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Markov-Perfect Optimal Fiscal Policy: The Case of Unbalanced Budgets

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Abstract

We study optimal time-consistent fiscal policy in a neoclassical economy with endogenous government spending, physical capital and public debt. We show that a dynamic complementarity between the households' consumption-savings decision and the government's policy decision gives rise to a multiplicity of expectations-driven Markov-perfect equilibria. The long-run levels of taxes, government spending and debt are not uniquely pinned down by economic fundamentals, but are determined by expectations over current and future policies. Accordingly, economies with identical fundamentals may significantly differ in their levels of public indebtedness.

Keywords: Optimal fiscal policy; Markov-perfect equilibrium; Time-consistent policy; expectation traps.

JEL Classification Numbers: E61; E62; H21; H63.

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1 Introduction

Developed economies with similar economic fundamentals display a sustained disparity in public indebtedness. According to OECD data,¹ the average debt-to-GDP ratio over the period 1991-2007 was below 20% in Australia, around 60% in Germany, France, and the United States, whereas Belgium, Italy and Japan showed ratios above 100%. Moreover, some countries have experienced prolonged periods of low levels of public debt followed by prolonged periods of high public debt. Can these observations be accounted for as the outcome of the government's welfare maximization problem in a standard, off-the-shelf neoclassical model of capital accumulation?

The present paper explores this question. We consider a simple model economy with three types of agents: firms, households, and a benevolent government. Firms are perfectly competitive; they hire physical capital and labor from households to produce a homogeneous good with a constant-returns-to-scale technology. Households enjoy consumption of a private good and of a government-provided, public good; they save in physical capital and government debt. The government decides on the level of spending in the public consumption good, on a uniform tax rate on labor income and asset returns, and on the issuance of public debt. The government lacks commitment to future policy choices and thus makes decisions sequentially. Specifically, we characterize optimal time-consistent fiscal policy in differentiable Markov-perfect equilibrium.

The main contribution of our analysis can be summarized as follows. In the class of economies outlined above, there exists a continuous multiplicity of Markov-perfect equilibrium policy rules, which in turn generate a multiplicity of steady-state values for allocations, prices and fiscal policy variables. The set of Markov-perfect equilibria includes the Ramsey equilibrium, which yields no distortionary taxation and negative debt in the long-run, as well as policy rules that generate positive levels of taxes and debt in the long run. For example, there exists a Markov-perfect equilibrium that features a steady-state income tax rate of 19% and a debt-GDP ratio of 56% under a standard calibration of parameter values.

Equilibrium multiplicity in our model is of the expectation trap type: households' actions, which are based on their fiscal policy expectations, condition the government's decision problem in a way such that it becomes optimal for the government to fulfill these expectations. That is, there is a dynamic complementarity between the decisions of households and the fiscal authority, which can render the expectation of positive taxes and debt self-fulfilling. We identify two properties of

¹See the OECD's Economic Outlook.

the equilibrium consumption function that are key to the multiplicity result: (i) households do not use information on the current income tax rate to pin down consumption, and (ii) public debt is not net wealth for the household sector. Private consumption is thus solely determined by the stock of physical capital and by expectations on current debt issuance and future fiscal policy. This has two important implications for the government when setting fiscal policy in a given time period. First, current income taxation is perceived as non-distortionary by the government, as not only distortions on past investment are overlooked but also current consumption is unaffected by taxes. Second, the continuation value of the current government does not depend on current debt issuance. This leads to a redundancy of policy instruments, as income taxation and debt issuance affect all trade-offs involved in the government's maximization problem equally. Equilibrium multiplicity follows directly from this redundancy, and is therefore generic in our environment (i.e., multiplicity does not depend on particular choices for the model's parameter values). We explain the underlying mechanism in greater detail below, exploring first a simple model economy that allows for closed-form solutions to the equilibrium policy functions. Finally, we use computational methods to show that our main findings hold also in the more general setup.

To the best of our knowledge, this is the first paper to show a generic, expectations-driven multiplicity of optimal discretionary fiscal policy in a standard, neoclassical economy. In models of optimal monetary policy under discretion, equilibrium multiplicity has recently been documented in several contributions. Albanesi et al. (2003) provide a theory based on expectation traps to account for the behavior of post-war US inflation. King and Wolman (2004) study discretionary monetary policy in a non-linear model with staggered prices; they show that a dynamic complementarity between the firms' pricing decision and the policy maker's money supply decision gives rise to two equilibria, one with high inflation and one with low inflation. Blake and Kirsanova (2012) show multiplicity of discretionary equilibria in linear-quadratic New-Keynesian economies. Specifically, they present an economy with capital accumulation (but without government debt) in which dynamic complementarities within the private sector can lead to equilibrium multiplicity, i.e., there can be countably many different private sector responses to a given policy decision.² By contrast, in our environment there generically arises a continuous multiplicity, due to a dynamic complementarity between the households' consumption-savings decision and the optimal fiscal policy set by the government.

²They also discuss an economy with government debt (but without capital) where the monetary authority minimizes a quadratic loss function while a fiscal authority employs a fixed, non-optimizing policy rule. If the fiscal feedback on debt is moderate, dynamic complementarities between firms' pricing decisions and the policy maker's interest rate decision give rise to the existence of two distinct equilibria.

It should also be emphasized that the multiplicity of long-run debt levels found in our paper differs from that shown in neoclassical economies without physical capital. For instance, Lucas and Stokey (1983) show that when a government with commitment chooses optimal fiscal policy, the long-run level of debt depends on initial debt. A related result is found by Krusell *et al.* (2006) under no commitment. These authors study non-differentiable Markov strategies and show that the equilibrium contains a large, countable set of long-run debt levels. Initial conditions pin down the element in this set to which the economy converges in a maximum of two periods. Thus, as in Lucas and Stokey (1983), countries with high initial levels of debt will have high debt forever, and vice versa. In contrast, long-run debt in our model is determined by expectations of current and future fiscal policy rather than the initial level of debt. Our model, if augmented with exogenous shocks to expectations, is thus consistent with debt dynamics featuring long spells of low public debt levels followed by spells of high public debt levels.

There is a vast literature on optimal fiscal policy in macroeconomics. Seminal contributions include Chamley (1986) and Judd (1985), who study optimal capital and labor tax decisions by a benevolent government that has full commitment to future policies. They show that the Ramsey equilibrium features no capital income taxation and a negative level of public debt in the long-run. In the short run, capital, which is in fixed supply, is taxed heavily in order to build up enough assets (negative debt) to finance future government expenditure from asset returns. This policy is, however, time-inconsistent, i.e. the government would revise it if it were allowed to re-optimize.³

Markov-perfect optimal taxation under a balanced-budget rule has been studied by Klein and Ríos-Rull (2003), Klein *et al.* (2008) and Ortigueira (2006), among others. Allowing for government debt, but abstracting from physical capital, Debortoli and Nunes (2011) find that long-run debt in Markov-perfect equilibrium converges to zero under standard parameter values. Martin (2009) and Diaz-Gimenez *et al.* (2008) also abstract from physical capital to study optimal monetary and debt policy. They find a unique Markov-perfect equilibrium policy which generates two steady states, one of which is stable and the other one unstable.

Dominguez (2007) and Reis (2011) study time-consistent optimal policy with history-dependent strategies. These authors find that the best sustainable equilibrium prescribes zero long-run capital taxation. A non-balanced budget constraint is key in obtaining this result, as it allows the government to increase its assets until the lack of commitment is no longer binding. Aiyagari *et al.* (2002) drop the complete markets assumption from the Stokey and Lucas (1993) framework,

³The Chamley-Judd is remarkably robust in models with commitment; see, for example, Atkeson *et al.* (1999).

which introduces a history dependence on the debt path as opposed to a contingency to future states. They show that when there are no exogenous bounds on debt the Ramsey planner in their economy lets public debt converge to a negative level.

Finally, theories as to why governments run deficits have also been offered in terms of political disagreement. When policymakers have different preferences on the type of public good that should be provided and alternate in office, public debt becomes a tool to influence the choices of future policymakers (see Alesina and Tabellini, 1990, and Persson and Svensson, 1989). Song *et al.* (2012) study optimal fiscal policy in a model where subsequent generations of agents (young and old) vote on policy. In the Markov-perfect political equilibrium, long-run debt depends crucially on the distortions brought about by taxation. When these distortions are large enough, debt converges to an interior value; otherwise debt accumulation depletes the economy. In our paper, we abstract both from political disagreement and alternation in office and highlight the role of expectation traps to understanding public deficits.

The remainder of this paper is organized as follows. Section 2 outlines our model economy. Section 3 briefly reviews optimal policy in the Ramsey equilibrium. Section 4 describes optimal policy under discretion and provides a closed-form characterization of Markov-perfect equilibria in a simplified version of our model. Section 5 employs computational methods to discuss the properties of Markov-perfect policy rules in a calibrated version of our general model. Section 6 concludes. Finally, there are three Appendixes detailing proofs and the numerical approach for the computation of the model.

2 The Model

Our framework is the standard, non-stochastic neoclassical model of capital accumulation, extended to include a benevolent government that provides a valued public good. In order to finance the provision of such public good the government can levy a tax on households' income and issue public debt. Thus, fiscal policy in each period consists of the expenditure on the public good, G_t , the tax rate on income, τ_t , and the issue of public debt, B_{t+1} , which matures in period $t + 1$.

We begin by describing the problem solved by each agent in this economy. We then characterize the fiscal policy set by the benevolent government lacking the ability to commit to future policies. To help compare our results with those arising under full commitment, we also present a brief review of fiscal policy in the Ramsey equilibrium.

2.1 Households

There is a continuum of homogeneous households with measure one. Each household supplies one unit of labor and chooses consumption and savings in order to maximize lifetime utility, subject to a budget constraint and initial endowments of physical capital and public debt,

$$\max_{\{c_t, k_{t+1}, b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(c_t, G_t), \quad (2.1)$$

s.t.

$$c_t + k_{t+1} + b_{t+1} = k_t + b_t + (1 - \tau_t) [w_t + (r_t - \delta) k_t + q_t b_t] \quad (2.2)$$

$$k_0 > 0 \text{ and } b_0 \text{ given.}$$

Small letters are used to denote individual variables and capital letters to denote economy-wide values. Function $U(\cdot)$ in equation (2.1) is the instantaneous utility function, which depends on the consumption of a private good, c_t , and the consumption of a public good, G_t . $U(\cdot)$ is assumed to be continuously differentiable, increasing and concave; and $0 < \beta < 1$ is the discount factor. Labor is supplied inelastically at a real wage rate w_t . Household's asset holdings are made up of physical capital, k_t , which is rented to firms at the rate r_t , and government bonds, b_t , which bear an interest denoted by q_t . Physical capital depreciates at a rate denoted by $0 < \delta < 1$. Household's total income, net of capital depreciation, is taxed at the rate τ_t . If the government is a net lender to the private sector, i.e. the household borrows from the government ($b_t < 0$), then taxable income is net of interest payments.

2.2 Firms

Firms are competitive and produce an aggregate good with a neoclassical production technology. Total production is given by,

$$Y_t = F(K_t, L_t) = F(K_t, 1) = f(K_t), \quad (2.3)$$

where K_t denotes the aggregate or economy-wide stock of capital. First-order conditions to profit maximization imply the typical demand and zero-profits equations,

$$r_t = f_K(K_t) \quad (2.4)$$

$$w_t = f(K_t) - r_t K_t. \quad (2.5)$$

2.3 Government

Fiscal policy involves expenditure in the public good and its financing through taxes and debt. The government is benevolent in the sense that it seeks to maximize households' lifetime utility, (2.1), subject to its budget constraint, to a feasibility restriction, and to the private sector's first-order conditions. In addition, government's policies may be conditioned by its lack of commitment. The budget constraint of the government is

$$G_t + (1 + q_t)B_t = B_{t+1} + \tau_t [w_t + (r_t - \delta)K_t + q_t B_t]. \quad (2.6)$$

The right-hand side of equation (2.6) represents government's revenues, which are made up of debt issues, B_{t+1} , and income taxation. The left-hand side is government's total expenditure, including the provision of the public good, and the repayment of outstanding financial liabilities.

3 Ramsey Optimal Fiscal Policy

This section presents a brief review of optimal fiscal policy in the Ramsey equilibrium of our model economy. In the Ramsey equilibrium, the benevolent government is assumed to have full commitment to future policies. Thus, it can credibly announce the whole sequence of expenditure in the public good, income taxes and debt issues from the first period onwards. This allows the government to anticipate the response of the private sector to its fiscal policy. Hence, the problem of the government in the Ramsey equilibrium is to choose sequences for taxes and public debt so that the competitive equilibrium maximizes social welfare [equation (2.1)].

Proposition 1 presents the optimal fiscal policy in the steady state of the Ramsey equilibrium for our economy.

Proposition 1: *In the steady state of the Ramsey equilibrium the income tax rate is zero and the government holds positive assets, i.e. $B < 0$.*

Proof: See Appendix I.

The results in Proposition 1 are well known in the literature of optimal fiscal policy, and hence we do not provide further details.

4 Markov-Perfect Optimal Fiscal Policy

In this section we drop the assumption of government commitment to future policies and study time-consistent optimal policies. More specifically, we focus on differentiable Markov-perfect equilibria of this economy populated by a continuum of households and a government that acts sequentially, foreseeing its future behavior when choosing current levels of the public good, income taxes and debt issues.

Following recent literature on Markovian policies, we assume that the government —although unable to commit to future policies— does commit to honoring the tax rate it announces for the current period and to repaying outstanding debt obligations. Commitment to the current tax rate implies an intra-period timing of actions that grants the government a first-mover advantage. That is, at the beginning of period t , the time- t government sets the tax rate for the period. Once that choice is publicly known, consumers choose consumption/savings and the composition of their portfolios, and the government chooses the provision of the public good (or equivalently, the issue of debt). Governments are thus (intra-period) *Stackelberg players* and can therefore anticipate the effects of current taxation on household's decisions.

In sum, under this timing of actions the time- t government has intra-period commitment to time- t taxes but not to debt issues. This timing fits well what we observe in real economies, where governments typically make decisions on taxes at discrete times but issue debt continuously. It is important to note, however, that this particular timing of actions bears no consequences for our multiplicity result. As we discuss in detail below, the alternative timing in which the government sets both the tax rate and debt issues at the same time as households choose consumption and savings also yields a multiplicity of Markov-perfect equilibria.

The optimization problem of a typical household

The household chooses (i) how much to consume and save; and (ii) how to allocate savings between physical capital and public debt. At the time the household makes these decisions the tax rate for the period is already known. However, the household must foresee both the current government's debt policy and future governments' fiscal policy.

The problem of a household that holds k and b of the physical and government assets, respectively, that has to pay taxes on current income at rate τ , that expects the current and future governments to issue new debt according to the policy $\psi_B : (K \times B \times \tau) \rightarrow B'$, and expects future

governments to set taxes according to the policy $\psi_\tau : (K \times B) \rightarrow \tau$, can be written as

$$\begin{aligned} v(k, b, K, B; \tau) &= \max_{c, k', b'} \{U(c, G) + \beta \tilde{v}(k', b', K', B')\} \\ &\quad s.t. \\ c + k' + b' &= k + b + (1 - \tau) [\omega(K) + [r(K) - \delta]k + q(K)b], \end{aligned} \quad (4.1)$$

where $\tilde{v}(k', b', K', B')$ is the continuation value as foreseen by the household. $\omega(K)$, $r(K)$ and $q(K)$ are pricing functions. The economy-wide stock of physical capital is expected to evolve according to the law $K' = \mathcal{H}(K, B, \tau)$. From the above maximization problem, it follows that the consumption function in a competitive equilibrium —where today's tax rate is τ , future taxes are set according to policy ψ_τ and current and futures issues of debt are set according to policy ψ_B — can be expressed in terms of K, B and τ , say $C(K, B, \tau)$, and must satisfy the following Euler equation:

$$U_c(C(K, B, \tau), G) = \beta U_c(C(K', B', \tau'), G') [1 + (1 - \tau') (f_K(K') - \delta)], \quad (4.2)$$

where $B' = \psi_B(K, B, \tau)$ and $\tau' = \psi_\tau(K', B')$. In equilibrium K' is given by,

$$K' = K + B + (1 - \tau) [f(K) - \delta K + q(K)B] - C(K, B, \tau) - B', \quad (4.3)$$

where G and G' are given by the time- t and time- $(t + 1)$ governments' budget constraints, respectively. Finally, the pricing functions $\omega(K)$ and $r(K)$ are given by (2.4) and (2.5), and $q(K)$ must satisfy the non-arbitrage condition between the two assets,

$$q(K') = f_K(K') - \delta. \quad (4.4)$$

The household's Euler equation, (4.2), has the usual interpretation: the marginal utility of consumption equals the present value of the last unit of income devoted to savings. Since physical capital and debt yield the same return in equilibrium, the supply of public debt determines the composition of the household's portfolio. This implies a one-to-one crowding out of investment in capital by public debt. Taxation, on the other hand, impinges on disposable income and thus may affect the level of consumption, in which case taxation would translate into a non-one-to-one crowding out of capital investment. The problem of the government is shown next.

The problem of the government

As explained above, the government's lack of commitment to future policies and our focus on Markov-perfect equilibria allows us to think of the government as a sequence of governments,

one for each time period. The time- t government sets the tax rate for the period and issues debt foreseeing the fiscal policy to be set by successive governments. Following the timing of actions established above, the government chooses the tax rate at the beginning of the period, taking into account the effect of τ on the level of consumption, as given by the consumption function, $C(K, B, \tau)$. In a second stage, the government sets debt issues simultaneously with the households' decision on consumption and savings. To highlight this timing of fiscal policy decisions, we write the problem of the period time- t government as a two-stage maximization problem. Given the initial choice for taxes, debt issues is the solution to

$$\begin{aligned}
V(K, B, \tau) &= \max_{B'} \left\{ U(C(K, B, \tau), G) + \beta \tilde{V}(K', B') \right\} & (4.5) \\
\text{s.t.} & \\
K' &= (1 - \delta)K + f(K) - C(K, B, \tau) - G \\
G &= \tau [f(K) - \delta K + q(K)B] + B' - [1 + q(K)] B, \\
q(K') &= f_K(K') - \delta,
\end{aligned}$$

where $V(K, B, \tau)$ is the value to the time- t government that has set the tax rate at τ and foresees the fiscal policy to be set by future governments. $\tilde{V}(K', B')$ is next-period value as foreseen by the time- t government. The debt policy that solves this problem can thus be written as $B'(K, B, \tau)$. Therefore, the tax rate set by the time- t government is the solution to

$$\begin{aligned}
W(K, B) &= \max_{\tau} \left\{ U(C(K, B, \tau), G) + \beta \tilde{V}(K', B'(K, B, \tau)) \right\} & (4.6) \\
\text{s.t.} & \\
K' &= (1 - \delta)K + f(K) - C(K, B, \tau) - G \\
G &= \tau [f(K) - \delta K + q(K)B] + B'(K, B, \tau) - [1 + q(K)] B, \\
q(K') &= f_K(K') - \delta.
\end{aligned}$$

The following proposition characterizes the fiscal policy set by the time- t government.

Proposition 2: *Tax and debt policies that solve the government's problem are solutions to the following Generalized Euler Equations:*

$$\frac{U_c C_\tau + U_G G_\tau}{G_\tau + C_\tau} = \beta \left[U'_c C'_{K'} + U'_{G'} G'_{K'} + \frac{U'_c C'_{\tau'} + U'_{G'} G'_{\tau'}}{G'_{\tau'} + C'_{\tau'}} (f'_{K'} + 1 - \delta - C'_{K'} - G'_{K'}) \right] \quad (4.7)$$

and

$$\frac{U_c C_\tau + U_G G_\tau}{G_\tau + C_\tau} = U_G + \beta \left[U'_{c'} C'_{B'} + U'_{G'} G'_{B'} - \frac{U'_{c'} C'_{\tau'} + U'_{G'} G'_{\tau'}}{G'_{\tau'} + C'_{\tau'}} (C'_{B'} + G'_{B'}) \right]. \quad (4.8)$$

Proof: See Appendix I.

Some comments on notation are in order. Function arguments in equations (4.7) and (4.8) have been omitted for expositional clarity. Subscripts denote the variable with respect to which the derivative is taken. A prime in a variable indicates next-period values, and a prime in a function indicates it is evaluated at next-period variables. Finally, $G_\tau = f - \delta K + qB$, $G_B = -1 - (1 - \tau)q$ and $G_K = \tau(f_K - \delta + q_K B) - q_K B$.

Before providing an interpretation of the two Generalized Euler Equations presented in Proposition 2, we offer the following definition of a Markov-perfect equilibrium in our economy:

Definition: A Markov-perfect equilibrium is a quadruplet of functions $C(K, B, \tau)$, $\psi_B(K, B, \tau)$, $\psi_\tau(K, B)$ and $W(K, B)$, such that:

- (i) Given ψ_B and ψ_τ , $C(K, B, \tau)$ solves the household's maximization problem.
- (ii) Given $C(K, B, \tau)$, ψ_B and ψ_τ solve the government's maximization problem. That is, $B' = \psi_B(K, B, \tau)$ and $\tau = \psi_\tau(K, B)$.
- (iii) $W(K, B)$ is the value function of the government.

An alternative definition of Markov-perfect equilibrium in an economy without debt has been suggested by Harald Uhlig [see Klein *et al.* (2008) for such definition]. In that framework, the problem of the government is set as choosing the level expenditure, G , and the stock of capital for the next period, K' , directly, and using the feasibility condition to express consumption as a function of K, K' , and G . In the current framework, we find our equilibrium definition above more transparent for two reasons: First, our timing of actions involves the government choosing debt simultaneously with the consumption/savings decision. This implies an inability of the government to anticipate the response of current consumption to debt issues. Under our timing of events it is thus more straightforward to set the problem of the government as choosing current policy rather than next-period's capital. Second, by defining the equilibrium in terms of a consumption function on capital, debt and the tax rate, it will allow us to stress important equilibrium properties of this consumption function.

The two Generalized Euler Equations, (4.7) and (4.8), which characterize Markov-perfect taxation and debt policies, respectively, have the following interpretation. Equation (4.7) establishes

that the tax rate has to equate the marginal value of taxation to the marginal value of investing in physical capital. Equation (4.8) establishes that debt issues have to equate the marginal value of issuing debt to the marginal value of investing in physical capital (and consequently to the marginal value of taxation). In a Markov-perfect equilibrium, the government is indifferent between using taxes or debt to finance the provision of the public good. Both equations involve only wedges between today and tomorrow, as subsequent wedges are implicitly handled optimally by an envelope argument. Consecutive governments, however, disagree on how much to tax tomorrow [the time- $(t + 1)$ government does not internalize the distortionary effects of its policy on time- t investment]. The current government thus takes into account the effect of its policy on tomorrow's initial conditions, K' and B' , in order to help compensate for that disagreement. Following this reasoning, one may interpret the different terms in (4.7) and (4.8) as follows.

The left-hand side of equation (4.7) is today's marginal utility of taxation *per* unit of savings crowded out. The numerator of this expression is the change in utility from a marginal increase in the tax rate, which is made up of the change in utility from the private good, $U_c C_\tau$, plus the change in utility from the public good, $U_G G_\tau$. The denominator is the amount of savings crowded out, or, equivalently, the change in consumption of the public and private good brought about by the increase in the tax rate.

The right-hand side of equation (4.7) is the marginal utility of investing in physical capital. An extra unit of investment today yields an increase in resources tomorrow by $f'_{K'} + 1 - \delta$. The breakdown of the value of these resources is: (i) $C'_{K'}$ of them are consumed as private good, yielding a value of $U'_c C'_{K'}$; (ii) $G'_{K'}$ corresponds to the increase in the provision of the public good obtained from the increase in the tax base, which yields a value of $U'_G G'_{K'}$; (iii) the remaining $f'_{K'} + 1 - \delta - C'_{K'} - G'_{K'}$ are taxed away, and the marginal value is the left-hand side of equation (4.7), updated one period ahead. Hence, the right-hand side of (4.7) results from adding up all these values and discounting.

Equation (4.8) is a non-arbitrage condition between taxation and public debt. Its interpretation is straightforward. The right-hand side is the value of issuing an extra unit of government debt today. The first term on the right-hand side is the value of today's extra public good financed with the increase in government debt. The second term is the present value of the implied changes in tomorrow's consumption of the private and public good, $C'_{B'}$ and $G'_{B'}$, respectively. Besides the direct effects on tomorrow's utility, these changes have an effect on tomorrow's taxation, which must be valued using the marginal utility of taxation. Equation (4.8) establishes that the value of issuing debt must equal the value of taxation (the left-hand side of the equation).

A re-arrangement of equation (4.8) offers an alternative interpretation of the non-arbitrage condition between taxes and bonds in terms of two wedges, $U_c - U_G$ and $U'_c - U'_{G'}$. Such a re-arrangement yields,

$$(U_c - U_G) \frac{C_\tau}{G_\tau + C_\tau} + \beta \left\{ (U'_c - U'_{G'}) \left(G'_{B'} + \frac{G'_{\tau'}}{G'_{\tau'} + C'_{\tau'}} K''_{B'} \right) \right\} = 0. \quad (4.9)$$

Equation (4.9) says that the value of using debt instead of taxes to finance the last unit of public expenditure equals zero in a Markov-perfect equilibrium. The first term is the net change in utility today of using debt instead of taxes per unit of forgone savings. The second term captures the change in future distortions induced by the extra unit of public debt.⁴ The way the current government trades off these two wedges when choosing B' depends on expectations on future government policy.

4.1 Markov-Perfect Equilibrium: Steady States

The *steady state of a Markov-perfect equilibrium* is defined as a list of infinite sequences for quantities $\{C_t, K_t\}$, prices $\{\omega_t, r_t, q_t\}$ and fiscal variables $\{G_t, \tau_t, B_t\}$, such that they are generated by Markov-perfect equilibrium policy rules and its values do not change over time, i.e. $K_{t+1} = K_t$, $B_{t+1} = B_t$, $\tau_{t+1} = \tau_t$ for all t , and the same is true for consumption and prices.

In this subsection we offer insights on the existence of a multiplicity of steady states, each of which is generated by a different Markov-perfect equilibrium policy. Evaluating equation (4.9) at the steady state of a Markov-perfect equilibrium yields,

$$(U_c - U_G) \left\{ \frac{C_\tau}{G_\tau + C_\tau} + \beta \left(G_B + \frac{G_\tau}{G_\tau + C_\tau} K'_B \right) \right\} = 0. \quad (4.10)$$

This equation suggests that there are two different types of tax and debt policies consistent with the existence of a steady state. The first one yields $U_c = U_G$, and corresponds to the policy prescribed by the long-run Ramsey outcome. As shown in Proposition 1, the Ramsey equilibrium prescribes zero income taxes and positive government asset holdings in the steady state. The provision of the public good is financed entirely from the returns on government assets. The next proposition proves that the long-run Ramsey outcome is the steady state of a Markov-perfect equilibrium.

⁴By way of clarity, the expression $K''_{B'}$ in this second term of the equation denotes the change in tomorrow's investment with respect to today's issue of debt. More specifically, if we combine the two restrictions in the maximization problem (4.6), we can write K' as a function, say H , of K, B, B' and τ . Thus, $K''_{B'}$ is the derivative of function H evaluated at K', B', B'', τ' , where $B'' = \psi_B(K', B', \tau')$ and $\tau' = \psi_\tau(K', B', \tau')$.

Proposition 3: *There exists a Markov-perfect equilibrium that generates the steady-state Ramsey outcome.*

Proof: See Appendix I.

In a related paper, Azzimonti-Renzo *et al.* (2006) study a model with differentiated taxes on capital and labor, and exogenous government expenditure. Within their framework, the authors find a Markov-perfect equilibrium which yields zero labor taxes from all initial conditions, K and B , and zero capital taxes from next-period onwards. As confirmed by our numerical computations, this result also holds in our model economy: when there are no exogenous bounds on income taxation, there exists a Markov-perfect equilibrium in which income taxes are zero after one period, and government assets converge to the long-run Ramsey value. Furthermore, for some initial conditions the initial income tax is negative, which amounts to a subsidy to households.

The second type of Markov-perfect tax and debt policies consistent with the existence of a steady state yield positive taxes in the long run. Under this type of policy $U_c \neq U_G$, and the second term on the left-hand side of equation (4.10) is zero. The next proposition presents an important feature of these Markov-perfect equilibria.

Proposition 4: *Along the steady state of a Markov-perfect equilibrium with positive distortions, government bonds are not net wealth, i.e., $C_B(K^*, B^*, \tau^*) = 0$.*

Proof: See Appendix I.

We show below that this result holds not only at the steady state but also along the entire equilibrium path. In the next section we obtain Markov-perfect equilibria in closed form under full capital depreciation, and find that government bonds are not net wealth in equilibrium. Our numerical results in Section 5 confirm that this result also holds under partial capital depreciation.

4.2 Indeterminacy of the Markov-perfect Equilibrium

In this section we show, by way of an example, the existence of a (continuous) multiplicity of Markov-perfect equilibria. Specifically, we consider the utility function $U(C, G) = \ln C + \theta \ln G$, and the production function $f(K) = K^\alpha$. Further, we assume full capital depreciation, $\delta = 1$, and no tax deductibility of depreciation. Multiplicity is shown using the guess-and-verify approach.

Let us conjecture that the household consumption function is given by $C(K) = a_1 K^\alpha$ and that government expenditure in the public good is $G(K) = a_2 K^\alpha$, where a_1 and a_2 are constants to be determined. For the sake of expositional clarity we first show the existence of the Markov-perfect

equilibrium with no distortions, i.e., one that yields the efficient allocations, and then show the existence of a multiplicity of Markov-perfect equilibria with distortions.

The Markov-perfect Equilibrium with no Distortions

Under our conjectured consumption and expenditure functions, the feasibility condition yields $K' = (1 - a_1 - a_2)K^\alpha$. In the equilibrium without distortions, Generalized Euler Equation (4.7) implies $U'_{C'} = U'_{G'}$, which under the conjectured functions yields

$$a_2 = a_1\theta. \quad (4.11)$$

The Generalized Euler Equation (4.8) becomes

$$\frac{\theta}{G} = \beta \frac{\theta}{G'} \alpha (K')^{\alpha-1}.$$

Plugging the expression for K' and the conjectured expenditure policy into this equation we obtain

$$1 - a_1 - a_2 = \beta\alpha. \quad (4.12)$$

It should be noted that the household's Euler equation, (4.2), holds under (4.11) and (4.12). From these latter two equations we have

$$a_1 = \frac{1 - \beta\alpha}{1 + \theta} \quad (4.13)$$

$$a_2 = \frac{\theta(1 - \beta\alpha)}{1 + \theta}. \quad (4.14)$$

We now derive the tax and debt policy functions that support these consumption and expenditure functions in a Markov-perfect equilibrium. We must find functions $\tau(K, B)$ and $B'(K, B)$ that solve the sequence of government's budget constraints under the conjectured functions shown above.

In the steady state of the Markov-perfect equilibrium with distortions we must have $\tau = 0$. Moreover, the steady-state values \hat{K} and \hat{B} must solve the steady-state versions of the feasibility condition and the government budget constraint

$$\hat{K} = (1 - a_1 - a_2)\hat{K}^\alpha \quad (4.15)$$

$$a_2\hat{K}^\alpha = [1 - \alpha\hat{K}^{\alpha-1}]\hat{B}, \quad (4.16)$$

where use of the non-arbitrage condition $q = f_K - 1/(1 - \tau)$ has been made to derive (4.16). The

solution to these two equations yields

$$\hat{K} = \left(\frac{1}{1 - a_1 - a_2} \right)^{\frac{1}{\alpha-1}} \quad (4.17)$$

$$\hat{B} = \frac{a_2 \hat{K}}{\hat{K}^{1-\alpha} - \alpha}. \quad (4.18)$$

Accordingly, the debt-to-capital ratio in the steady state of this Markov-perfect equilibrium is

$$\frac{\hat{B}}{\hat{K}} = - \frac{(1 - \beta\alpha)\theta}{(1 + \theta)(1 - \beta)\alpha},$$

from where it is apparent that debt is negative in the steady state.

We now obtain the dynamics towards this steady state and derive the functions $\tau(K, B)$ and $B'(K, B)$. Using our conjectured functions and the non-arbitrage condition, the budget constraint for the period- $(t + 1)$ government is,

$$\frac{B''}{(K')^\alpha} = a_2 + \frac{1}{\beta} \frac{B'}{K^\alpha}. \quad (4.19)$$

Since $\beta < 1$, this equation is unstable. Therefore, we must have that

$$\frac{B''}{(K')^\alpha} = \frac{B'}{K^\alpha} = \frac{\hat{B}}{\hat{K}^\alpha}.$$

Then, using the steady-state values for physical capital and debt shown above along with the values for a_1 and a_2 in (4.13) and (4.14), we obtain the debt policy function

$$B' = - \frac{\beta\theta(1 - \beta\alpha)}{(1 + \theta)(1 - \beta)} K^\alpha.$$

Plugging this policy into the period- t government's budget constraint we find the tax rate set by this government

$$\tau(K, B) = \frac{a_2 + \alpha \frac{B}{K} - \frac{\beta\theta(1 - \beta\alpha)}{(1 + \theta)(1 - \beta)}}{1 + \alpha \frac{B}{K}}.$$

Finally, it is straightforward to show that this tax function evaluated at K' and B' yields a tax rate equal to zero, i.e., $\tau(K', B') = 0$, which confirms that the tax rate jumps to its steady-state value in one period.

The Markov-perfect Equilibrium with Distortions

We now show that there is a multiplicity of Markov-perfect equilibria where $U'_{C'} \neq U'_{G'}$. We again guess that consumption and expenditure policy functions are given by: $C(K) = a_1 K^\alpha$ and

$G(K) = a_2 K^\alpha$, where a_1 and a_2 are parameters to be determined. Let us look for solutions where the wedge between marginal utilities of private and public good consumption is proportional to the inverse of output, that is,

$$U'_{C'} - U'_{G'} = \frac{a_3}{(K')^\alpha},$$

where a_3 is a constant. We will show that there is a continuum of values for a_3 consistent with a Markov-perfect equilibrium. This equation yields,

$$a_2 - \theta a_1 = a_1 a_2 a_3. \quad (4.20)$$

Now, making use of the conjectured consumption function the household's Euler equation becomes,

$$\frac{1}{a_1 K^\alpha} = \beta \frac{1}{a_1 (K')^\alpha} \left[(1 - \tau') \alpha (K')^{\alpha-1} \right],$$

which, after using the feasibility condition, $K' = (1 - a_1 - a_2) K^\alpha$, yields,

$$\tau' = 1 - \frac{1 - a_1 - a_2}{\beta \alpha}. \quad (4.21)$$

Hence, under our conjectures the tax rate in equilibrium is constant from $t+1$ onwards. Plugging the conjectures into the Generalized Euler Equation (4.8) we get

$$\frac{\theta}{a_2 K^\alpha} = \beta \left[\left(\frac{1}{a_1 (K')^\alpha} - \frac{\theta}{a_2 (K')^\alpha} \right) a_1 \alpha (K')^{\alpha-1} + \frac{\theta}{a_2 (K')^\alpha} \alpha (K')^{\alpha-1} \right].$$

Under the assumed wedge between marginal utilities this equation gives,

$$\frac{\theta}{a_2 K^\alpha} = \beta \left[\frac{a_3}{(K')^\alpha} a_1 \alpha (K')^{\alpha-1} + \frac{\theta}{a_2 (K')^\alpha} \alpha (K')^{\alpha-1} \right].$$

Further, using the feasibility condition we obtain,

$$(1 - a_1 - a_2) \frac{\theta}{\beta} = a_1 a_2 a_3 \alpha + \theta \alpha. \quad (4.22)$$

Note that given a_3 , equations (4.20) and (4.22) can be solved for a_1 and a_2 .

From the budget constraint of the period- $(t+1)$ government, we obtain debt issues by this government as

$$B'' = (a_2 - \tau') (K')^\alpha + (1 - \tau') \alpha (K')^{\alpha-1} B',$$

where τ' is the constant shown in (4.21). Dividing by $(K')^\alpha$ we get

$$\frac{B''}{(K')^\alpha} = (a_2 - \tau') + (1 - \tau') \alpha \frac{B'}{K'},$$

and using the implied law of motion for capital on the right-hand side,

$$\frac{B''}{(K')^\alpha} = (a_2 - \tau') + \frac{1}{\beta} \frac{B'}{K^\alpha}.$$

Since $\beta < 1$, we must have that $\frac{B''}{(K')^\alpha} = \frac{B'}{K^\alpha} = \frac{\hat{B}}{K^\alpha} = \frac{\beta}{\beta-1}(a_2 - \tau')$. From this we obtain the period- t debt policy function as

$$B' = \frac{\beta}{\beta-1}(a_2 - \tau')K^\alpha.$$

Plugging this policy function into the budget constraint of the period- t government we find its tax policy function as

$$\tau(K, B) = \frac{a_2 + \alpha \frac{B}{K} - \frac{\beta}{\beta-1}(a_2 - \tau')}{1 + \alpha \frac{B}{K}}.$$

It is readily seen that this policy function, when evaluated at K' and B' , yields the constant obtained in (4.21) for all values of a_3 . Consequently, any value of a_3 yielding solutions to a_1 and a_2 from (4.20) and (4.22) that fall within the feasible range for all economic variables is consistent with a Markov-perfect equilibrium. In Appendix I we show that these solutions satisfy the second-order optimality conditions. That is, no government has an incentive to deviate.

REMARKS: A couple of remarks on the Markov-perfect equilibria presented above are in order. **(1)** The household consumption function is independent of current taxes. That is, in equilibrium households do not use the tax rate set by the period- t government to pin down the level of consumption. This implies that the assumption of government's within-period commitment to taxes is not binding in equilibrium. Hence, the Markov-perfect equilibria shown above also arise when this commitment is removed, and households and the government are assumed to make all their decisions simultaneously within the period. (A formal characterization of Markov-perfect equilibria under this alternative timing of actions with simultaneous moves is presented in Appendix II.) **(2)** The value function does not depend on the level of public debt. This follows immediately from the fact that both the consumption function and government expenditure in the public good are independent of the level of debt. The tax policy function does depend, however, on the debt level in equilibrium.

The two equilibrium properties described in (1) and (2) above provide the key to understanding the indeterminacy of Markov-perfect equilibrium policy rules. Note first that the period- t government can provide a fixed level of the public good, say \hat{G} , using different combinations of taxes and debt issues. In particular, the set of pairs (τ, B') that yield \hat{G} is given by the government budget constraint,

$$\hat{G} = \tau [f(K) - \delta K + q(K)B] + B' - [1 + q(K)] B.$$

Since consumption is independent of taxes in equilibrium —property (1)—, period- t taxation is non distortionary from the standpoint of the period- t government. Further, the government’s continuation value does not depend on the level of debt —from property (2). Since both τ and B' are non distortionary from the standpoint of the period- t government, this government has more policy instruments than policy goals, thus rendering one of the instruments redundant. As households pin down consumption based on expectations on government expenditure (to forecast the aggregate stock of capital in period $t+1$) and on next-period taxes, there always exists a pair of welfare-maximizing policies for the period- t government, (τ, B') , so that household’s expectations are fulfilled. Technically, this implies that the government’s objective function, under any level of consumption pinned down by the household, reaches its maximum at a one-dimensional space in (τ, B') . Hence, whatever expectations households may have on current debt issuance (and future policies), there is a welfare-maximizing policy that fulfills these expectations.

Proposition 5 below presents the indeterminacy result and proves the existence of a “discontinuity at infinity”. I.e., the set of Markov-perfect equilibria changes discontinuously when the economy’s time horizon is extended from finite to infinite. In particular, we show that steady states implied by Markov-perfect equilibria with positive distortions are not the limit of a finite-horizon economy’s Markov-perfect equilibrium as the time horizon goes to infinity. On the contrary, the steady state of the Markov-perfect equilibrium without distortions is the limit of the finite-horizon Markov-perfect equilibrium.

Proposition 5: *There exists a continuum of Markov-perfect equilibria. Given K_0 and B_0 , the Markov-perfect equilibrium is (globally) indeterminate. Equilibrium multiplicity results from a discontinuity at infinity. That is, if T is the economy’s planning horizon there is a unique Markov-perfect equilibrium featuring $\tau_T = 0$ and $B_T < 0$. If $T = \infty$ there is a multiplicity of Markov-perfect equilibria.*

Proof: See Appendix I.

We now calibrate and numerically solve our model economy with tax-deductible, partial capital depreciation, and show the existence of a multiplicity of Markov-perfect equilibria with distortions. In doing so, we show that the two properties of Markov-perfect equilibrium policy rules discussed in (1) and (2) above are not specific to our particular example economy, but also hold in the general model with partial capital depreciation. Finally, we also confirm numerically that the assumption of within-period commitment to taxes is not binding.

5 Markov-Perfect Equilibria in a Calibrated Economy

In this section, we study Markov-perfect equilibria in a calibrated version of the model economy presented in Section 2. Markov-perfect equilibria are found as the solutions to the three functional equations defined by the household's Euler equation, eq. (4.2), and the two Generalized Euler equations, eqs. (4.7) and (4.8). Our algorithm⁵ computes approximate solutions to these functional equations and then checks that there are no profitable deviations. That is, no government wants to deviate from the policy prescribed in each of these solutions. Special attention will be devoted first to Markov-perfect equilibria featuring income taxation and positive debt issues in the long run.

Functional Forms and Parameter Values The instantaneous utility function is assumed to be of the CES form in the composite good $c_t G_t^\theta$, that is,

$$U(c, G) = \frac{(c G^\theta)^{1-\sigma} - 1}{1-\sigma}, \quad (5.1)$$

where $0 < \theta < 1$, and $1/\sigma$ denotes the elasticity of intertemporal substitution of the composite good. The production technology is characterized by the standard Cobb-Douglas function, with α denoting the capital's share of income, i.e.

$$f(K) = AK^\alpha, \quad A > 0. \quad (5.2)$$

Parameter values are set as follows. The constant in the production function, A , and the inverse of the elasticity of intertemporal substitution, σ , are both set equal to one. The value of α is set at 0.36, which is the capital's share of income in the US economy; the depreciation rate of capital is set at 0.09, which is a standard value in macroeconomic models; β is set at 0.96, and θ is 0.2. These parameter values are in line with those used in the macro literature. Importantly, the multiplicity of equilibria discussed below does not hinge on this particular set of parameter values. Multiplicity is a generic result in our model economy.

Steady States Figure 1 plots the ray of steady states in the space of fiscal variables (τ, B) . Each point in this ray corresponds to a steady state generated by a Markov-perfect equilibrium. The point $(0, -5.3090)$ in this Figure is the steady state generated by the Markov-perfect equilibrium that yields the Ramsey outcome.

[Insert Figure 1 around here]

⁵See Appendix III for a sketch of the numerical procedure.

Table 1 below presents allocations and fiscal variables in three of these steady states, along with allocations in the efficient solution (equilibrium with lump sum taxes). Column [1] presents the efficient allocations. Columns [2] to [4] present three steady states implied by different Markov-perfect equilibrium policy rules. Column [2] corresponds to the Markov-perfect equilibrium with no distortions. Allocations and fiscal policy variables in this steady state coincide with those of the Ramsey equilibrium. (Note that allocations in the Ramsey equilibrium are the efficient ones.) In the Markov-perfect equilibrium of column [2] the government does not distort long-run investment and sets income taxes equal to zero. Public expenditure is financed entirely from the income generated by the assets owned by the government. That is, negative public debt (positive asset holdings) is the only source of income for the government in this steady state. Columns [3] and [4] correspond to steady states generated by Markov-perfect equilibrium policy rules with positive distortions. The steady state in [3] features a tax rate of 19% and a debt-to-GDP ratio of 56%.

Table 1
Steady-State Equilibria

	Markov-perfect			
	[1]	[2]	[3]	[4]
	Efficient	No distortions	Positive distortions	Positive distortions
Y	1.7608	1.7608	1.6934	1.7053
K	4.8144	4.8144	4.3201	4.4045
C	1.1063	1.1063	1.1017	1.1044
G	0.2213	0.2213	0.2032	0.2045
G/C	0.2	0.2	0.1844	0.1852
τ	indet.	0	0.1905	0.1562
B/Y	indet.	-3.015	0.5639	0
W	-5.0157	-5.0157	-5.5525	-5.4533

Notes: Steady-state values for the efficient solution and three Markov-perfect equilibria.

Transitional Dynamics We now present Markov-perfect equilibrium policy rules for consumption, savings, government expenditure in the public good, taxes and debt issues. To simplify the exposition, we restrict attention to equilibrium policy rules yielding the steady state with positive distortions in column [3] of Table 1. Equilibrium policy rules yielding the steady state in column [2] (no distortions) are sufficiently well known from the literature on Ramsey optimal

policy, and are not therefore discussed here. We start by computing equilibrium policy rules within a relatively small subset of the state space containing the steady state. Then, we expand this subset so that we can show equilibrium dynamics between this steady state and the one without distortions.

Figure 1 shows the tax policy rule, $\psi_\tau(K, B)$. The tax rate increases both with capital and debt. Figure 2 shows the debt policy rule, $\psi_B(K, B, \psi_\tau(K, B))$. Debt issues decrease sharply with capital, indicating that, in this equilibrium, capital-rich economies rely relatively less on public debt to finance government expenditure. Figure 3 shows public expenditure. The private-good consumption function is displayed in Figure 4.

[Insert Figures 2 to 5 around here]

The stability of the steady-state implied by these policy rules is illustrated in Figures 5 to 7. Net investment in physical capital, $K' - K$, is presented in Figure 5. In Figure 6 we plot the change in the level of debt, $B' - B$. Finally, Figure 7 presents the two loci, $K' = K$ and $B' = B$. The point in which these two loci intersect corresponds to the steady-state values for K and B . The arrows indicate the direction of the trajectories starting in the different regions of the state space.

[Insert Figures 6, 7 and 8 around here]

In order to provide accuracy measures of our numerical computations, Figures 8, 9 and 10 plot relative residuals in the household's Euler equation and the two Generalized Euler equations, respectively. Errors outside collocation points are very small, actually less than 0.001 of 1 per cent, and satisfy relatively well the equi-oscillation property: The sign of the errors alternates between positive and negative. Overestimation and underestimation alternate between collocation points and each error function achieves its extreme points about ten times. This property of the errors indicates that our approximations are close to being optimal, in the sense that there are no better polynomials to approximate the unknown functions.

[Insert Figures 9, 10, 11 around here]

Public Debt is not Net Wealth in a Markov-perfect Equilibrium We have shown in Proposition 4 that at the steady state of a Markov-perfect equilibrium with long-run distortions

public debt is not net worth. In the example economy of Section 4 we showed that this result holds true along the entire equilibrium path (property 2 in the remarks). Here, we confirm numerically that this is also the case in our calibrated economy with partial capital depreciation. In our numerical solution, absolute values of the derivative of the consumption function with respect to debt, $C_B(K, B, \tau)$, where τ is evaluated at $\psi_\tau(K, B)$, are below 2×10^{-6} for all K and B in our subset of the state space. We have also computed the derivative of the consumption function with respect to the tax rate, $C_\tau(K, B, \tau)$ evaluated at $\psi_\tau(K, B)$. In the example economy we showed that this derivative is zero in a Markov-perfect equilibrium (property 1 in the remarks). Our numerical solution yields absolute values for this derivative below 1×10^{-4} . Finally, absolute values of the derivative of the value function with respect to public debt in our numerical solution are below 1×10^{-5} . This extends the result found in the example economy with full capital depreciation.

Second-order Conditions: Non-profitable Deviations In models with no commitment, the concavity of the government's maximization problem is not guaranteed by the concavity of the utility and production functions. We must check that the period- t government is indeed maximizing utility by setting the tax rate and debt issues prescribed by our computed solutions. In other words, we must show that there are no profitable deviations for the period- t government, when all subsequent governments are expected to set their policy according to the computed policy rules and households expect the current and future governments to do so. That is, the maximization problem of the period- t government must be concave in τ and B' when current utility and the continuation value are evaluated at the computed policy rules, $C(K, B, \tau)$, $\tau' = \psi_\tau(K', B')$, and $B'' = \psi_B(K', B', \tau')$.

Figures 12 to 14 present our results. Figure 12 plots the value of the period- t government at $K = K^*$ and $B = B^*$ (steady-state values in column [3] of Table 1), as a function of τ and B' . This figure shows that the maximization problem of the period- t government is concave in the considered subset of the state space. Figure 13 is a contour map of Figure 12. Figure 14 plots the isoline where the value of the period- t government attains its maximum, along with the tax rate and debt issues prescribed by our computed policy rules evaluated at $K = K^*$ and $B = B^*$. That is, the maximum of the period- t government problem is attained by a one-dimensional space in (τ, B') which includes the policy prescribed by the computed Markov-perfect equilibrium rules, (τ^*, B^*) . This shows that the period- t government has no incentives to deviate.

[Insert Figures 12, 13 and 14 around here]

5.1 Equilibrium Dynamics between Steady States

Our numerical results above display equilibrium dynamics of one Markov-perfect equilibrium with positive distortions within a small subset of the state space containing the steady state of this equilibrium. In this section, we use the aforementioned properties of the equilibrium consumption function to expand the computation of this Markov-perfect equilibrium to a larger subset of the state space containing also the steady state of the Markov-perfect equilibrium without distortions (column [2] in Table 1). In particular, the fact that public debt is not household's net wealth in a Markov-perfect equilibrium allows us to approximate the unknown consumption function without having to condition on the stock of debt. We can then solve for a Markov-perfect equilibrium within an arbitrarily large subset of the state space.

Figures 15 to 19 display transitional dynamics to steady state [3] from initial conditions to capital and debt given by steady state [2] (the steady state generated by the Markov-perfect equilibrium without distortions and by the Ramsey equilibrium). Let us denote these initial conditions by (K^{RO}, B^{RO}) . At this state, debt is negative, which means the government is endowed with financial claims on the private sector. If households expect current and future governments to employ the policy rules of the Markov-perfect equilibrium without distortions, the economy stays at (K^{RO}, B^{RO}) (dotted line in the Figures). Conversely, if households expect that the current and future governments will employ the policy rules associated with the Markov-perfect equilibrium with distortions, the economy converges to the steady state given in column [3] of Table 1 (solid lines in the Figures). The intuition behind this result is as follows. If households expect taxes and debt to be positive from next-period onwards, they increase current consumption. Since households do not use the current tax rate to pin down consumption, the policy problem of the period-0 government is to optimally trade off public consumption, G , and savings, K' and B' . Under expectations that future governments will use the policy rules of the equilibrium with distortions, the continuation value for the period-0 government is independent of the level of debt, B' . As public consumption crowds out capital accumulation one-to-one, the optimal policy in period 0 hence implements $U_G = \beta \tilde{V}'_K$. Moreover, the period-0 government is indifferent between any combination of τ and B' that delivers the desired levels of G (and K'). An optimal current policy is therefore to implement G *via* a negative tax rate (which implies a lump-sum transfer to households) and issue debt exactly as expected by households. The expectation of positive taxes and debt is hence self-fulfilling.

[Insert Figures 15, 16, 17, 18 and 19 around here]

To further illustrate this mechanism, Figure 20 plots the welfare level of adopting different policies. Welfare for the period-0 government of setting the policy prescribed by the equilibrium with distortions, when households and future governments are expected to also follow policy rules of the equilibrium with distortions, is higher than that of setting the policy prescribed by the equilibrium without distortions. In Figure 20 we plot welfare for the period-0 government holding policy rules for households and future governments fixed at those of the equilibrium with distortions. Welfare under the policies of the equilibrium with distortions is 5.0633, which is the level indicated by the horizontal line in Figure 20. The curve beneath the horizontal line gives welfare under different deviations. We consider deviations where the tax rate is set at zero and debt issues range between -5.32 and -5.3 . Welfare of the particular deviation where the period-0 government sets $\tau = 0$ and $B' = B^{RO}$, that is, the tax rate and debt level associated with the policy rules of the equilibrium without distortions, is given by the square dot in the Figure. This dot is below the horizontal line and hence the government obtains less welfare than under no deviation. In sum, the period-0 government has no incentive to deviate from the policies prescribed by the considered equilibrium with distortions. The other dot on the curve, corresponding to $\tau = 0$ and $B' = -5.3151$, yields the same level of welfare as the one attained by policy rules of the equilibrium with distortions. The existence of this equal-welfare pair of taxes and debt issues is explained by the one-dimensional space in (τ, B') that solves the government maximization problem. We showed this space both in the example economy and, numerically, in Figure 14. Finally, we have also considered deviations where B' is kept fixed and τ is let to change, and found similar results.

[Insert Figure 20 around here]

6 Conclusions

This paper studies Markov-perfect optimal fiscal policy in a neoclassical economy with infinitely-lived households. We extend a recent literature on time-consistent policies to economies where the government can also issue debt to finance government expenditures and households hold physical capital and public debt in their portfolios. Previous studies on Markov-perfect policy abstract from either public debt, by assuming a government's period-by-period balanced budget constraint, or from physical capital, assuming that labor is the only factor of production.

We find a multiplicity of Markov-perfect equilibria. Equilibrium multiplicity in our model is generated by expectation traps: Households' consumption decisions based on expectations about

fiscal policy influence government policy so that those expectations are fulfilled. In shaping these dynamic complementarities between households and the government, two properties of the consumption function play a key role. First, the tax rate announced at the beginning of a given period is not used by the households to pin down consumption for the period. Second, government debt is not net wealth for the household. These two properties imply a redundancy of policy instruments and hence a multiplicity of equilibria.

Our results illustrate how in environments where the government is unable to commit, tax rates and public debt become determined by expectations. In the set of equilibria found in our model, the highest level of social welfare is attained by the Markov-perfect equilibrium that yields the allocations and policy of the Ramsey equilibrium. This result has a number of implications in terms of institutional design. In particular, it has implications for the ongoing debate on whether fiscal policy should be conducted by an independent, committed agency, as it is the case with monetary policy. The introduction of caps on government deficits is only a first step toward limiting the potential negative welfare effects of expectation traps. In sum, our findings in this paper extend to fiscal policy the same concerns about coordination failures raised in the literature on optimal discretionary monetary policy.

7 Appendices

This section contains three Appendices. Appendix I presents the proofs of the Propositions and the analysis of second-order optimality conditions for the economy of our example in Section 4. Appendix II presents the characterization of Markov-perfect equilibria under the alternative timing of actions, namely simultaneous moves. Appendix III briefly describes the numerical algorithm used in the computation of Markov-perfect equilibria.

Appendix I: Proofs

Proof of Proposition 1:

A government with full commitment sets infinite sequences $\{G_t, \tau_t, B_{t+1}\}_{t=0}^{\infty}$ so that the im-

plied competitive equilibrium maximizes welfare. That is,

$$\max_{\{G_t, \tau_t, B_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(C_t, G_t) \quad (7.1)$$

s.t.

$$C_t + K_{t+1} + G_t = f(K_t) + (1 - \delta)K_t \quad (7.2)$$

$$G_t + [1 + r_t - \delta]B_t = B_{t+1} + \tau_t[(r_t - \delta)(K_t + B_t) + \omega_t] \quad (7.3)$$

$$U_c(C_t, G_t) = \beta U_c(C_{t+1}, G_{t+1})[1 + (1 - \tau_{t+1})(r_{t+1} - \delta)], \quad t = 0 \dots \infty, \quad (7.4)$$

K_0 and B_0 are given.

After defining new variables $\tilde{r}_t \equiv (1 - \tau_t)(r_t - \delta)$ and $\tilde{\omega}_t \equiv (1 - \tau_t)\omega_t$, and formulating the problem of the government as choosing after-tax rental prices, the first-order condition with respect to K_{t+1} (by using the primal approach) can be written as,

$$\Gamma_t = \beta [\Lambda_{t+1}(r_{t+1} - \tilde{r}_{t+1}) + \Gamma_{t+1}(1 + r_{t+1} - \delta)], \quad (7.5)$$

where Γ_t and Λ_t are Lagrange multipliers. Using the Euler equation, equation (7.4) in a steady-state equilibrium becomes,

$$(\Gamma + \Lambda)(r - \tilde{r}) = 0, \quad (7.6)$$

from which it follows that $\tau = 0$ in the steady-state equilibrium, and, consequently, $B < 0$.

Proof of Proposition 2:

The first-order condition to B' in government's maximization problem (4.5) is given by,

$$U_G - \beta \tilde{V}'_{K'} + \beta \tilde{V}'_{B'} = 0. \quad (7.7)$$

The first-order condition to τ in government's maximization (4.6) is,

$$U_c C_\tau + U_G (G_\tau + B'_\tau) - \beta \tilde{V}'_{K'} (C_\tau + G_\tau + B'_\tau) + \beta \tilde{V}'_{B'} B'_\tau = 0, \quad (7.8)$$

which, after making use of (7.7), simplifies to,

$$U_c C_\tau + U_G G_\tau - \beta \tilde{V}'_{K'} (C_\tau + G_\tau) = 0. \quad (7.9)$$

Envelope conditions, along with $W(K, B) = \tilde{V}(K, B)$, yield,

$$W_K = U_c C_K + U_G G_K + \beta W'_{K'} [1 + f_K - \delta - C_K - G_K] \quad (7.10)$$

$$W_B = U_c C_B + U_G G_B + \beta W_{K'}' [-C_B - G_B]. \quad (7.11)$$

Forwarding these envelope conditions one period and using the above first-order conditions, we obtain the two Generalized Euler Equations, (4.7) and (4.8), presented in Proposition 2.

Proof of Proposition 3:

As shown in Proposition 1, in the steady state of the Ramsey equilibrium income taxes are zero and the government holds negative debt (assets) to finance the provision of the public good. The government does not rely on distortionary taxation, and the efficiency condition, $U_c = U_G$, is attained. In this proof we show that the system of equations characterizing the steady states of Markov-perfect equilibria has a solution with these properties.

Let us start by assuming that $U_c = U_G$. Then, from (7.9) it follows that $U_c = \beta W_{K'}'$. From (7.11) it is then easy to see that $W_B = 0$. Finally, equation (7.10) becomes,

$$\frac{1}{\beta} = 1 + f_K - \delta, \quad (7.12)$$

which, along with the consumer's Euler equation, implies that $\tau = 0$.

Proof of Proposition 4:

The proof follows directly from (4.10) along with the non-arbitrage, Euler and feasibility conditions. In a Markov-perfect equilibrium with positive distortions $U_c \neq U_G$. Thus, from (4.10),

$$\frac{C_\tau}{G_\tau + C_\tau} + \beta \left(G_B + \frac{G_\tau}{G_\tau + C_\tau} K_B' \right) = 0. \quad (7.13)$$

Using the feasibility condition, this is equivalent to

$$\frac{C_\tau}{G_\tau + C_\tau} \left(\frac{1}{\beta} + C_B + G_B \right) = C_B. \quad (7.14)$$

Then, plugging $G_B = -(1 - \tau)q - 1$ and the non-arbitrage condition, $q = f_K - \delta$, into equation (7.14) and using the household's Euler equation, it follows that C_B must be equal to zero at the steady state of a Markov-perfect equilibrium with positive distortions.

Second-order Optimality Conditions in the Example Economy

We show that the solutions obtained from feasibility and first-order conditions in our example economy also satisfy second-order conditions for optimality and are thus Markov-perfect equilibria. We do so by showing that none of the successive governments has incentives to deviate from any of these solutions if households and future governments are expected to follow them. In

particular, we show that the maximization problem of the period-0 government is concave under any of the solutions found above and that, hence, there are no profitable deviations.

Let us first construct the continuation value for the time-0 government when households are expected to set consumption according to the Markov strategy $C(K) = a_1 K^\alpha$, and future governments are expected to set expenditure in the public good according to $G(K) = a_2 K^\alpha$. The continuation value for the time-zero government under these policies is

$$\tilde{V}(K_1) = \sum_{t=1}^{\infty} \beta^{t-1} [\ln(a_1 K_t^\alpha) + \theta \ln(a_2 K_t^\alpha)], \quad (7.15)$$

which can be written as

$$\tilde{V}(K_1) = \sum_{t=1}^{\infty} \beta^{t-1} [\ln a_1 + \alpha \ln K_t + \theta \ln a_2 + \theta \alpha \ln K_t].$$

Rearranging this expression, we obtain

$$\tilde{V}(K_1) = \sum_{t=1}^{\infty} \beta^{t-1} [\ln a_1 + \theta \ln a_2 + (1 + \theta)\alpha \ln K_t]$$

or

$$\tilde{V}(K_1) = [\ln a_1 + \theta \ln a_2] \sum_{t=1}^{\infty} \beta^{t-1} + (1 + \theta)\alpha \sum_{t=1}^{\infty} \beta^{t-1} \ln K_t$$

or, equivalently

$$\tilde{V}(K_1) = \frac{\ln a_1 + \theta \ln a_2}{1 - \beta} + (1 + \theta)\alpha \sum_{t=1}^{\infty} \beta^{t-1} \ln K_t.$$

Under the assumed policies, the law of motion for capital for $t \geq 2$ is

$$K_t = (1 - a_1 - a_2)^{\frac{\alpha^{t-1} - 1}{\alpha - 1}} K_1^{\alpha^{t-1}}.$$

Therefore, the continuation value becomes,

$$\tilde{V}(K_1) = \frac{\ln a_1 + \theta \ln a_2}{1 - \beta} + (1 + \theta)\alpha \sum_{t=1}^{\infty} \beta^{t-1} \left(\frac{\alpha^{t-1} - 1}{\alpha - 1} \ln(1 - a_1 - a_2) + \alpha^{t-1} \ln K_1 \right).$$

Rearranging we obtain

$$\tilde{V}(K_1) = \frac{\ln a_1 + \theta \ln a_2}{1 - \beta} + (1 + \theta)\alpha \left[\ln(1 - a_1 - a_2) \sum_{t=1}^{\infty} \beta^{t-1} \frac{\alpha^{t-1} - 1}{\alpha - 1} + \ln K_1 \sum_{t=1}^{\infty} \beta^{t-1} \alpha^{t-1} \right],$$

or

$$\tilde{V}(K_1) = \frac{\ln a_1 + \theta \ln a_2}{1 - \beta} + (1 + \theta)\alpha \left[\ln(1 - a_1 - a_2) \left(\frac{1}{(1 - \beta)(1 - \alpha)} + \sum_{t=1}^{\infty} \beta^{t-1} \frac{\alpha^{t-1}}{\alpha - 1} \right) + \ln K_1 \sum_{t=1}^{\infty} (\beta\alpha)^{t-1} \right],$$

or

$$\tilde{V}(K_1) = \frac{\ln a_1 + \theta \ln a_2}{1 - \beta} + (1 + \theta)\alpha \left[\beta \frac{\ln(1 - a_1 - a_2)}{(1 - \beta\alpha)(1 - \beta)} + \frac{\ln K_1}{1 - \beta\alpha} \right],$$

or, rearranging,

$$\tilde{V}(K_1) = \gamma + \frac{(1 + \theta)\alpha}{1 - \beta\alpha} \ln K_1, \quad (7.16)$$

where

$$\gamma \equiv \frac{\ln a_1 + \theta \ln a_2}{1 - \beta} + \frac{(1 + \theta)\alpha\beta}{(1 - \beta\alpha)(1 - \beta)} \ln(1 - a_1 - a_2). \quad (7.17)$$

Then, given a_1 and q_0 , the maximization problem for the time-zero government is,

$$\max_{\tau_0, G_0, B_1} \left\{ \ln C(K_0) + \theta \ln G_0 + \beta \left(\gamma + \frac{(1 + \theta)\alpha}{1 - \beta\alpha} \ln K_1 \right) \right\} \quad (7.18)$$

s.t.

$$K_1 = K_0^\alpha - a_1 K_0^\alpha - G_0 \quad (7.19)$$

$$G_0 = \tau_0(K_0^\alpha + q_0 B_0) + B_1 - (1 + q_0)B_0 \quad (7.20)$$

which is a concave maximization problem. The redundancy of policy instruments can be seen from the first-order conditions with respect to τ_0 and B_1 , respectively:

$$\frac{\theta(K_0^\alpha + q_0 B_0)}{\tau_0(K_0^\alpha + q_0 B_0) + B_1 - (1 + q_0)B_0} = \frac{(1 + \theta)\beta\alpha(K_0^\alpha + q_0 B_0)}{(1 - \beta\alpha)[K_0^\alpha - a_1 K_0^\alpha - \tau_0(K_0^\alpha + q_0 B_0) - B_1 + (1 + q_0)B_0]}$$

$$\frac{\theta}{\tau_0(K_0^\alpha + q_0 B_0) + B_1 - (1 + q_0)B_0} = \frac{(1 + \theta)\beta\alpha}{(1 - \beta\alpha)[K_0^\alpha - a_1 K_0^\alpha - \tau_0(K_0^\alpha + q_0 B_0) - B_1 + (1 + q_0)B_0]}.$$

If we divide the first of these equations by $K_0^\alpha + q_0 B_0$ we obtain the second. As a consequence, we can only pin down a one-dimensional space in (τ_0, B_1) that solves the maximization problem of the period-zero government. This government has no incentive to deviate from any of the solutions found above because the policy implied by these solutions belongs to the one-dimensional space in (τ_0, B_1) . More specifically, given a particular a_1 and a_2 the period-zero's government policy under no deviation is given by,

$$a_2 K_0^\alpha = \tau_0(K_0^\alpha + q_0 B_0) + B_1 - (1 + q_0)B_0,$$

and any of the two equations above.

Proof of Proposition 5:

Here we prove that the finite-horizon economy with final period T has a unique Markov-perfect equilibrium with $\tau_T = 0$ and $B_T < 0$. Without loss of generality we assume full capital depreciation. The proof, although algebraically tedious, is straightforward.

In the finite-horizon economy with final period T , households consume all their resources in period T and then $K_{T+1} = B_{T+1} = 0$. The problem of the time- T government is then,

$$\max_{\tau_T} \quad \{\ln C_T + \theta \ln G_T\}$$

s.t.

$$K_T^\alpha = C_T + G_T \quad (7.21)$$

$$G_T = \tau_T [K_T^\alpha + q_T B_T] - (1 + q_T) B_T. \quad (7.22)$$

The first-order condition to this problem is,

$$\frac{1}{C_T} = \theta \frac{1}{G_T}, \quad (7.23)$$

from where it follows that $G_T = \theta C_T$. Using the feasibility condition, (7.21), we obtain,

$$C_T = \frac{1}{1 + \theta} K_T^\alpha \quad (7.24)$$

$$G_T = \frac{\theta}{1 + \theta} K_T^\alpha. \quad (7.25)$$

In period $T - 1$, the households' Euler equation is,

$$\frac{1}{C_{T-1}} = \beta \frac{1}{C_T} (1 - \tau_T) \alpha K_T^{\alpha-1}, \quad (7.26)$$

and the non-arbitrage condition between the two assets is

$$q_T = \alpha K_T^{\alpha-1} - 1/(1 - \tau_T). \quad (7.27)$$

The fiscal policy chosen by the time- $(T - 1)$ government is obtained as the solution to the following maximization problem,

$$\max_{\tau_{T-1}, B_T} \quad \{\ln C_{T-1} + \theta \ln G_{T-1} + \beta (\ln C_T + \theta \ln G_T)\}$$

s.t.

$$K_{T-1}^\alpha = K_T + C_{T-1} + G_{T-1} \quad (7.28)$$

$$G_{T-1} = B_T + \tau_{T-1} [K_{T-1}^\alpha + q_{T-1} B_{T-1}] - (1 + q_{T-1}) B_{T-1} \quad (7.29)$$

and equations (7.22), (7.24), (7.25), (7.26) and (7.27).

Before deriving the first-order conditions, we find it appropriate to plug (7.24) into (7.26), to write the household Euler equation as,

$$K_T = (1 + \theta) \beta \alpha (1 - \tau_T) C_{T-1}. \quad (7.30)$$

From the budget constraint of the government in T , equation (7.22), from the level of expenditure in T , equation (7.25), and from the non-arbitrage condition in $T - 1$, equation (7.27), we obtain that,

$$1 - \tau_T = \frac{1}{(1 + \theta)[1 + \alpha \frac{B_T}{K_T}]}. \quad (7.31)$$

Plugging this expression into (7.30) we get,

$$K_T = \beta\alpha C_{T-1} - \alpha B_T, \quad (7.32)$$

which, along with the feasibility condition and the budget constraint of the government in $T - 1$, yields,

$$C_{T-1} = \frac{K_{T-1}^\alpha - \tau_{T-1}[K_{T-1}^\alpha + q_{T-1}B_{T-1}] - (1 - \alpha)B_T + (1 + q_{T-1})B_{T-1}}{1 + \beta\alpha}. \quad (7.33)$$

This is the level of consumption in period $T - 1$ as a function of the policy chosen in period $T - 1$, i.e. τ_{T-1} and B_T . Note, however, that before deriving the first-order conditions of the period- $(T - 1)$ government we must substitute B_T by its expectation, say B_T^e , in (7.33), in order to embed our timing of actions. The period- $(T - 1)$ government chooses taxes before the household's consumption-savings decision, but chooses debt issues simultaneously with the household decision. Therefore, this government cannot anticipate the effect of debt issues on the level of consumption in $T - 1$.

The first-order conditions with respect to τ_{T-1} and B_T are, respectively,

$$-\frac{1}{C_{T-1}} \frac{1}{1 + \beta\alpha} + \frac{\theta}{G_{T-1}} = \frac{\beta^2 \alpha^2}{1 + \beta\alpha} \frac{1}{C_T} K_T^{\alpha-1} \quad (7.34)$$

$$\frac{\theta}{G_{T-1}} = \beta\alpha \frac{1}{C_T} K_T^{\alpha-1}. \quad (7.35)$$

Plugging one of these equations into the other, we get

$$\frac{1}{C_{T-1}} = \beta \frac{1}{C_T} \alpha K_T^{\alpha-1}. \quad (7.36)$$

Using the household Euler equation, (7.26), it immediately follows that $\tau_T = 0$. Finally, from the budget constraint of the period- T government we obtain that $B_T < 0$.

Appendix II: Alternative Timing of Actions: Simultaneous Moves

This Appendix characterizes Markov-perfect optimal policy in an economy where the government sets its policy —both taxes and debt issues— at the same time households choose consumption and savings. In contrast to the timing of events studied above, the tax rate for

the current period is not announced before households make their decision, implying that they must now forecast this rate as well as debt issues. The consumption function that solves the household's Euler equation can now be expressed in terms of K and B , say $C(K, B)$,

$$U_c(C(K, B), G) = \beta U_c(C(K', B'), G') [1 + (1 - \tau') (f_K(K') - \delta)], \quad (7.37)$$

where $B' = \psi_B(K, B)$ and $\tau' = \psi_\tau(K', B')$. And K' is given by,

$$K' = K + B + (1 - \tau) [f(K) - \delta K + q(K)B] - C(K, B) - B'. \quad (7.38)$$

Since households do not know the tax rate for the current period, they anticipate that the government will set τ as a function of K and B , $\psi_\tau(K, B)$.

The government is no longer an intra-period *Stackelberg player*, and it sets τ and B' taking as given the consumption function and the policy of future governments. As will become apparent below, the functional equations characterizing a Markov-perfect equilibrium are now substantially simpler.

The problem of the current government can then be written as,

$$V(K, B) = \max_{\tau, B'} \left\{ U(C(K, B), G) + \beta \tilde{V}(K', B') \right\} \quad (7.39)$$

s.t.

$$K' = (1 - \delta)K + f(K) - C(K, B) - G$$

$$G = \tau [f(K) - \delta K + q(K)B] + B' - [1 + q(K)] B,$$

and equation (4.4).

By assuming that savings is the residual variable, the government is unable to affect current private-good consumption, thus leaving consumption of the public good and the continuation value as its only trade-off. First-order conditions with respect to τ and B' are, respectively, $U_G = \beta \tilde{V}'_{K'}$ and $G_{B'} (\beta \tilde{V}'_{K'} - U_G) = \beta \tilde{V}'_{B'}$. From these two conditions it is straightforward to see that $\tilde{V}'_{B'} = 0$. Using the envelope conditions we obtain the two generalized Euler equations,

$$(U'_{C'} - U'_{G'}) C'_{B'} = 0 \quad (7.40)$$

$$U_G = \beta [U'_{C'} C'_{K'} + U'_{G'} (1 + f'_{K'} - \delta - C'_{K'})]. \quad (7.41)$$

Under this alternative timing, the possibility of multiple Markov-perfect equilibria becomes readily apparent from equation (7.40). Both a Markov-perfect equilibrium with no distortions,

$U'_{C'} = U'_{G'}$, and Markov-perfect equilibria with distortions and with $C'_{B'} = 0$ satisfy this condition. The first of these equilibria yields a steady state that coincides with the Ramsey outcome. The second type of equilibria yield steady states with positive taxes.

Appendix III: Numerical Approach

This appendix outlines the algorithm for the computation of Markov-perfect equilibria. The first challenge in the computation of the three unknown functions $C(K, B, \tau)$, $\psi_\tau(K, B)$, and $\psi_B(K, B, \tau)$ stems from the presence of the derivatives of the consumption function in the two generalized Euler equations, (4.7) and (4.8). In a steady state, these derivatives must be solved for, thus making the number of unknowns exceed the number of equations.

The computational method is an application of a projection method which approximates the three unknown functions with a combination of Chebyshev polynomials. Within the class of orthogonal polynomials, Chebyshev polynomials stand out for their efficiency to approximate smooth functions.⁶ The unknown coefficients in the approximate functions are then obtained so that they satisfy the three Euler equations at some collocation points within a subset of the state space, $[K_{min}, K_{max}] \times [B_{min}, B_{max}]$.

Thus, we approximate functions for consumption, taxes and the issue of debt by⁷:

$$\hat{C}(K, B, \tau; \vec{a}) = \sum_{i=0}^{n_k^c} \sum_{j=0}^{n_b^c} \sum_{\ell=0}^{n_\tau^c} a_{ij\ell} \phi_{ij\ell}(K, B, \tau) \quad (7.42)$$

$$\hat{\psi}_\tau(K, B, \vec{d}) = \sum_{i=0}^{n_k^b} \sum_{j=0}^{n_b^b} d_{ij} \phi_{ij}(K, B) \quad (7.43)$$

$$\hat{\psi}_B(K, B, \vec{h}) = \sum_{i=0}^{n_k^r} \sum_{j=0}^{n_b^r} h_{ij} \phi_{ij}(K, B), \quad (7.44)$$

where $\phi_{ij\ell}(K, B, \tau)$ and $\phi_{ij}(K, B)$ are tensor products of univariate Chebyshev polynomials, which form the multidimensional basis for approximation. For instance, $\phi_{ij}(K, B) = \phi_i(K)\phi_j(B)$, with $\phi_i(K)$ denoting the Chebyshev polynomial of order i in K and $\phi_j(B)$ the Chebyshev polynomial of order j in B . Since Chebyshev polynomials are only defined in the interval $[-1, 1]$, K and B

⁶For a complete characterization of their properties and a rigorous exposition of projection techniques see Judd (1992, 1998). For a previous application of these ideas to the computation of Markovian optimal taxes see Ortigueira (2006).

⁷Since the derivative of $\psi_B(K, B, \tau)$ with respect to τ is not needed to solve our system of functional equations, we approximate the debt policy rule at equilibrium by a function of K and B alone, replacing τ by $\psi_\tau(K, B)$.

must be re-scaled accordingly, using the chosen $K_{\min}, K_{\max}, B_{\min}, B_{\max}$. That is,

$$\phi_{ij}(K, B) = \phi_i \left(\frac{2(K - K_{\min})}{K_{\max} - K_{\min}} - 1 \right) \times \phi_j \left(\frac{2(B - B_{\min})}{B_{\max} - B_{\min}} - 1 \right). \quad (7.45)$$

Vectors \vec{a} , \vec{d} , \vec{h} in (7.42) – (7.44) are the unknown coefficients, which are pinned down by imposing that $\hat{C}(K, B, \tau; \vec{a})$, $\hat{\psi}_\tau(K, B, \vec{d})$ and $\hat{\psi}_B(K, B, \vec{h})$ satisfy the three Euler equations and the laws of motion at a number of collocation points. The number of collocation points is set so that the number of equations equals the number of unknown coefficients. In our exercise we choose Chebyshev collocation. It should be noted that the approximation of the debt policy, equation (7.44), embeds already the approximation of the tax policy in terms of K and B . On the other hand, the approximation of the consumption function, (7.42), must be done in terms of K , B and τ , in order to obtain the derivatives of the consumption function which show up in the Generalized Euler equations. Note, however, that we compute the consumption function, and its derivatives, only along the equilibrium path, i.e., for $\tau = \hat{\psi}_\tau(K, B)$. That is, both the consumption function and its derivatives are computed as functions of K and B .

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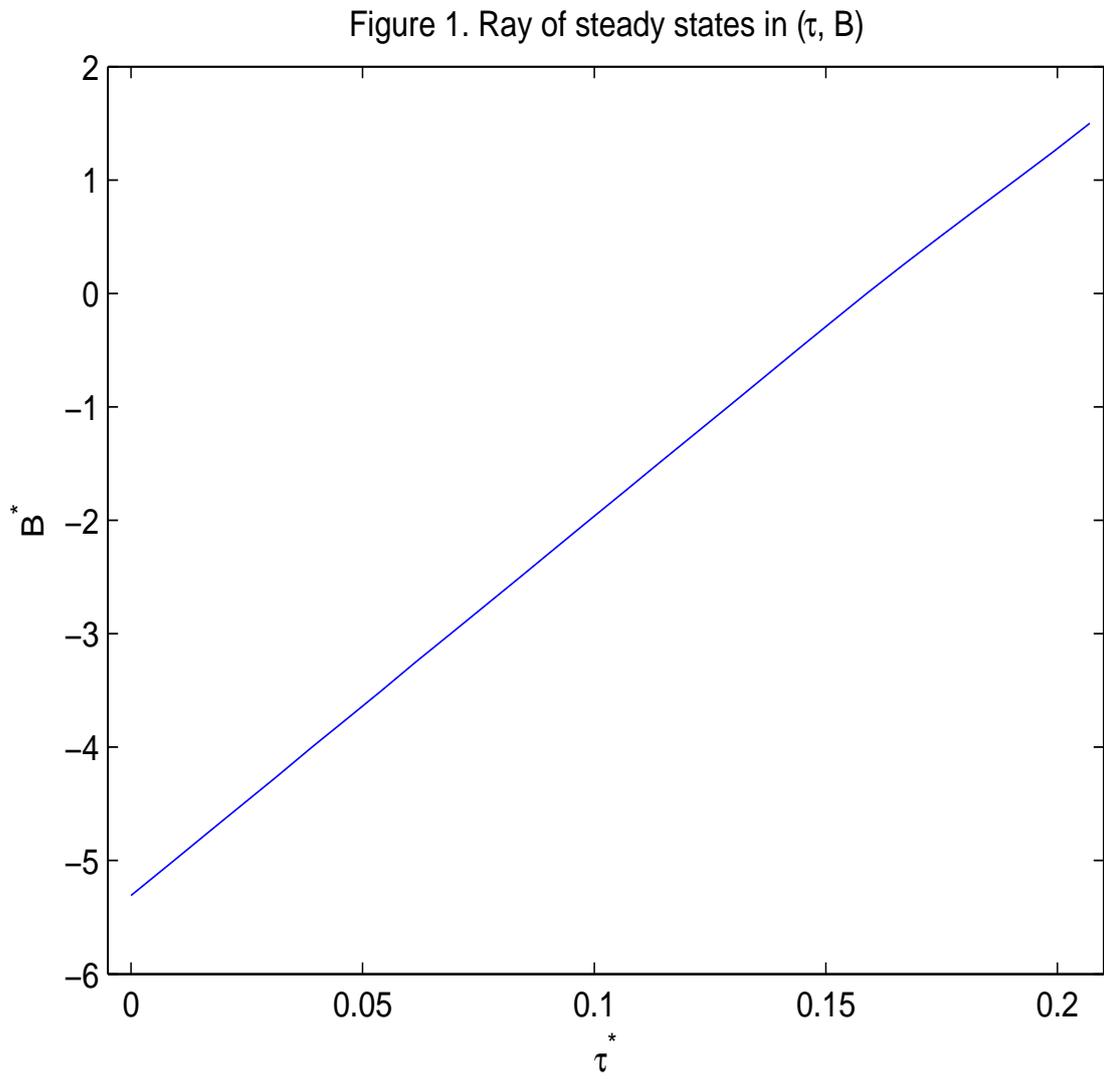
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Notes: Figure 1 plots the ray of steady states in the space of taxes and debt levels.

Policy Functions

Figure 2. Tax Policy Function

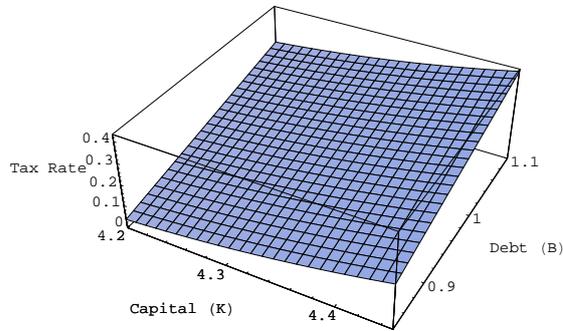


Figure 3. Debt Policy Function

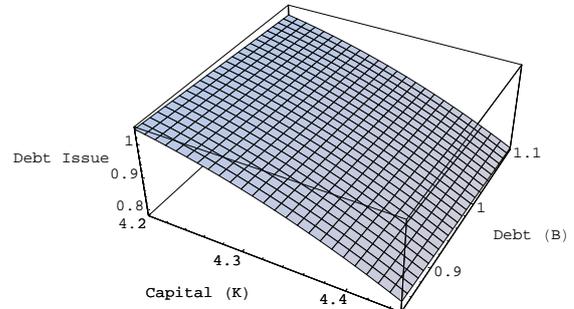


Figure 4. Gov. Expenditure Function

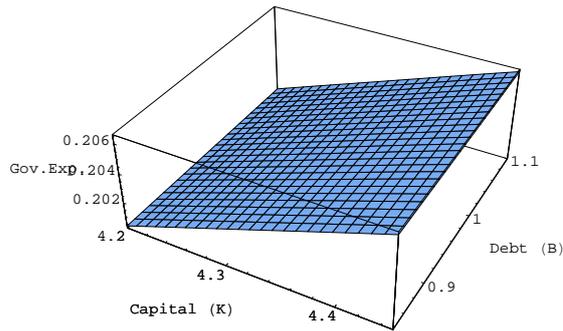
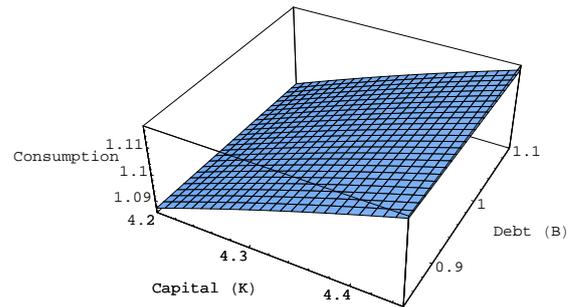


Figure 5. Consumption Function



Notes: Figures 2 to 5 show policy functions in the Markov-perfect equilibrium generating steady state [3] in Table 1. The government's tax policy is shown in Figure 2. The government's debt policy is shown in Figure 3. The government's spending policy is displayed in Figure 4. Finally, the private consumption function is shown in Figure 5.

Figure 6. Net Investment, $K' - K$

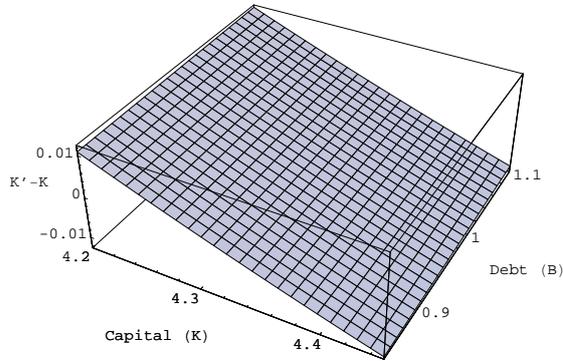


Figure 7. Change in Public Debt, $B' - B$

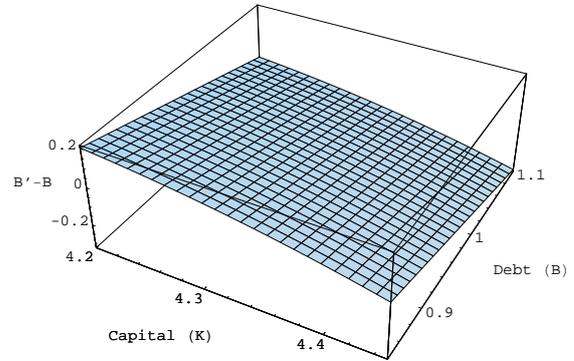
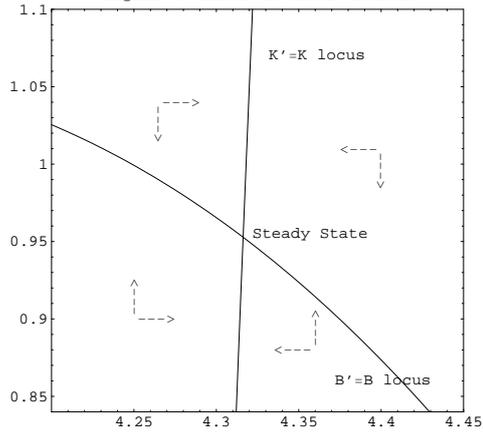


Figure 8. The $K' = K$ and $B' = B$ Loci



Notes: Figures 6 to 8 show equilibrium dynamics around steady state [3] in Table 1. Figure 6 shows net investment; Figure 7 shows the change in government debt; and Figure 8 shows the $K' = K$ and $B' = B$ loci.

Errors

Figure 9. Euler Equation Errors

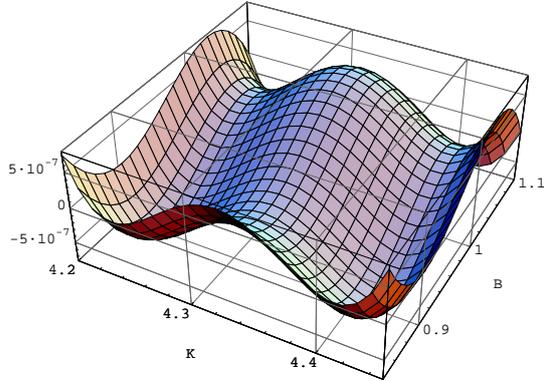


Figure 10. Generalized Euler Equation 1 Errors

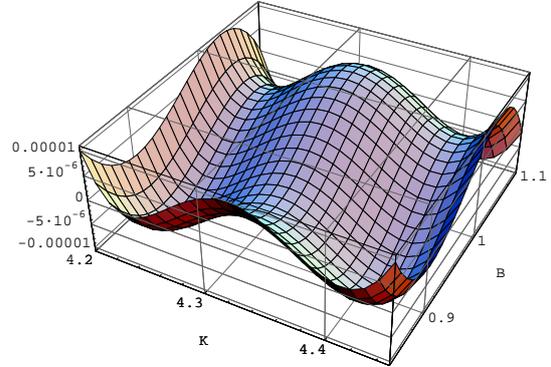
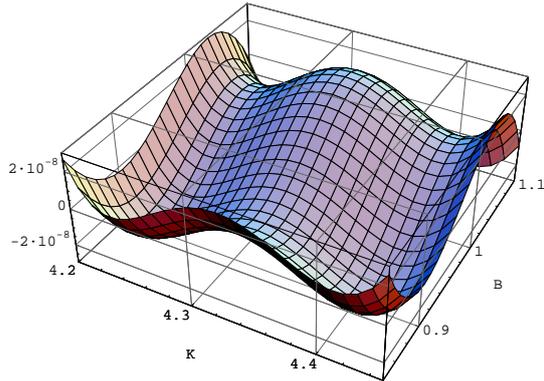
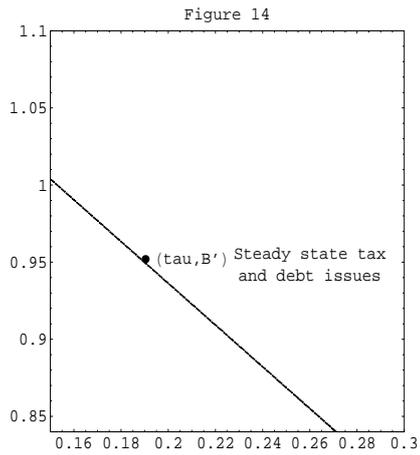
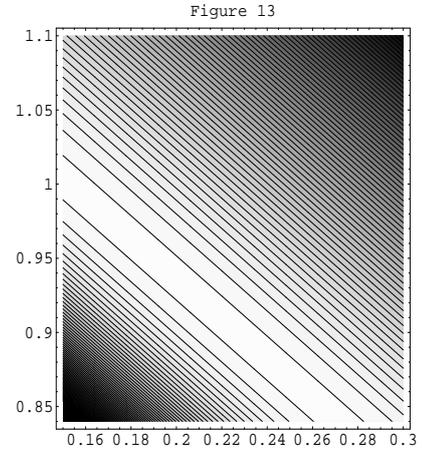
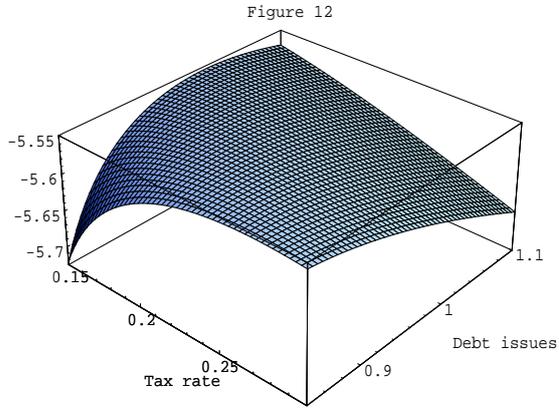


Figure 11. Generalized Euler Equation 2 Errors

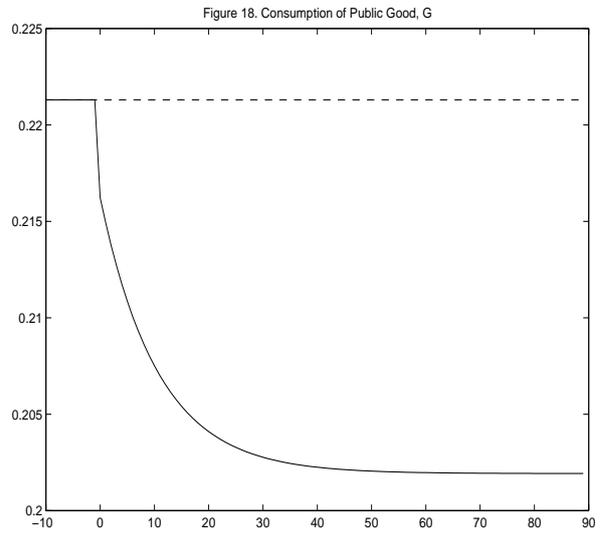
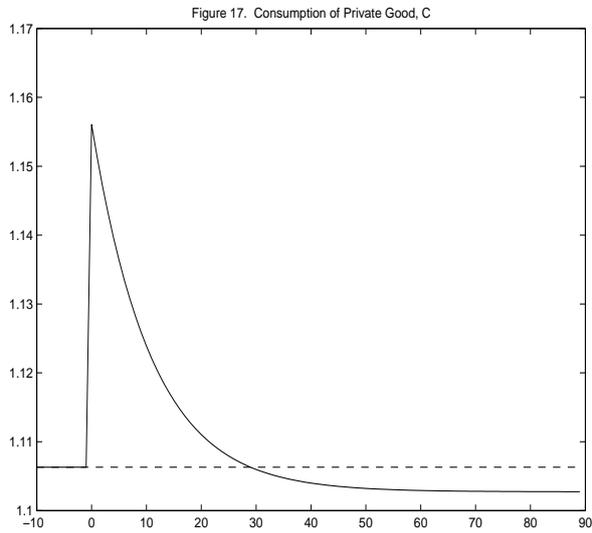
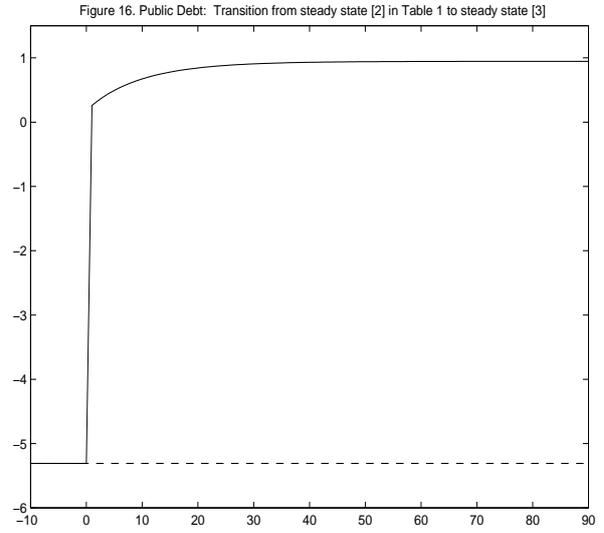
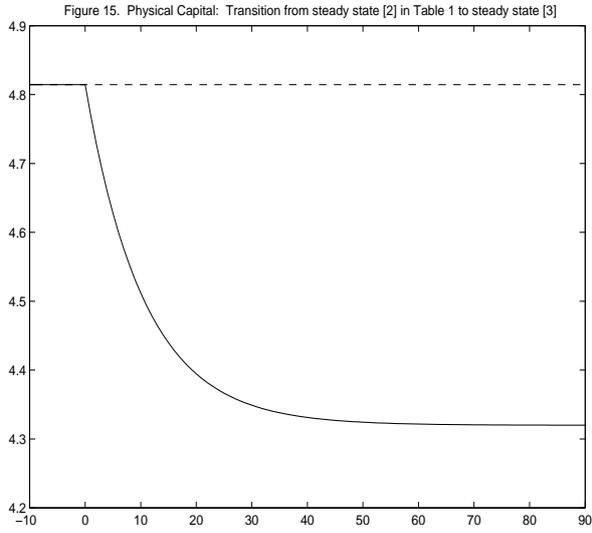


Notes: Figures 9 to 11 show relative errors of Chebyshev collocation for the Euler equation (Figure 9), and the two Generalized Euler equations for the government (Figures 10 and 11).

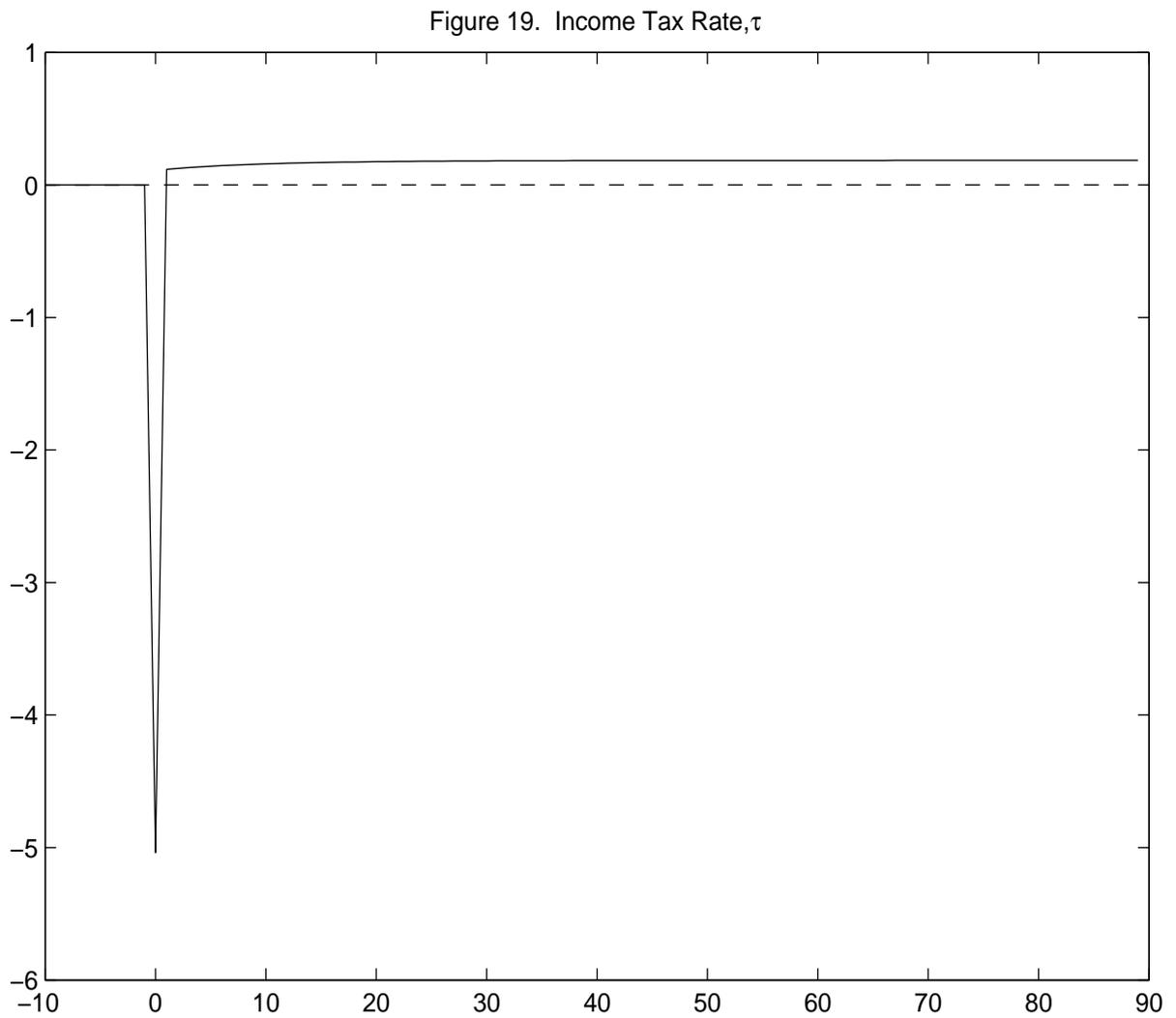


Notes: Figure 12 plots the value of the period- t government at (K^*, B^*) —steady state [3] in Table 1—, as a function of the period- t tax rate and debt issues. Figure 13 is a contour map of the function in Figure 12. Figure 14 plots the isoline where the value of the period- t government attains its maximum. The dot in Figure 14 corresponds to the tax rate and debt issues prescribed by the policies of the Markov-perfect equilibrium generating steady state [3].

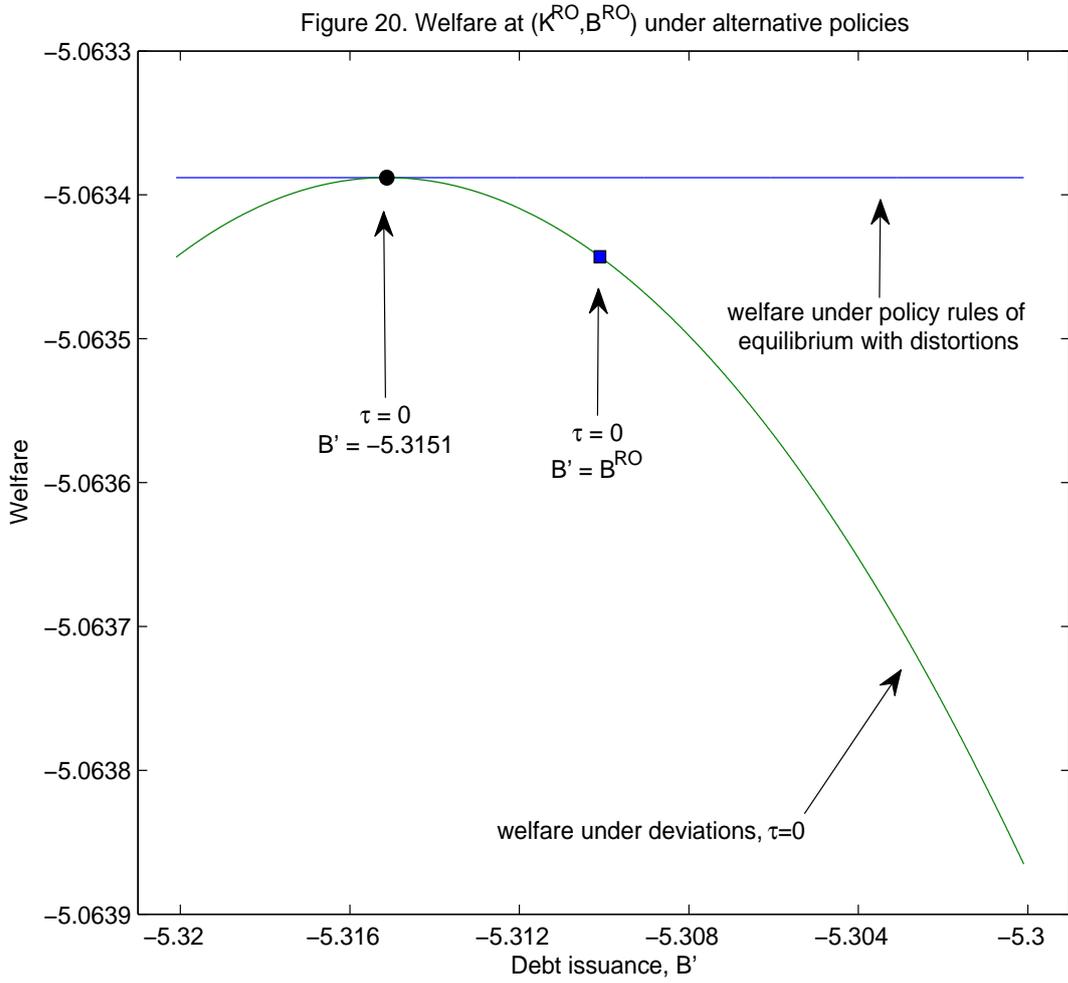
Transitional Dynamics



Notes: Figures 15-18 plot equilibrium dynamics for physical capital, public debt and private and public consumption from initial conditions (K_0, B_0) given by steady-state values in column [2] of Table 1, (the steady state with no distortions) and converging to steady-state [3].



Notes: Figure 19 plots the income tax rate along a Markov-perfect equilibrium from initial conditions (K_0, B_0) given by steady-state values in column [2] of Table 1, (the steady state with no distortions) and converging to steady-state values in column [3].



Notes: Figure 20 plots welfare at initial conditions (K^{RO}, B^{RO}) —the steady-state values of the Ramsey outcome— under alternative policies for the period-zero government. The horizontal line marks the level of welfare of adopting policy rules of the equilibrium with distortions yielding steady state [3] in Table 1.