite because the principal section $\mathcal{C}_r, r = 0, \ldots, m$ of $\mathcal{C}$ satisfies

$$\det(\mathcal{C}_r) = P_1^2 \cdots P_r^2 \det \begin{pmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 \end{pmatrix}_{(r+1)\times(r+1)} > 0.$$

Let $\overline{\mu}(\mathcal{C})$ be the equilibrium solution for the corresponding vector potential problem. We have

Theorem 1. *Let* $(\sigma_0;\sigma_1,\ldots,\sigma_m) \in \mathbf{Reg}, (s_1,\ldots,s_m) = \mathcal{N}(\sigma_1,\ldots,\sigma_m)$, *and consider a sequence of multi-indices* $\Lambda = \Lambda(p_0,\ldots,p_m)$. *Let* $(A_{\mathbf{n},0},\, A_{\mathbf{n},1},\ldots,\, A_{\mathbf{n},m}), \mathbf{n} \in \Lambda$, *be the associated sequence of type I Fourier-Padé with respect to* σ_0 *for the Nikishin system of functions* $(\widehat{s}_1,\ldots,\widehat{s}_m)$. *Then,*

$$\lim_{\mathbf{n}\in\Lambda} |L_{\mathbf{n},0}(z)|^{1/|\mathbf{n}|} = G_0(z)\,, \tag{5}$$

uniformly on each compact subset of $\mathbb{C}\setminus(\Delta_0\cup\Delta_1)$, *where*

$$G_0(z) = \exp\left(P_1 V^{\overline{\mu}_1}(z) - V^{\overline{\mu}_0}(z) - 2\sum_{k=1}^{m} \frac{\omega_k^{\overline{\mu}}}{P_k}\right).$$

$\overline{\mu} = \overline{\mu}(\mathcal{C}) = (\overline{\mu}_0,\ldots,\overline{\mu}_m)$ *is the equilibrium vector measure and* $(\omega_0^{\overline{\mu}},\ldots,\omega_m^{\overline{\mu}})$ *is the system of equilibrium constants for the vector potential problem determined by the interaction matrix* $\mathcal{C}$ *defined in* (4) *on the system of intervals* $\Delta_j, j = 0,\ldots,m$.

The corresponding result for Hermite-Padé approximants of Nikishin systems appears in Section 7, Chapter 5 of [7] (see also [6]).

Besides normality, in Section 2 we obtain the orthogonality relations satisfied by the different polynomials involved in the construction. Section 3 is devoted to the study of an extremal problem that allows to prove Theorem 3 in Section 4 of which Theorem 1 is a corollary.

2. Normality and orthogonality relations

Set

$$s_{j,k} = \langle \sigma_j,\ldots,\sigma_k\rangle, \quad 1 \le j < k \le m, \qquad s_{j,j} = \sigma_j\,.$$

We denote

$$L_{\mathbf{n},j}(z) = \sum_{k=j}^{m} A_{\mathbf{n},k}(z)\widehat{s}_{j+1,k}(z), \qquad j = 0,\ldots,m\,.$$

$(\widehat{s}_{j+1,j}(z) \equiv 1, L_{\mathbf{n},m} \equiv A_{\mathbf{n},m})$.

In [5], E. M. Nikishin introduced the following definition.

Definition 1. *A set of continuous real functions* $u_0(x),\ldots,\, u_m(x)$ *defined on an interval* Δ, *is called an AT-system for the index* $\mathbf{n} \in \mathbb{Z}_+^{m+1}$, *if for any polynomials* $h_0,\ldots,h_m$ *such that* $\deg(h_i) \le n_i - 1$, $i = 0,\ldots,m$, *not all simultaneously* *identically equal to zero, the function*

$$H(x) = h_0(x)u_0(x) + \cdots + h_m(x)u_m(x),$$

has at most $|\mathbf{n}| - 1$ *zeros on* Δ *(*$\deg h_j \leq -1$ *means that* $h_j \equiv 0$*).*

Let $\mathbb{Z}_+^{m+1}(*)$ be the set of multi-indices given by

$$\mathbb{Z}_+^{m+1}(*) = \{\mathbf{n} \in \mathbb{Z}_+^{m+1} : \nexists\, i < k < j \text{ such that } n_i < n_j < n_k\}.$$

In connection with AT-systems, in [2] U. Fidalgo and G. López proved

Lemma 2. *Let* $\mathbf{n} \in \mathbb{Z}_+^{m+1}(*)$ *and* $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$*, then* $(1, \widehat{s}_1, \ldots, \widehat{s}_m)$ *defines an AT-system with respect to* $\mathbf{n}$ *on any interval disjoint from* Δ_1*.*

Notice that for each $j \in \{0, \ldots, m-1\}$, $(s_{j+1,j+1}, \ldots, s_{j+1,m}) = \mathcal{N}(\sigma_{j+1}, \ldots, \sigma_m)$ and using Lemma 2 it follows that for $\mathbf{n} \in \mathbb{Z}_+^{m+1}(\bullet) \subset \mathbb{Z}_+^{m+1}(*)$ the linear form $L_{\mathbf{n},j}$ cannot have more that $N_j - 1$ zeros on Δ_j, where

$$N_j = n_j + \cdots + n_m\,.$$

Obviously, the same is true for the polynomial $L_{\mathbf{n},m} \equiv A_{\mathbf{n},m}$.

The definition of Fourier-Padé approximant implies that

$$\int x^k L_{\mathbf{n},0}(x) d\sigma_0(x) = 0\,, \qquad k = 0, \ldots, |\mathbf{n}| - 2\,. \tag{6}$$

Since the function $L_{\mathbf{n},0}(x)$ is continuous on Δ_0, from (6) we have that $L_{\mathbf{n},0}(x)$ has at least $|\mathbf{n}| - 1$ sign changes in the interior of Δ_0. This and the previous remark indicate that it has exactly $|\mathbf{n}| - 1$ sign changes in the interior of Δ_0; thus, all the zeros of $L_{\mathbf{n},0}(x)$ in Δ_0 are simple and lie in its interior. In connection with intervals of the real line, the interior refers to the Euclidean topology of $\mathbb{R}$. In short we shall see that $L_{\mathbf{n},0}(x)$ has no other zeros in $\overline{\mathbb{C}} \setminus \Delta_1$. Before proving this, let us turn to the question of normality.

Proposition 2.1. *Let* $\mathbf{n} \in \mathbb{Z}_+^{m+1}(\bullet)$ *and* $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$*. Then,* $\mathbf{n}$ *is normal and* $(A_{\mathbf{n},0}, \ldots, A_{\mathbf{n},m})$ *is uniquely determined except for a constant factor.*

Proof. Let us assume that there exists $j \in \{0, \ldots, m\}$ such that $\deg A_{\mathbf{n},j} \leq n_j - 2$. Then $\mathbf{n} - \mathbf{e}^j \in \mathbb{Z}_+^{m+1}(*)$, where $\mathbf{e}^j$ denotes the $m+1$ dimensional unit vector with all components equal to zero except the component $j+1$ which equals 1. According to Lemma 2 the linear form $L_{\mathbf{n},0}$ has at most $|\mathbf{n}| - 2$ zeros on Δ_0 but we pointed out before that it has at least $|\mathbf{n}| - 1$ sign changes on this interval. This contradiction yields that for all $j \in \{0, \ldots, m\}$, $\deg A_{\mathbf{n},j} = n_j - 1$, which implies normality.

Now let us assume that $(A_{\mathbf{n},0}, \ldots, A_{\mathbf{n},m})$ and $(A'_{\mathbf{n},0}, \ldots, A'_{\mathbf{n},m})$ solve i)-ii) and these vectors are not collinear. According to what we just proved, for all $j \in \{0, \ldots, m\}$, $\deg A_{\mathbf{n},j} = \deg A'_{\mathbf{n},j} = n_j - 1$. Take $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\deg(A_{\mathbf{n},0} - \lambda A'_{\mathbf{n},0}) \leq n_0 - 2$. Obviously, the vector $(A_{\mathbf{n},0} - \lambda A'_{\mathbf{n},0}, \ldots, A_{\mathbf{n},m} - \lambda A'_{\mathbf{n},0})$ is not identically equal to zero and also solves i)-ii) which is not possible since all non trivial solutions must have all components of maximal degree. □

Because of Proposition 2.1, we can assume that $(A_{\mathbf{n},0}, A_{\mathbf{n},1}, \ldots, A_{\mathbf{n},m})$ is normalized so that $A_{\mathbf{n},m}$ is a monic polynomial of degree $n_m - 1$. In the rest of the paper we take this normalization in order to determine the linear forms $L_{\mathbf{n},j}$ in a unique manner.

In the sequel, we assume that $\mathbf{n} \in \mathbb{Z}_+^{m+1}(\bullet)$. For $j = 0, \ldots, m$, let $Q_{\mathbf{n},j}$ be the monic polynomial whose zeros are those of the linear form $L_{\mathbf{n},j}$ in the region $\overline{\mathbb{C}} \setminus \Delta_{j+1}$ counting multiplicities ($\Delta_{m+1} = \emptyset$). In particular, $L_{\mathbf{n},m} = A_{\mathbf{n},m} = Q_{\mathbf{n},m}$. From the previous proposition, if $n_m \geq 1$, ∞ is not a zero of any one of these linear forms; thus, ∞ cannot be an accumulation point of zeros of them. Though it is not the case, in principle, some of these linear forms may have an infinite number of zeros which may accumulate on the boundary of the corresponding region of meromorphicity. In that case, for the time being, $Q_{\mathbf{n},j}$ denotes a formal infinite product.

Proposition 2.2. *Let* $\mathbf{n} \in \mathbb{Z}_+^{m+1}(\bullet), n_m \geq 1$, *and* $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$. *Then,* $\deg Q_{\mathbf{n},j} = N_j - 1, j = 0 \ldots, m$, *all its zeros are simple and lie in the interior of* Δ_j. *Moreover,*

$$\int x^\nu L_{\mathbf{n},j}(x) \frac{d\sigma_j(x)}{Q_{\mathbf{n},j-1}(x)} = 0\,, \qquad \nu = 0, \ldots, N_j - 2\,, \tag{7}$$

$(Q_{\mathbf{n},-1} \equiv 1)$.

Proof. We proceed by induction on j. For $j = 0$, (7) is (6) and this implies that $L_{\mathbf{n},0}$ has $N_0 - 1 = |\mathbf{n}| - 1$ simple zeros in the interior of Δ_0. Therefore, $\deg Q_{\mathbf{n},0} \geq N_0 - 1$. If $\deg Q_{\mathbf{n},0} = N_0 - 1$ we conclude with the initial step. Suppose that $\deg Q_{\mathbf{n},0} \geq N_0$ (including the possible case that $\deg Q_{\mathbf{n},0} = \infty$). Choose N_0 zeros of $Q_{\mathbf{n},0}$ and denote the monic polynomial with these N_0 zeros by $Q^*_{\mathbf{n},0}$.

Notice that

$$\frac{L_{\mathbf{n},0}}{Q^*_{\mathbf{n},0}} \in \mathcal{H}(\overline{\mathbb{C}} \setminus \Delta_1)$$

is analytic in the indicated region and

$$\frac{z^\nu L_{\mathbf{n},0}}{Q^*_{\mathbf{n},0}} = \mathcal{O}\left(\frac{1}{z^2}\right), \qquad \nu = 0,\ldots,N_1-1\,.$$

Let Γ_1 be a closed, smooth, Jordan curve that surrounds Δ_1 such that all the zeros of $Q^*_{\mathbf{n},0}$ lie in the unbounded connected component of the complement of Γ_1. By Cauchy's Theorem, Cauchy's Integral Formula and Fubini's Theorem, we have

$$0 = \frac{1}{2\pi i}\int_{\Gamma_1} z^\nu \frac{L_{\mathbf{n},0}(z)}{Q^*_{\mathbf{n},0}(z)}dz = \frac{1}{2\pi i}\int_{\Gamma_1} z^\nu \frac{\sum_{k=1}^m A_{\mathbf{n},k}(z)\widehat{s}_k(z)}{Q^*_{\mathbf{n},0}(z)}dz =$$
$$\frac{1}{2\pi i}\int_{\Gamma_1} z^\nu \sum_{k=1}^m A_{\mathbf{n},k}(z)\int\frac{\widehat{s}_{2,k}(x)d\sigma_1(x)}{z-x}\frac{dz}{Q^*_{\mathbf{n},0}(z)} = \int x^\nu L_{\mathbf{n},1}(x)\frac{d\sigma_1(x)}{Q^*_{\mathbf{n},0}(x)},$$

with $\nu = 0,\ldots,N_1-1$. This implies that $L_{\mathbf{n},1}$ has at least N_1 zeros on Δ_1. According to Lemma 2 this linear form can only have N_1-1 zeros on this interval. This implies that our initial assumption is false and $\deg Q_{\mathbf{n},0} = N_0 - 1$.

Assume that the statement is true for some $j \in \{0,\ldots,m-1\}$ and let us show that it holds for $j+1$. Indeed, since $\deg Q_{\mathbf{n},j} = N_j - 1$ and its zeros are simple and lie in the interior of Δ_j then

$$\frac{L_{\mathbf{n},j}}{Q_{\mathbf{n},j}} \in \mathcal{H}(\overline{\mathbb{C}}\setminus\Delta_{j+1}), \qquad \frac{z^\nu L_{\mathbf{n},j}}{Q_{\mathbf{n},j}} = \mathcal{O}\left(\frac{1}{z^2}\right), \qquad \nu = 0,\ldots,N_{j+1}-2\,.$$

Let Γ_{j+1} be a closed, smooth, Jordan curve that surrounds Δ_{j+1} such that Δ_j lies in the unbounded connected component of the complement of Γ_{j+1}. By Cauchy's Theorem, Cauchy's Integral Formula and Fubini's Theorem, it follows that

$$0 = \frac{1}{2\pi i}\int_{\Gamma_{j+1}} z^\nu \frac{L_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j}(z)}dz = \frac{1}{2\pi i}\int_{\Gamma_{j+1}} z^\nu \frac{\sum_{k=j+1}^m A_{\mathbf{n},k}(z)\widehat{s}_{j+1,k}(z)}{Q_{\mathbf{n},j}(z)}dz =$$
$$\frac{1}{2\pi i}\int_{\Gamma_{j+1}} z^\nu \sum_{k=j+1}^m A_{\mathbf{n},k}(z)\int\frac{\widehat{s}_{j+2,k}(x)d\sigma_{j+1}(x)}{z-x}\frac{dz}{Q_{\mathbf{n},j}(z)} = \int x^\nu L_{\mathbf{n},j+1}(x)\frac{d\sigma_{j+1}(x)}{Q_{\mathbf{n},j}(x)},$$

with $\nu = 0,\ldots,N_{j+1}-2$. We have obtained (7) for $j+1$.

Formula (7) for $j+1$ implies that $Q_{\mathbf{n},j+1}$ has at least $N_{j+1}-1$ simple zeros in the interior of Δ_{j+1}. If $\deg Q_{\mathbf{n},j+1} = N_{j+1}-1$ we have finished the proof (for example, this is the case when $j+1 = m$ because $L_{\mathbf{n},m} \equiv A_{\mathbf{n},m}$). Let us suppose that $\deg Q_{\mathbf{n},j+1} \geq N_{j+1}$ (including the possible case that $\deg Q_{\mathbf{n},j+1} = \infty$, and of course $j \leq m-2$). Choose N_{j+1} zeros of $Q_{\mathbf{n},j+1}$ and denote the monic polynomial with these N_{j+1} zeros by $Q^*_{\mathbf{n},j+1}$. Then

$$\frac{L_{\mathbf{n},j+1}}{Q^*_{\mathbf{n},j+1}} \in \mathcal{H}(\overline{\mathbb{C}}\setminus\Delta_{j+2}), \qquad \frac{z^\nu L_{\mathbf{n},j+1}}{Q^*_{\mathbf{n},j+1}} = \mathcal{O}\left(\frac{1}{z^2}\right), \qquad \nu = 0,\ldots,N_{j+2}-1\,.$$

Let Γ_{j+2} be a closed, smooth, Jordan curve that surrounds Δ_{j+2} such that Δ_{j+1} and all the zeros of $Q^*_{\mathbf{n},j+1}$ lie in the unbounded connected component of the complement of Γ_{j+2}. By Cauchy's Theorem, Cauchy's Integral Formula and Fubini's Theorem, we have

$$0 = \frac{1}{2\pi i}\int_{\Gamma_{j+2}} z^\nu \frac{L_{\mathbf{n},j+1}(z)}{Q^*_{\mathbf{n},j+1}(z)}dz = \frac{1}{2\pi i}\int_{\Gamma_{j+2}} z^\nu \frac{\sum_{k=j+2}^m A_{\mathbf{n},k}(z)\widehat{s}_{j+2,k}(z)}{Q^*_{\mathbf{n},j+1}(z)}dz =$$

$$\frac{1}{2\pi i}\int_{\Gamma_{j+2}} z^\nu \sum_{k=j+2}^m A_{\mathbf{n},k}(z)\int \frac{\widehat{s}_{j+3,k}(x)d\sigma_{j+2}(x)}{z-x}\frac{dz}{Q^*_{\mathbf{n},j+1}(z)} =$$

$$\int x^\nu L_{\mathbf{n},j+2}(x)\frac{d\sigma_{j+2}(x)}{Q^*_{\mathbf{n},j+1}(x)},$$

with $\nu = 0,\dots,N_{j+2}-1$. This implies that $L_{\mathbf{n},j+2}$ has at least N_{j+2} zeros on Δ_{j+2}. According to Lemma 2 this linear form can only have $N_{j+2}-1$ zeros on this interval. This implies that our initial assumption is false; therefore, $\deg Q_{\mathbf{n},j+1} = N_{j+1}-1$ as we needed to prove. □

Proposition 2.3. *Let* $\mathbf{n} \in \mathbb{Z}_+^{m+1}(\bullet), n_m \geq 1,$ *and* $(s_1,\dots,s_m) = \mathcal{N}(\sigma_1,\dots,\sigma_m)$. *Then, for each* $j = 0,\dots,m-1$ *and each polynomial* $q, \deg q \leq N_{j+1}-1$,

$$\frac{q(z)L_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j}(z)} = \int \frac{q(x)L_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j}(x)}\frac{d\sigma_{j+1}(x)}{z-x}. \tag{8}$$

Proof. From Proposition 2.2 for any $q, \deg q \leq N_{j+1}-1$,

$$\frac{qL_{\mathbf{n},j}}{Q_{\mathbf{n},j}} \in \mathcal{H}(\overline{\mathbb{C}}\setminus\Delta_{j+1}), \quad \frac{qL_{\mathbf{n},j}}{Q_{\mathbf{n},j}} = \mathcal{O}\left(\frac{1}{z}\right), z\to\infty.$$

Let Γ_{j+1} be a closed, smooth, Jordan curve that surrounds Δ_{j+1} such that Δ_j and z lie in the unbounded connected component of the complement of Γ_{j+1}. By Cauchy's Integral Formula, Cauchy's Theorem, and Fubini's Theorem, it follows that

$$\frac{q(z)L_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j}(z)} = \frac{1}{2\pi i}\int_{\Gamma_{j+1}} \frac{q(\zeta)L_{\mathbf{n},j}(\zeta)}{Q_{\mathbf{n},j}(\zeta)}\frac{d\zeta}{z-\zeta} =$$

$$\frac{1}{2\pi i}\int_{\Gamma_{j+1}} \frac{q(\zeta)\sum_{k=j+1}^m A_{\mathbf{n},k}(\zeta)\widehat{s}_{j+1,k}(\zeta)}{Q_{\mathbf{n},j}(\zeta)}\frac{d\zeta}{z-\zeta} =$$

$$\int \sum_{k=j+1}^m \frac{1}{2\pi i}\int_{\Gamma_{j+1}} \frac{q(\zeta)A_{\mathbf{n},k}(\zeta)}{Q_{\mathbf{n},j}(\zeta)(z-\zeta)}\frac{d\zeta}{\zeta-x}\widehat{s}_{j+2,k}(x)d\sigma_{j+1}(x) =$$

$$\int \frac{q(x)L_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j}(x)}\frac{d\sigma_{j+1}(x)}{z-x},$$

and we have obtained (8). □

3. Extremal Problems

Let $\{\mu_l\} \subset \mathcal{M}(\mathcal{K})$ be a sequence of measures, where $\mathcal{K}$ is a compact subset of the complex plane and $\mu \in \mathcal{M}(\mathcal{K})$. We write

$$* \lim_{l} \mu_l = \mu\,, \qquad \mu \in \mathcal{M}(\mathcal{K})\,,$$

if for every continuous function $f \in \mathcal{C}(\mathcal{K})$

$$\lim_{l} \int f d\mu_l = \int f d\mu\,;$$

that is, when the sequence of measures converges to μ in the weak star topology. Given a polynomial q_l of degree $l \geq 1$, we denote the associated normalized zero counting measure by

$$\nu_{q_l} = \frac{1}{l} \sum_{q_l(x)=0} \delta_x\,,$$

where δ_x is the Dirac measure with mass 1 at x (in the sum the zeros are repeated according to their multiplicity).

In order to prove our main result we need Theorem 3.3.3 of [8]. We present it in the form stated in [3] which is more adequate for our purpose. In [3], it was proved under stronger assumptions on the measure.

Lemma 3. *Let $\{\phi_l\}, l \in \Lambda \subset \mathbb{Z}_+$, be a sequence of positive continuous functions on a bounded closed interval $\Delta \subset \mathbb{R}$, $\sigma \in \mathbf{Reg} \cap \mathcal{M}(\Delta)$, and let $\{q_l\}, l \in \Lambda$, be a sequence of monic polynomials such that $\deg q_l = l$ and*

$$\int x^k q_l(x) \phi_l(x) d\sigma(x) = 0, \qquad k = 0, \ldots, l-1.$$

Assume that

$$\lim_{l \in \Lambda} \frac{1}{2l} \log \frac{1}{|\phi_l(x)|} = v(x),$$

uniformly on Δ. Then

$$* \lim_{l \in \Lambda} \nu_{q_l} = \overline{\nu},$$

and

$$\lim_{l \to \infty} \left(\int |q_l(x)|^2 \phi_l(x) d\mu(x) \right)^{1/2l} = e^{-\omega},$$

where $\overline{\nu} \in \mathcal{M}_1(\Delta)$ is the unique solution of the extremal problem

$$V^{\overline{\nu}}(x) + v(x) \begin{cases} = \omega, & x \in supp(\overline{\nu})\,, \\ \geq \omega, & x \in \Delta\,, \end{cases}$$

in the presence of the external field v.

Using this result, we can obtain the asymptotic limit distribution of the zeros of the polynomials $Q_{\mathbf{n},j}$, $j = 0, \ldots, m$.

Theorem 2. *Let* $(\sigma_0; \sigma_1, \ldots, \sigma_m) \in \mathbf{Reg}, (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$, *and consider the sequence of multi-indices* $\Lambda = \Lambda(p_0, \ldots, p_m)$. *Then*

$$* \lim_{\mathbf{n}\in\Lambda} \nu_{Q_{\mathbf{n},j}} = \overline{\mu}_j, \quad j = 0, \ldots, m, \tag{9}$$

where $\overline{\mu} = \overline{\mu}(\mathcal{C}) \in \mathcal{M}_1$ *is the vector equilibrium measure determined by the matrix* $\mathcal{C}$ *in* (4) *on the system of intervals* $F_j = \Delta_j$, $j = 0, \ldots, m$.

Proof. The unit ball in the cone of positive Borel measures is weakly compact; therefore, it is sufficient to show that each one of the sequences of measures $\{\nu_{Q_{\mathbf{n},j}}\}$, $\mathbf{n} \in \Lambda$, $j = 0, \ldots, m$, has only one accumulation point which coincides with the corresponding component of the vector measure $\overline{\mu}(\mathcal{C})$. Let $\Lambda' \subset \Lambda$ be a subsequence of multi-indices such that for each $j = 0, \ldots, m$

$$* \lim_{\mathbf{n}\in\Lambda'} \nu_{Q_{\mathbf{n},j}} = \nu_j.$$

Notice that $\nu_j \in \mathcal{M}_1(\Delta_j)$, $j = 0, \ldots, m$. Therefore,

$$\lim_{\mathbf{n}\in\Lambda'} |Q_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp(-P_j V^{\nu_j}(z)), \tag{10}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_j$, where $P_j = p_j + \cdots , p_m$.

Because of the normalization adopted on $A_{\mathbf{n},m}$, $L_{\mathbf{n},m} = Q_{\mathbf{n},m}$; consequently, when $j = m$, (7) takes the form

$$\int x^\nu Q_{\mathbf{n},m}(x) \frac{d|\sigma_m|(x)}{|Q_{\mathbf{n},m-1}(x)|} = 0\,, \qquad \nu = 0, \ldots, N_m - 2\,.$$

(By $|\sigma|$ we denote the total variation of the measure σ.) According to (10)

$$\lim_{\mathbf{n}\in\Lambda'} \frac{1}{2N_m} \log|Q_{\mathbf{n},m-1}(x)| = -\frac{P_{m-1}}{2P_m} V^{\nu_{m-1}}(x)\,,$$

uniformly on Δ_m. Using Lemma 3, it follows that ν_m is the unique solution of the extremal problem

$$V^{\nu_m}(x) - \frac{P_{m-1}}{2P_m} V^{\nu_{m-1}}(x) \begin{cases} = \omega_m, & x \in \operatorname{supp}(\nu_m)\,, \\ \geq \omega_m, & x \in \Delta_m\,, \end{cases} \tag{11}$$

and

$$\lim_{\mathbf{n}\in\Lambda'} \left(\int \frac{Q^2_{\mathbf{n},m}(x)}{|Q_{\mathbf{n},m-1}(x)|} d|\sigma_m|(x) \right)^{1/2N_m} = e^{-\omega_m}\,. \tag{12}$$

Let us show by induction on decreasing values of j, that for all $j \in \{0,\ldots,m\}$

$$(13)\quad V^{\nu_j}(x)-\frac{P_{j-1}}{2P_j}V^{\nu_{j-1}}(x)-\frac{P_{j+1}}{2P_j}V^{\nu_{j+1}}(x)+\frac{P_{j+1}}{P_j}\omega_{j+1}\begin{cases} =\omega_j, & x\in \operatorname{supp}(\nu_j)\,,\\ \geq \omega_j, & x\in\Delta_j\,,\end{cases}$$

where $P_{-1}=P_{m+1}=0$, and

$$(14)\qquad \lim_{\mathbf{n}\in\Lambda'}\left(\int \frac{Q^2_{\mathbf{n},j}(x)}{|Q_{\mathbf{n},j-1}(x)|}\frac{|L_{\mathbf{n},j}(x)|}{|Q_{\mathbf{n},j}(x)|}d|\sigma_j|(x)\right)^{1/2N_j}=e^{-\omega_j}\,,$$

where $Q_{\mathbf{n},-1}\equiv 1$. For $j=m$ these relations are non other than (11)-(12) and the initial induction step is settled. Let us assume that the statement is true for $j+1\in\{1,\ldots,m\}$ and let us prove it for j.

It is easy to see that the orthogonality relations (7) can be expressed as

$$\int x^{\nu}Q_{\mathbf{n},j}(x)\frac{|Q_{\mathbf{n},j+1}(x)L_{\mathbf{n},j}(x)|}{|Q_{\mathbf{n},j}(x)|}\frac{d|\sigma_j|(x)}{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|}=0\,,\qquad \nu=0,\ldots,N_j-2\,.$$

On account of (8) with $q=Q_{\mathbf{n},j+1}$, this can be further transformed into

$$\int x^{\nu}Q_{\mathbf{n},j}(x)\left(\int\frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|}\frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|}\frac{d|\sigma_{j+1}|(t)}{|x-t|}\right)\frac{d|\sigma_j|(x)}{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|}=0\,,$$

for $\nu=0,\ldots,N_j-2$.

Relation (10) implies that

$$(15)\qquad \lim_{\mathbf{n}\in\Lambda'}\frac{1}{2N_j}\log|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|=-\frac{P_{j-1}}{2P_j}V^{\nu_{j-1}}(x)-\frac{P_{j+1}}{2P_j}V^{\nu_{j+1}}(x)\,,$$

uniformly on Δ_j. (Since $Q_{\mathbf{n},-1}\equiv 1$, when $j=0$ we only get the second term on the right hand side of this limit.)

Set

$$\rho_{\mathbf{n},j+1}=\int\frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|}\frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|}d|\sigma_{j+1}|(t).$$

It follows that for all $x\in\Delta_j$

$$\frac{\rho_{\mathbf{n},j+1}}{\delta^*_{j+1}}\leq\int\frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|}\frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|}\frac{d|\sigma_{j+1}|(t)}{|x-t|}\leq\frac{\rho_{\mathbf{n},j+1}}{\delta_{j+1}},$$

where $0<\delta_{j+1}=\inf\{|x-t|:t\in\Delta_{j+1},x\in\Delta_j\}\leq\max\{|x-t|:t\in\Delta_{j+1},x\in\Delta_j\}=\delta^*_{j+1}<\infty$. Taking into consideration these inequalities, from the induction hypothesis we obtain that

$$(16)\qquad \lim_{\mathbf{n}\in\Lambda'}\left(\int\frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|}\frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|}\frac{d|\sigma_{j+1}|(t)}{|x-t|}\right)^{1/2N_j}=e^{-P_{j+1}\omega_{j+1}/P_j}$$

Taking (15) and (16) into account, Lemma 3 yields that ν_j is the unique solution of the extremal problem (13) and

$$\lim_{\mathbf{n}\in\Lambda'}\left(\int\int \frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|}\frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|}\frac{d|\sigma_{j+1}|(t)}{|x-t|}\frac{Q^2_{\mathbf{n},j}(x)d|\sigma_j|(x)}{|Q_{\mathbf{n},j-1}(x)Q_{\mathbf{n},j+1}(x)|}\right)^{1/2N_j} = e^{-\omega_j}$$

According to (8) with $q = Q_{\mathbf{n},j+1}$

$$\frac{1}{|Q_{\mathbf{n},j+1}(x)|}\int \frac{Q^2_{\mathbf{n},j+1}(t)}{|Q_{\mathbf{n},j}(t)|}\frac{|L_{\mathbf{n},j+1}(t)|}{|Q_{\mathbf{n},j+1}(t)|}\frac{d|\sigma_{j+1}(t)}{|x-t|} = \frac{|L_{\mathbf{n},j}(x)|}{|Q_{\mathbf{n},j}(x)|}, \quad x\in\Delta_j\,,$$

which allows to reduce the previous formula to (14) thus concluding the induction proof.

Now, we can rewrite (13) multiplying through by P_j^2 and taking the constant term on the left to the right to obtain the system of boundary value equations

$$P_j^2 V^{\nu_j}(x) - \frac{P_{j-1}P_j}{2}V^{\nu_{j-1}}(x) - \frac{P_jP_{j+1}}{2}V^{\nu_{j+1}}(x)\begin{cases} = \omega_j', & x\in \operatorname{supp}(\nu_j)\,,\\ \geq \omega_j', & x\in\Delta_j\,,\end{cases} \tag{17}$$

for $j = 0,\ldots,m$, where $\omega_j' = P_j^2\omega_j - P_jP_{j+1}\omega_{j+1}$. (The terms with P_{-1} and P_{m+1} do not appear when $j = 0$ and $j = m$, respectively.) By Lemma 1, $(\nu_0,\ldots,\nu_m) = (\overline{\mu}_0,\ldots,\overline{\mu}_m)$ and $(\omega_0',\ldots,\omega_m') = (\omega_0^{\overline{\mu}},\ldots,\omega_m^{\overline{\mu}})$ for any convergent subsequence showing the existence of the limits in (9) as stated. □

4. Proof of Theorem 1

Theorem 1 is a consequence of the following more general result.

Theorem 3. *Let* $(\sigma_0;\sigma_1,\ldots,\sigma_m)\in\mathbf{Reg}$, $(s_1,\ldots,s_m) = \mathcal{N}(\sigma_1,\ldots,\sigma_m)$, *and consider a sequence of multi-indices* $\Lambda = \Lambda(p_0,\ldots,p_m)$. *Let* $(A_{\mathbf{n},0},\, A_{\mathbf{n},1},\ldots,\, A_{\mathbf{n},m})$, $\mathbf{n}\in\Lambda$, *be the associated sequence of type I Fourier-Padé approximants for the Nikishin system of Markov functions* $(\widehat{s}_1,\ldots,\widehat{s}_m)$ *normalized so that for all* $\mathbf{n}$, $A_{\mathbf{n},m}$ *is monic. Then, for* $j = 0,\ldots,m$

$$\lim_{\mathbf{n}\in\Lambda}|L_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = G_j(z), \tag{18}$$

uniformly on each compact subset of $\mathbb{C}\setminus(\Delta_j\cup\Delta_{j+1})$, *where*

$$G_j(z) = \exp\left(P_{j+1}V^{\overline{\mu}_{j+1}}(z) - P_jV^{\overline{\mu}_j}(z) - 2\sum_{k=j+1}^{m}\frac{\omega_k^{\overline{\mu}}}{P_k}\right), \quad j = 0,\ldots,m-1,$$

and

$$G_m(z) = \exp\left(-P_mV^{\overline{\mu}_m}(z)\right).$$

$\overline{\mu} = \overline{\mu}(\mathcal{C}) = (\overline{\mu}_0, \ldots, \overline{\mu}_m)$ *is the equilibrium vector measure and* $(\omega_0^{\overline{\mu}}, \ldots, \omega_m^{\overline{\mu}})$ *is the system of equilibrium constants for the vector potential problem determined by the interaction matrix* $\mathcal{C}$ *defined in* (4) *on the system of intervals* $\Delta_j, j = 0, \ldots, m$.

Proof. If $j = m, L_{\mathbf{n},m} = Q_{\mathbf{n},m}$ and (9) directly implies that

$$\lim_{\mathbf{n}\in\Lambda} |L_{\mathbf{n},m}(z)|^{1/|\mathbf{n}|} = \exp\left(-P_m V^{\overline{\mu}_m}(z)\right),$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_m$. For $j \in \{0, \ldots, m-1\}$, using (8) with $q = Q_{\mathbf{n},j+1}$, we obtain

$$L_{\mathbf{n},j}(z) = \frac{Q_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j+1}(z)} \int \frac{Q^2_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j}(x)} \frac{L_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j+1}(x)} \frac{d\sigma_{j+1}(x)}{z-x}. \tag{19}$$

From (9), it follows that

$$\lim_{\mathbf{n}\in\Lambda} \left|\frac{Q_{\mathbf{n},j}(z)}{Q_{\mathbf{n},j+1}(z)}\right|^{1/|\mathbf{n}|} = \exp\left(P_{j+1} V^{\overline{\mu}_{j+1}}(z) - P_j V^{\overline{\mu}_j}(z)\right), \tag{20}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_j \cup \Delta_{j+1}$ (we also use that the zeros of $Q_{\mathbf{n},j}$ and $Q_{\mathbf{n},j+1}$ lie in Δ_j and Δ_{j+1}, respectively). It remains to find the $|\mathbf{n}|$th root asymptotic behavior of the integral.

Fix a compact set $\mathcal{K} \subset \mathbb{C} \setminus \Delta_{j+1}$. It is easy to verify that (for the definition of $\rho_{\mathbf{n},j+1}$ see proof of Theorem 2 above)

$$C_1 \rho_{\mathbf{n},j+1} \leq \left|\int \frac{Q^2_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j}(x)} \frac{L_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j+1}(x)} \frac{d\sigma_{j+1}(x)}{z-x}\right| \leq C_2 \rho_{\mathbf{n},j+1},$$

where

$$C_1 = \frac{\min\{\max\{|u-x|, |v| : z = u+iv\} : z \in \mathcal{K}, x \in \Delta_{j+1}\}}{\max\{|z-x|^2 : z \in \mathcal{K}, x \in \Delta_{j+1}\}} > 0$$

and

$$C_2 = \frac{1}{\min\{|z-x| : z \in \mathcal{K}, x \in \Delta_{j+1}\}} < \infty.$$

Taking into account (14)

$$\lim_{\mathbf{n}\in\Lambda} \left|\int \frac{Q^2_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j}(x)} \frac{L_{\mathbf{n},j+1}(x)}{Q_{\mathbf{n},j+1}(x)} \frac{d\sigma_{j+1}(x)}{z-x}\right|^{1/|\mathbf{n}|} = e^{-2P_{j+1}\omega_{j+1}}. \tag{21}$$

From (19)-(21), we obtain

$$\lim_{\mathbf{n}\in\Lambda} |L_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp\left(P_{j+1} V^{\overline{\mu}_{j+1}}(z) - P_j V^{\overline{\mu}_j}(z) - P_{j+1}\omega_{j+1}\right).$$

It rests to show that for all $j = 0, \ldots, m, P_j \omega_j = \sum_{k=j}^m \frac{\omega_k^{\overline{\mu}}}{P_k}$.

At the end of the proof of Theorem 2, we saw that

$$\omega_m^{\overline{\mu}} = P_m^2 \omega_m \,, \quad \omega_j^{\overline{\mu}} = P_j^2 \omega_j - P_j P_{j+1} \omega_{j+1} \,, \quad j = 0, \ldots, m-1 \,.$$

From the first relation it follows that $P_m \omega_m = \omega_m^{\overline{\mu}}/P_m$. Let us show that the rest of the relations hold using induction on decreasing values of j. Suppose that the formula is true for some $j+1 \in \{1, \ldots, m\}$. Then, according to the formulas displayed above

$$P_j \omega_j = \frac{\omega_j^{\overline{\mu}}}{P_j} + P_{j+1}\omega_{j+1}$$

and using the induction hypothesis the result immediately follows. □

Set

$$U_j^{\overline{\mu}}(z) = P_j V^{\overline{\mu}_j}(z) - P_{j+1} V^{\overline{\mu}_{j+1}}(z) + 2 \sum_{k=j+1}^{m} \frac{\omega_k^{\overline{\mu}}}{P_k}, \qquad j = 0, \ldots, m-1,$$

and

$$U_{-1}^{\overline{\mu}}(z) = -P_0 V^{\overline{\mu}_0}(z), \qquad U_m^{\overline{\mu}}(z) = P_m V^{\overline{\mu}_m}(z).$$

Hence, $G_j(z) = \exp(-U_j(z)), j = 0, \ldots, m$. We have that for $j = 0, \ldots, m$

$$\frac{P_j}{2}(U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z)) = -\frac{P_{j+1}P_j}{2} V^{\overline{\mu}_{j+1}}(z) + P_j^2 V^{\overline{\mu}_j}(z) - \frac{P_j P_{j-1}}{2} V^{\overline{\mu}_{j-1}}(z) - \omega_j^{\overline{\mu}},$$

$(P_{-1} = P_{m+1} = 0)$.

From the equilibrium property (see Lemma 1 and (17)), it follows that

$$U_j^{\overline{\mu}}(x) - U_{j-1}^{\overline{\mu}}(x) = 0, \qquad x \in \operatorname{supp}(\overline{\mu}_j),$$

On the other hand,

$$U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) = \begin{cases} \mathcal{O}((p_j - p_{j-1}) \log 1/|z|), z \to \infty, & p_{j-1} > p_j, \\ \mathcal{O}(1), z \to \infty, & p_{j-1} = p_j. \end{cases} \tag{22}$$

Let us analyze separately these two cases.

If $p_j = p_{j-1}$, the second part of (22) implies that $U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z)$ is subharmonic in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\overline{\mu}_j)$; consequently, $U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) \leq 0$ on Δ_j and $U_j^{\overline{\mu}}(z) < U_{j-1}^{\overline{\mu}}(z)$ on $\overline{\mathbb{C}} \setminus \Delta_j$.

When $p_{j-1} > p_j$, the first part of (22) entails that in a neighborhood of $z = \infty, U_j^{\overline{\mu}}(z) > U_{j-1}^{\overline{\mu}}(z)$. Let $\gamma_j = \{z \in \mathbb{C} : U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) = 0\}$. The equilibrium condition implies that $\gamma_j \supset \operatorname{supp}(\overline{\mu}_j)$ and the initial remark of this sentence indicates that γ_j is bounded. Consider any bounded component of the complement of γ_j. On it, $U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z)$ is subharmonic and on its boundary $U_j^{\overline{\mu}}(z) - U_{j-1}^{\overline{\mu}}(z) = 0$. Thus, on any bounded component of the complement of γ_j we have that $U_j^{\overline{\mu}}(z) < U_{j-1}^{\overline{\mu}}(z)$. From the first remark of this sentence it follows that on the unbounded component of the complement of $\gamma_j, U_j^{\overline{\mu}}(z) > U_{j-1}^{\overline{\mu}}(z)$.

Fix $j \in \{0, \dots, m\}$. For each $k \in \{j, \dots, m\}$ define

$$D_k^j = \{z \in \mathbb{C} \setminus \Delta_j : U_k^{\overline{\mu}}(z) < U_i^{\overline{\mu}}(z), i = j, \dots, m, i \neq k\}$$

Let

$$\zeta_j(z) = \min\{U_k^{\overline{\mu}}(z) : k = j, \dots, m\}$$

Corollary 1. *Let* $(\sigma_0; \sigma_1, \dots, \sigma_m) \in \mathbf{Reg}, (s_1, \dots, s_m) = \mathcal{N}(\sigma_1, \dots, \sigma_m)$, *and consider a sequence of multi-indices* $\Lambda = \Lambda(p_0, \dots, p_m)$. *Let* $(A_{\mathbf{n},0}, A_{\mathbf{n},1}, \dots, A_{\mathbf{n},m}), \mathbf{n} \in \Lambda$, *be the associated sequence of type I Fourier-Padé approximants for the Nikishin system of Markov functions* $(\widehat{s}_1, \dots, \widehat{s}_m)$ *normalized so that for all* $\mathbf{n}, A_{\mathbf{n},m}$ *is monic. Then, for* $j = 0, \dots, m$

$$\lim_{\mathbf{n} \in \Lambda} |A_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp(-\zeta_j(z)), \qquad z \in \cup_{k=j}^m D_k^j, \tag{23}$$

and

$$\limsup_{\mathbf{n} \in \Lambda} |A_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} \leq \exp(-\zeta_j(z)), \qquad z \in \mathbb{C} \setminus (\Delta_j \cup_{k=j}^m D_k^j). \tag{24}$$

uniformly on each compact subset of the indicated set, where $\overline{\mu} = \overline{\mu}(\mathcal{C}) = (\overline{\mu}_0, \dots, \overline{\mu}_m)$ *is the equilibrium vector measure and* $(\omega_0^{\overline{\mu}}, \dots, \omega_m^{\overline{\mu}})$ *is the system of equilibrium constants for the vector potential problem determined by the interaction matrix* $\mathcal{C}$ *defined in* (4) *on the system of intervals* $\Delta_j, j = 0, \dots, m$. *In particular, if* $p_0 = \cdots = p_m = 1/(m+1)$ *then*

$$\lim_{\mathbf{n} \in \Lambda} |A_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp(-U_m(z)), \qquad z \in \mathbb{C} \setminus \cup_{k=j}^m \Delta_k. \tag{25}$$

Proof. For $j = m$, $L_{\mathbf{n},m} = A_{\mathbf{n},m}, D_m^m = \mathbb{C} \setminus \Delta_m$ and $\zeta_m = U_m$. Therefore, (23) reduces to (18), whereas (24) is satisfied by exclusion since $\mathbb{C} \setminus (\Delta_m \cup D_m^m) = \emptyset$. Let us assume that (23)-(24) hold for some $j + 1 \in \{1, \dots, m\}$ and let us prove that it is also true for j.

Notice that

$$A_{\mathbf{n},j}(z) = L_{\mathbf{n},j}(z) - \sum_{k=j+1}^{m} A_{\mathbf{n},k}(z) \widehat{s}_{j+1,k}(z).$$

Obviously $\zeta_j(z) = \min(U_j(z), \zeta_{j+1}(z))$. Taking (18) and (23) (for $j+1$) into consideration, on D_k^j the term containing $A_{\mathbf{n},k}$ (or $L_{\mathbf{n},j}$ if $k = j$) dominates the sum and (23) immediately follows (notice that $\widehat{s}_{j+1,k}(z) \neq 0, z \in \mathbb{C} \setminus \Delta_{j+1}$). On the complement of $\cup_{k=j}^m D_k^j$ there is no dominating term and all we can conclude from the previous equality is (24).

Let $p_0 = \cdots = p_m = 1/(m+1)$. In this case, on $\mathbb{C} \setminus \cup_{k=j}^m \Delta_k$ we have that $U_m(z) < U_{m-1}(z) < \cdots < U_j(z)$ and (25) follows from (23). □

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Universidad Carlos III, Spain and Universidad La Rioja, Spain