Continuous Optimization

Extending pricing rules with general risk functions

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A B S T R A C T

The paper addresses pricing issues in imperfect and/or incomplete markets if the risk level of the hedging strategy is measured by a general risk function. Convex Optimization Theory is used in order to extend pricing rules for a wide family of risk functions, including Deviation Measures, Expectation Bounded Risk Measures and Coherent Measures of Risk. Necessary and sufficient optimality conditions are provided in a very general setting. For imperfect markets the extended pricing rules reduce the bid ask spread. The findings are particularized so as to study with more detail some concrete examples, including the Conditional Value at Risk and some properties of the Standard Deviation. Applications dealing with the valuation of volatility linked derivatives are discussed.

1. Introduction

General risk functions are becoming more and more important in finance. Since the paper of Artzner et al. (1999) introduced the axioms and properties of their "Coherent Measures of Risk", many authors have extended the discussion. Hence, it is not surprising that the recent literature presents many interesting contributions focusing on new methods for measuring risk levels. Among others, Goovaerts et al. (2004) have introduced the Consistent Risk Measures, and Rockafellar et al. (2006a) have defined the Deviations and the Expectation Bounded Risk Measures.

Many classical financial problems have been revisited by using new risk functions. So, Mansini et al. (2007) deal with Portfolio Choice Problems with complex risk measures, Alexander et al. (2006) compare the minimization of the Value at Risk (*VaR*) and the Conditional Value at Risk (*CVaR*) for a portfolio of derivatives, Calafiore (2007) studies "robust" efficient portfolios if risk levels are given by Standard Deviations and absolute deviations, and Schied (2007) deals with Optimal Investment with Convex Risk Measures.

The extension of pricing rules to the whole space in incomplete markets is a major topic in finance. Several papers have used Coherent Measures of Risk to price and hedge under incompleteness, though the article by Nakano (2004) seems to be an interesting approach that also incorporates previous and significant contributions of other authors. Another line of research is related to the concept of "good deal", introduced in the seminal paper by Cochrane and Saa Requejo (2000). A good deal is not an arbitrage but is close to an arbitrage, so the absence of good deal may be an adequate assumption if it is used for pricing.

In recent papers Jaschke and Küchler (2001) and Staum (2004) extended the notion of good deal so as to involve coherent measures of risk, and they introduced the "coherent prices" as upper and lower bounds that every extension of the pricing rule to the whole space must respect. They allowed for imperfections in the initial market and also studied existence properties and other classical issues. Later Cherny (2006) also dealt with pricing issues with risk measures in incomplete markets, though it is not the major focus of the article.

The present paper considers an initial incomplete and maybe imperfect market and deals with the Expectation Bounded Risk Measures and the Deviation Measures of Rockafellar et al. (2006a) in order to extend the pricing rule to the whole space. As we will see, the Representation Theorems of Risk Measures provided by the authors above are very appropriate to simplify the Mathematical Programming Problems leading to Optimal Hedging Strategies and prices, which permits us to introduce new pricing rules satisfying adequate properties and easy to compute in practice.

The paper's outline is as follows: Section 2 will present notations and the basic conditions and properties of the initial pricing rule π to be extended and the risk function ρ to be used. Since the risk function is not differentiable in general, the optimization problem giving the optimal hedging strategy and the pricing rule extension is not differentiable either, and Section 3 will be devoted to overcome this caveat. Actually, the minimization of risk

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functions may be very complicated in practice, as pointed out by Rockafellar et al. (2006b) and Ruszczynski and Shapiro (2006), among others. Thus, a major objective of this paper is to yield necessary and sufficient optimality conditions that will allow us to solve the minimization problem we have to deal with in order to price and hedge new assets.

We will use Representation Theorems of Risk Measures so as to transform the initial optimal hedging problem in a minimax problem. Later, following an idea developed in Balbás et al. (forthcoming), the minimax problem is equivalent to a new convex optimization problem in Banach spaces. In particular, the dual variable belongs to the set of probabilities on the Borel σ algebra of the sub gradient of ρ. Since this fact would provoke high degree of complexity when dealing with the optimality conditions of the hedging problem, Theorem 2 is one of the most important results in this section, because it guarantees that the optimal dual solution will be a Dirac Delta, and thus we can leave the use of general probability measures in order to characterize the optimal solutions. The section ends by proving its second important result. Theorem 4 yields simple necessary and sufficient optimality conditions as well as guarantees the existence of Stochastic Discount Factors of π into the sub gradient of ρ.

Section 4 starts by introducing the extension π_ρ of π. π_ρ is given by four equivalent expressions. The first expression is generated by the dual problem, while the remaining ones are related to the primal. Theorem 7 shows the interesting properties of π_ρ, that is convex, continuous, and bounded by π and ρ. Furthermore, π_ρ is a genuine extension of π if the initial market is free of frictions, and reduces the transaction costs caused by π otherwise. We have proved the theorem by using the dual expression of π_ρ. However, it may be worth to remark that the proof may also be constructed by using the primal expressions, *i.e.*, the duality theory of Section 3 does not have to be used to establish Theorem 7.

Theorem 9 states that the Stochastic Discount Factors of π and π_ρ that belong to the sub gradient of ρ coincide, which enables us to prevent the existence of arbitrage opportunities for π_ρ in Corollary 10. The section ends by proving that π_ρ outperforms the classical extension of pricing rules in incomplete (and maybe imperfect) markets if ρ is coherent.

Section 5 considers a General Deviation Measure and focuses on this particular case. Special attention is paid to the Standard Deviation, since it is often used in finance to extend pricing rules (see Schweizer, 1995, or Luenberger, 2001, among others). Some relationships between the proposed extension and other classical ones are analyzed. Section 6 deals with the *CVaR*, since it is becoming a very popular Coherent and Expectation Bounded Risk Measure that respects the second order Stochastic Dominance (Ogryczak and Ruszczynski, 2002) and has been used by several authors in different types of Portfolio Choice Problems. Theorem 13 characterizes the proposed extension in this special case and its Corollary 14 focuses on some particular situations.

Section 7 attempts to summarize how π_ρ may perform in some practical situations. Illustrative numerical examples are yielded, and applications to price volatility linked derivatives are discussed. The last section of the paper points out the most important conclusions.

2. Preliminaries and notations

Consider the probability space $(\Omega, \mathcal{F}, \mu)$ composed of the set of "states of the world" Ω, the σ algebra $\mathcal{F}$ and the probability measure μ. Consider also a couple of conjugate numbers $p \in [1, \infty)$ and $q \in (1, \infty]$ (*i.e.*, $1/p + 1/q = 1$). As usual L^p (L^q) denotes the Banach space of $\mathbb{R}$ valued measurable functions y on Ω such that $\mathbb{E}(|y|^p) < \infty$, $\mathbb{E}()$ representing the mathematical expectation ($\mathbb{E}(|y|^q) < \infty$, or y essentially bounded if $q = \infty$). According to the Riesz Representation Theorem, we have that L^q is the dual space of L^p.

Consider a time interval $[0, T]$, a subset $\mathcal{T} \subset [0, T]$ of trading dates containing 0 and T, and a filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ providing the arrival of information and such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. In general, $(S_t)_{t \in \mathcal{T}}$ will denote an adapted stochastic price process.

Let us assume that $Y \subset L^p$ is a convex cone composed of super replicable pay offs, *i.e.*, for every $y \in Y$ there exists at least one self financing portfolio whose final pay off is $S_T \geqslant y$. Denote by $\mathcal{S}(y)$ the family of such self financing portfolios, and suppose that there exists

$$\pi(y) = Inf\{S_0; (S_t)_{t \in \mathcal{T}} \in \mathcal{S}(y)\} \tag{1}$$

for every $y \in Y$. We will say that $\pi(y)$ is the price of y. The market will be said to be complete if for every $y \in L^p$ there exists $(S_t)_{t \in \mathcal{T}} \in \mathcal{S}(y)$ such that $S_T = y$, and incomplete otherwise. Besides, the market will be said to be perfect if Y is a subspace of L^p and $\pi : Y \to \mathbb{R}$ is linear, and imperfect otherwise. In general, we will impose the natural conditions, sub additivity

$$\pi(y_1 + y_2) \leqslant \pi(y_1) + \pi(y_2) \tag{2}$$

for every $y_1, \ y_2 \in Y$, and positive homogeneity

$$\pi(\alpha y) = \alpha \pi(y) \tag{3}$$

for every $y \in Y$ and $\alpha \geqslant 0$. Consequently, π is a convex function. Finally, we will assume the existence of a riskless asset that does not generate any friction, *i.e.*, almost surely constant random variables $y = k$ belong to Y for every $k \in \mathbb{R}$, and there exists a risk free rate $r_f \geqslant 0$ such that

$$\pi(k) = ke^{-r_f T}, \tag{4}$$

holds. It is easy to see that (4) leads to

$$\pi(y + k) = \pi(y) + ke^{-r_f T} \tag{5}$$

for every $y \in Y$ and $k \in \mathbb{R}$. Indeed $\pi(y + k) \leqslant \pi(y) + ke^{-r_f T}$ is clear, and

$$\pi(y) = \pi(y + k - k) \leqslant \pi(y + k) + \pi(-k) = \pi(y + k) - ke^{-r_f T}$$

implies the opposite inequality.

Though it is not formally needed, previous literature often uses a finite or at best countable set of states Ω for static or discrete time dynamic models (see for instance Duffie, 1996), since the mathematical exposition is significantly simplified. The simplification is not feasible if continuous time pricing models are involved.

When Ω is finite then L^p and L^q will be $\mathbb{R}^n$, n denoting the cardinal of Ω. Thus, the completeness of the market holds if every vector (final pay off) in $\mathbb{R}^n$ may be replicated by combining the available assets. If the model is also static ($\mathcal{T} = \{0, T\}$, *i.e.*, there are only two trading dates) then the completeness of the market holds if and only if the number of independent assets equals n. An interesting interpretation is discussed, for instance, in Tapiero (2004), where it is said that the market is incomplete if the number of assets that make up a portfolio is less than the market risk sources (plus one if we do not use the risk free asset in the portfolio). As already said, this framework simplifies the degree of mathematics in the analysis, although it does not apply for many very important financial models (the Black and Scholes model, for instance).

From a financial perspective, an imperfect market is one in which market participants do not have full access to information about the securities and in which buyers are not immediately matched with sellers. The most important mathematical consequence is that the pricing rule of the market is not linear any more. The approaches by Lakner (1998) and Grorud and Pontier (2001), among others, provide a theoretical framework for thinking about imperfect markets.

Let

$\rho : L^p \rightarrow \mathbb{R}$

be the general risk function that a trader uses in order to control the risk level of his final wealth at T. Denote by

$$\Delta_\rho = \{z \in L^q; \quad -\mathbb{E}(yz) \leqslant \rho(y), \ \forall y \in L^p\}. \tag{6}$$

The set Δ_ρ is obviously convex. We will assume that Δ_ρ is also $\sigma(L^q, L^p)$ compact and

$$\rho(y) = \operatorname{Max}\{-\mathbb{E}(yz) : z \in \Delta_\rho\}, \tag{7}$$

holds for every $y \in L^p$. Furthermore, we will also impose

$$\Delta_\rho \subset \{z \in L^q; \mathbb{E}(z) = 1\}. \tag{8}$$

These are quite natural assumptions closely related to the Representation Theorems of Risk Measures stated in Rockafellar et al. (2006a). Following their ideas, and bearing in mind the Representation Theorem 2.4.9 in Zalinescu (2002) for convex functions, it is easy to prove that the $\sigma(L^q, L^p)$ compactness of Δ_ρ and the fulfillment of (7) and (8) hold if ρ is continuous and satisfies:

(a)

$$\rho(y + k) = \rho(y) - k \tag{9}$$

for every $y \in L^p$ and $k \in \mathbb{R}$.

(b)

$$\rho(\alpha y) = \alpha \rho(y) \tag{10}$$

for every $y \in L^p$ and $\alpha > 0$.

(c)

$$\rho(y_1 + y_2) \leqslant \rho(y_1) + \rho(y_2) \tag{11}$$

for every $y_1, y_2 \in L^p$.

(d)

$$\rho(y) \geqslant -\mathbb{E}(y) \tag{12}$$

for every $y \in L^p$.[1,2]

It is easy to see that if ρ is continuous and satisfies Properties (a)–(d) above then it is also coherent in the sense of Artzner et al. (1999) if and only if

$$\Delta_\rho \subset L^q_+ = \{z \in L^q; \mu(z \geqslant 0) = 1\}. \tag{13}$$

Particular interesting examples are the Conditional Value at Risk (*CVaR*) of Rockafellar et al. (2006a), the Dual Power Transform (*DPT*) of Wang (2000) and the Wang Measure (Wang, 2000), among many others. Furthermore, following the original idea of Rockafellar et al. (2006a) to identify their Expectation Bounded Risk Measures and their Deviation Measures, it is easy to see that

$$\rho(y) = \sigma(y) - \mathbb{E}(y) \tag{14}$$

is continuous and satisfies (a)–(d) if $\sigma : L^p \rightarrow \mathbb{R}$ is a continuous (or lower semi-continuous) deviation, that is, if σ satisfies (b) and (c),

(e)

$$\sigma(y + k) = \sigma(y) \tag{15}$$

for every $y \in L^p$ and $k \in \mathbb{R}$, and

(f)

$$\sigma(y) \geqslant 0 \tag{16}$$

for every $y \in L^p$.

[1] Actually, the properties above are almost similar to those used by Rockafellar et al. (2006a) in order to introduce their Expectation Bounded Risk Measures. These authors also impose (a)–(d), work with $p = 2$, allow for $\rho(y) = \infty$, and impose $\rho(y) > -\mathbb{E}(y)$ if y is not constant.

[2] According to Theorem 2.2.20 in Zalinescu (2002), if ρ satisfies (a)–(d) then ρ is continuous if and only if ρ is lower semi-continuous.

Particular examples are the p-deviation given by $\sigma(y) = [\mathbb{E}(|\mathbb{E}(y) - y|^p)]^{1/p}$, or the downside p-semi-deviation given by $\sigma(y) = [\mathbb{E}(|\operatorname{Max}\{\mathbb{E}(y) - y, 0\}|^p)]^{1/p}$, among many others.

Denote by $g \in L^p$ a new pay-off that we are interested in pricing and hedging. If the trader sells g for $Pe^{-r_f T}$ dollars and buys $y \in Y$ in order to hedge the global position, then he will choose $x = (P, y)$ so as to solve

$$\begin{cases} \operatorname{Min} \rho(y - g) + P, \\ \pi(y) \leqslant Pe^{-r_f T}, \\ P \in \mathbb{R}, \quad y \in Y. \end{cases} \tag{17}$$

If (P_0, y_0) solves (17) then

$$(\rho(y_0 - g) + P_0)e^{-r_f T} \tag{18}$$

will be the (ask) price of g, composed of the cost of the hedging strategy $P_0 e^{-r_f T}$ plus the initial capital requirement $\rho(y_0 - g)e^{-r_f T}$ that the trader should provide.

The ask price (18) does not consider any utility function. On the contrary, it only focuses on the capital needed by the trader (the seller). Nevertheless, there are many relationships between utility functions and risk functions, as pointed out by Ogryczak and Ruszczynski (1999, 2002), and Biagini and Fritelli (2005), among others. In Section 7 we will also present some comments about it.

If we fix an arbitrary $P \in \mathbb{R}$ then there is only one decision variable $y \in Y$ in (17). Henceforth this simplified problem will be denoted by (17–P).

3. Optimal hedging: primal and dual problems and optimality conditions

In general ρ will be non-differentiable and therefore so will be Problems (17) and (17–P). To overcome this caveat we follow the method proposed in Balbás et al. (2009). So, bearing in mind (7), Problem (17–P) is equivalent to

$$\begin{cases} \operatorname{Min} \theta + P, \\ \theta + \mathbb{E}(yz) - \mathbb{E}(gz) \geqslant 0, \quad \forall z \in \Delta_\rho, \\ \pi(y) \leqslant Pe^{-r_f T}, \\ \theta \in \mathbb{R}, \quad y \in Y \end{cases} \tag{19}$$

in the sense that y solves (17–P) if and only if there exists $\theta \in \mathbb{R}$ such that (θ, y) solves (19), in which case

$\theta = \rho(y - g)$

holds. Notice that the objective of (19) is differentiable and even linear. The first constraint is valued on the Banach space $\mathcal{C}(\Delta_\rho)$ of real-valued and continuous functions on the ($weak^*$) compact space Δ_ρ. Since its dual space is $\mathcal{M}(\Delta_\rho)$, the space of inner regular real-valued σ-additive measures on the Borel σ-algebra of Δ_ρ (endowed with the $weak^*$ topology), the Lagrangian function

$\mathcal{L} : \mathbb{R} \times Y \times \mathbb{R} \times \mathcal{M}(\Delta_\rho) \rightarrow \mathbb{R}$

becomes

$$\mathcal{L}(\theta, y, \lambda, \nu) = \theta\left(1 - \int_{\Delta_\rho} d\nu(z)\right) - \int_{\Delta_\rho} \mathbb{E}(yz)\, d\nu(z) + \int_{\Delta_\rho} \mathbb{E}(zg) d\nu(z) + \lambda\pi(y) - \lambda P e^{-r_f T}.$$

Following Luenberger (1969) the element $(\lambda, \nu) \in \mathbb{R} \times \mathcal{M}(\Delta_\rho)$ is dual feasible if and only if it belongs to the non-negative cone $\mathbb{R}_+ \times \mathcal{M}_+(\Delta_\rho)$ and

$\operatorname{Inf}\{\mathcal{L}(\theta, y, \lambda, \nu) : \theta \in \mathbb{R}, \ y \in Y\} > -\infty$

in which case the infimum above equals the dual objective on (λ, ν). Hence, bearing in mind (2) and (3), the dual problem of (19) becomes

$$\begin{cases} \text{Max } \int_{\Delta_\rho} \mathbb{E}(gz)\, d\nu(z) + P(1 \quad \lambda e^{\ r_f T}), \\ \lambda\pi(y) \quad \int_{\Delta_\rho} \mathbb{E}(yz)\, d\nu(z) \geqslant 0, \quad \forall y \in Y, \\ \lambda \in \mathbb{R}_+, \quad \nu \in \mathscr{P}(\Delta_\rho), \end{cases} \tag{20}$$

$\mathscr{P}(\Delta_\rho)$ denoting the set composed of those elements in $\mathscr{M}(\Delta_\rho)$ that are probabilities.

$\mathscr{P}(\Delta_\rho)$ is convex, and the theorem of Alaoglu easily leads to the compactness of $\mathscr{P}(\Delta_\rho)$ when endowed with the $\sigma(\mathscr{M}(\Delta_\rho), \mathscr{C}(\Delta_\rho))$ topology (Luenberger, 1969). Besides, given $z \in \Delta_\rho$ we will denote by $\delta_z \in \mathscr{P}(\Delta_\rho)$ the usual Dirac delta that concentrates the mass on $\{z\}$, *i.e.*, $\delta_z(\{z\}) \quad 1$ and $\delta_z(\Delta_\rho \setminus \{z\}) \quad 0$. It is known that the set of extreme points of $\mathscr{P}(\Delta_\rho)$ is given by

$$\text{ext}(\mathscr{P}(\Delta_\rho)) \quad \{\delta_z; z \in \Delta_\rho\}, \tag{21}$$

though we will not have to draw on this result. The optimal value of dual problems in the finite dimensional case is attained in a ex treme feasible solution, which, along with (21), suggest that the solution of (20) could be achieved in $\{\delta_z; z \in \Delta_\rho\}$. Let us show that this conjecture is correct. First we provide an instrumental lemma whose statement and complete proof may be found in Balbás et al. (forthcoming).

Lemma 1 (*Mean Value Theorem*). *Let $\nu \in \mathscr{P}(\Delta_\rho)$. Then there exists $z_\nu \in \Delta_\rho$ such that*

$$\int_{\Delta_\rho} \mathbb{E}(yz)\, d\nu(z) \quad \mathbb{E}(yz_\nu) \tag{22}$$

holds for every $y \in L^p$.

Theorem 2. *If $(\lambda, \nu) \in \mathbb{R}_+ \times \mathscr{P}(\Delta_\rho)$ solves (20) then there exists $z \in \Delta_\rho$ such that (λ, δ_z) solves (20).*

Proof. Consider (λ, ν) solving (20) and take $z_\nu \in \Delta_\rho$ satisfying (22). Then, for every $y \in Y$ we have that

$$0 \leqslant \lambda\pi(y) \quad \int_{\Delta_\rho} \mathbb{E}(yz)\, d\nu(z) \quad \lambda\pi(y) \quad \mathbb{E}(yz_\nu)$$
$$\lambda\pi(y) \quad \int_{\Delta_\rho} \mathbb{E}(yz)\, d\delta_{z_\nu}(z),$$

and

$$P(1 \quad \lambda e^{\ r_f T}) + \int_{\Delta_\rho} \mathbb{E}(gz)\, d\nu(z) \quad \lambda P \quad \mathbb{E}(gz_\nu)$$
$$P(1 \quad \lambda e^{\ r_f T}) + \int_{\Delta_\rho} \mathbb{E}(gz)\, d\delta_{z_\nu}(z),$$

which proves that $(\lambda, \delta_{z_\nu})$ is (20) feasible and the objective values of (20) in (λ, ν) and $(\lambda, \delta_{z_\nu})$ are identical. □

Remark 1. The latter theorem leads to significant consequences. In particular, we can consider the alternative and far simpler dual problem

$$\begin{cases} \text{Max}\, \mathbb{E}(gz) + P(1 \quad \lambda e^{\ r_f T}), \\ \lambda\pi(y) \quad \mathbb{E}(yz) \geqslant 0, \quad \forall y \in Y, \\ z \in \Delta_\rho, \quad \lambda \in \mathbb{R}_+, \end{cases} \tag{23}$$

where $z \in \Delta_\rho$ is playing the role of $\nu \in \mathscr{P}(\Delta_\rho)$. □

Proposition 3. *Let be $z \in \Delta_\rho$. The inequality $\lambda\pi(y) \quad \mathbb{E}(yz) \geqslant 0$ for every $y \in Y$ can only hold for $\lambda \quad e^{r_f T}$.*

Proof. Indeed, the inequality leads to $\lambda e^{\ r_f T} \quad \mathbb{E}(z) \geqslant 0$ if $y \quad 1$, and $\lambda e^{\ r_f T} \quad \mathbb{E}(z) \leqslant 0$ if $y \quad 1$, so the conclusion is obvious because $\mathbb{E}(z) \quad 1$ for every $z \in \Delta_\rho$ (see (8)). □

Remark 2. The previous proposition enables us to simplify (23) once again. The equivalent problem will be

$$\begin{cases} \text{Max}\, \mathbb{E}(gz), \\ \pi(y)e^{r_f T} \quad \mathbb{E}(yz) \geqslant 0, \quad \forall y \in Y, \\ z \in \Delta_\rho, \end{cases} \tag{24}$$

where the λ variable has been removed.

Notice that (4) implies that (17 P) is feasible, and therefore so is (19). Since we are dealing with infinite dimensional Banach spaces the so called "duality gap" between (19) and (24) might arise.[3] To prevent this pathological situation we will give the next theorem and impose a very weak assumption with clear economic interpreta tion. We will also connect the statement (b) of the theorem below with classical key notions in Asset Pricing Theory.

Theorem 4. *The three following conditions are equivalent:*

(a) *There exist $P_0 \in \mathbb{R}$ and $g_0 \in L^p$ such that (19) is not unbounded, i.e., there are no sequences $(y_n) \subset Y$ of feasible solutions such that $\rho(y_n \quad g_0) \to \quad \infty$.*

(b) *The (24) feasible set*

$$D_f \quad \{z \in \Delta_\rho; \pi(y)e^{r_f T} \quad \mathbb{E}(yz) \geqslant 0, \forall y \in Y\} \tag{25}$$

is non void.

(c) *Problem (19) is not unbounded for every $P \in \mathbb{R}$ and $g \in L^p$. Fur thermore, in the affirmative case (19) and (24) are feasible and bounded, (24) attains its optimal value, the dual maximum equals the primal infimum, and the following Karush Kuhn Tucker conditions*

$$\begin{cases} \theta^* + \mathbb{E}(y^* z^*) \quad \mathbb{E}(gz^*) \quad 0, \\ \theta^* + \mathbb{E}(y^* z) \quad \mathbb{E}(gz) \geqslant 0, \quad \forall z \in \Delta_\rho, \\ \pi(y^*) \quad Pe^{\ r_f T} \quad 0, \\ \pi(y^*)e^{r_f T} \quad \mathbb{E}(y^* z^*) \quad 0, \\ \pi(y)e^{r_f T} \quad \mathbb{E}(yz^*) \geqslant 0, \quad \forall y \in Y \\ \theta \in \mathbb{R}, y^* \in Y, \ z^* \in \Delta_\rho, \end{cases} \tag{26}$$

are necessary and sufficient so as to guarantee that (θ^*, y^*) and z^* solve (19) and (24) respectively.

Proof

(a)⇒(b) Suppose that we prove the fulfillment of the Slater Qual ification for (19) (Luenberger, 1969), *i.e.*, the existence of $(\theta_0, y_0) \in \mathbb{R} \times Y$ such that

$$\begin{cases} \theta_0 + \mathbb{E}(y_0 z) \quad \mathbb{E}(g_0 z) > 0, \quad \forall z \in \Delta_\rho, \\ \pi(y_0) < P_0 \end{cases}$$

holds. Then Condition (a) implies that (20) (and therefore (24)) must be feasible (Luenberger, 1969).In order to show the fulfillment of the Slater Qualification notice that (4) implies that (17) is always feasible, and therefore so is (19). Moreover, given a (19) feasible solution (θ, y), and bearing in mind (8), the element $(\theta_0, y_0) \quad (\theta + 2, y \quad 1)$ satisfies the primal constraints as strict inequalities.

[3] If we only deal with a finite set of states Ω then, as already said, L^p has a finite dimension, but the duality gap may also exist unless the market is perfect and therefore the pricing rule π is linear (Luenberger, 1969).

(b)⇒(c) If D_f is not empty then (24) is feasible and therefore so is (20). Thus (19) cannot be unbounded because it is easy to verify that the primal objective is never lower than the dual one (see also Luenberger (1969)).

(c) ⇒ (a) Obvious.Moreover, in the affirmative case (26) provides sufficient optimality conditions because (19) is a convex problem, and these conditions are also necessary because, as shown in the implication (a) ⇒ (b), the Slater Qualification holds (Luenberger, 1969). Finally, this Qual ification also implies that the dual maximum is attained and equals the primal infimum. □

Assumption 1. Hereafter we will assume the existence of $P_0 \in \mathbb{R}$ and $g_0 \in L^p$ such that (19) is not unbounded. Thus, Conditions (b) and (c) in the theorem above also hold.

Remark 3 (*Example illustrating that the fulfillment of Assumption 1 is not guaranteed.*[4]). Consider $\Omega \quad \{\omega_1\omega_2\}$, $\mu(\omega_1) \quad 0.1$, $\mu(\omega_2) \quad 0.9$, and

$$\pi(\alpha(1,1)+\beta(1,0)) \quad \begin{cases} \alpha+0.7\beta, & \text{if } \beta \geqslant 0, \\ \alpha+0.4\beta, & \text{if } \beta < 0. \end{cases}.$$

The example indicates that the risk free rate vanishes and the risky asset with pay off (1,0) has a bid price equal to 0.4 and an ask price equal to 0.7. Suppose that

$$\Delta_\rho \quad \{(z_1,z_2);\ 0.1z_1+0.9z_2 \quad 1 \text{ and } 0 \leqslant z_i \leqslant 2.5,\ i \quad 1,2\}.$$

It will be seen in Section 6 that Δ_ρ corresponds to the Conditional Value at Risk with 0.6 60% as the level of confidence. The condi tions defining the set D_f are

$$\begin{cases} 0.4 \leqslant 0.1z_1 \leqslant 0.7 \\ 0.1z_1+0.9z_2 \quad 1 \\ 0 \leqslant z_i \leqslant 2.5, \quad i \quad 1,2, \end{cases}$$

and therefore D_f is void.

Remark 4. Since Condition (b) holds D_f (see (25)) is not empty, and its elements will be called "Stochastic Discount Factors (*SDF*) of (π,ρ)". Notice that

$$\mathbb{E}(yz) \quad \pi(y)e^{r_fT} \tag{27}$$

holds for every $y \in Y$ and every $z \in D_f$ if the market is perfect, since $y \in Y$ for every $y \in Y$ and consequently

$$\pi(y)e^{r_fT}+\mathbb{E}(yz) \quad \pi(\ y)e^{r_fT} \quad \mathbb{E}(\ yz) \geqslant 0$$

must also hold.[5]

Expression (27) leads to

$$\pi(y) \quad e^{\ r_fT}\mathbb{E}(yz) \quad e^{\ r_fT}\mathbb{E}_{\mu_z}(y), \tag{28}$$

i.e., the current price of any asset equals the present value of its ex pected pay off once modified with the "distortion variable" z, or the present value of its expected pay off if the expectation is computed with the "risk neutral probability measure" μ_z such that

$$z \quad \frac{d\mu_z}{d\mu}.$$

Eq. (28) is closely related to the First Fundamental Theorem of Asset Pricing. Notice that μ_z is actually a probability owing to (8), and will be equivalent to μ as long as

$$\mu(z>0) \quad 1.$$

See Duffie (1996) or De Wagenaere and Wakker (2001), among many others, for further details about the Fundamental Theorem of Asset Pricing and risk neutral or martingale measures in both perfect and imperfect markets.

4. Pricing rules

This section will be devoted to extend the pricing rule π to the whole space L^p. First we present a proposition without proof, since it is trivial. One only must bear in mind that (24) does not depend on P.

Proposition 5. *The optimal value of (17) equals the optimal value of (19) and (17 P) for every $P \in \mathbb{R}$. It also equals the optimal value of (24).*

As a consequence of the previous proposition we can introduce the first pricing rule we are going to deal with. Indeed, we will define

$$\pi_\rho(g) \quad e^{\ r_fT}\text{Max}\{\mathbb{E}(gz);\ z \in \Delta_\rho \text{ and } \pi(y)e^{r_fT} \quad \mathbb{E}(yz) \geqslant 0,\ \forall y \in Y\} \tag{29}$$

for every $g \in L^p$. Obviously, given $g \in L^p$, the latter proposition also shows that

$$\pi_\rho(g) \quad e^{\ r_fT}\text{Inf}\ \{\rho(y \quad g)+P;\ \pi(y) \leqslant Pe^{\ r_fT},\ P \in \mathbb{R},\ y \in Y\}, \tag{30}$$

$$\pi_\rho(g) \quad e^{\ r_fT}\text{Inf}\ \{\rho(y \quad g)+P;\ \pi(y) \leqslant Pe^{\ r_fT},\ y \in Y\} \tag{31}$$

for every fixed $P \in \mathbb{R}$, and

$$\pi_\rho(g) \quad e^{\ r_fT}\mathit{Inf}\ \{\rho(y \quad g);\ \pi(y) \leqslant 0,\ y \in Y\}. \tag{32}$$

Next let us see that the independence of π_ρ and the solution of (24) with respect to P, obvious consequence of the form of (24), is also fulfilled by the optimal hedging portfolios, *i.e.*, by the solution of (19).

Proposition 6. *Suppose that (θ^*, y^*) solves (19) for $P \in \mathbb{R}$ and $g \in L^p$. Then $(\theta^* \quad \alpha, y^*+\alpha)$ solves (19) for $P+\alpha \in \mathbb{R}$ and $g \in L^p$.*[6]

Proof. The proof is quite easy and consequently we will simplify the exposition. Just consider a dual solution z^*, that does not depend on P as pointed out by (24), and bear in mind that (θ^*, y^*) and z^* satisfy (26) for (P,g). Then use (5) and (8) so as to verify that $(\theta^* \quad \alpha, y^*+\alpha)$ and z^* satisfy (26) for $(P+\alpha, g)$. □

Next let us present the interesting properties of the extension π_ρ above. It conserves the properties of π, reduces the bid ask spread and is a genuine extension of π if we deal with a perfect market.

[4] The existence of duality gaps and the lack of primal solutions or Lagrange or Karush–Kuhn–Tucker multipliers is not so rare in financial problems. See for instance Jim et al. (2008) for noteworthy counter-examples in portfolio selection.

[5] Actually, many authors only use the term "Stochastic Discount Factor" if $p \quad 2$, (27) holds and z may be replicated. In such a case, the existence and uniqueness of z in an arbitrage free market may be proved. Furthermore, z is closely related to the "Market Portfolio" that allows us to measure the systematic risk of every asset, and to establish the classical relationship between the expected asset return and its systematic risk, *i.e.*, the classical expressions of the Capital Asset Pricing Model (Duffie, 1996 or Cochrane, 2001). In this paper we use the term "Stochastic Discount Factor" in less restrictive sense. It is sufficient the fulfillment of $z \in \Delta_\rho$ and

$$\pi(y)e^{r_fT} \quad \mathbb{E}(yz) \geqslant 0, \quad \forall y \in Y$$

[6] Notice that this fact simplifies (26), in the sense that the equation $\pi(y^*) \quad Pe^{\ r_fT}$ may be removed.

Theorem 7

(A) $\pi_\rho(g) \leqslant \rho(-g)e^{-r_fT}$ *for every* $g \in L^p$.
(B) π_ρ *is sub additive and positively homogeneous (and therefore convex).*
(C) π_ρ *is continuous.*
(D) $\pi_\rho(y) \leqslant \pi(y)$ for every $y \in Y$.[7]
(E) *If* g *and* $-g$ *belong to* Y *and* $\pi(-g) = -\pi(g)$ *then* $\pi_\rho(g) = \pi(g)$. *In particular,* $\pi_\rho(k) = ke^{-r_fT}$ *for every* $k \in \mathbb{R}$. *If the market is perfect then* π_ρ *extends* π *to the whole space* L^p.
(F) *If* ρ *is a coherent risk measure then* π_ρ *is increasing.*[8]

Proof

(A) (32) implies that

$$\pi_\rho(g) = e^{-r_fT}\text{Inf}\,\{\rho(y-g)e^{-r_fT}; \pi(y) \leqslant 0,\ y \in Y\}.$$

Hence, for $y = 0$, $\pi_\rho(g) \leqslant \rho(-g)e^{-r_fT}$.

(B) (29) shows that

$$\pi_\rho(g_1+g_2) = \text{Max}\,\{\mathbb{E}((g_1+g_2)z)e^{-r_fT}; z \in D_f\}.$$

If $z_{g_1+g_2} \in D_f$ denotes the dual feasible solution where the maximum is attained, then

$$\pi_\rho(g_1+g_2) = \mathbb{E}((g_1+g_2)z_{g_1+g_2})e^{-r_fT} = \mathbb{E}((g_1)z_{g_1+g_2})e^{-r_fT} + \mathbb{E}((g_2)z_{g_1+g_2})e^{-r_fT}.$$

If $z_{g_1} \in D_f$ and $z_{g_2} \in D_f$ are the obvious, bearing in mind that D_f does not depend on g (see (25)) we have

$$\mathbb{E}((g_1)z_{g_1+g_2})e^{-r_fT} + \mathbb{E}((g_2)z_{g_1+g_2})e^{-r_fT} \leqslant \mathbb{E}(g_1z_{g_1})e^{-r_fT} + \mathbb{E}(g_2z_{g_2})e^{-r_fT} = \pi_\rho(g_1) + \pi_\rho(g_2).$$

On the other hand, if $\alpha > 0$ we have

$$\pi_\rho(\alpha g) = \mathbb{E}(\alpha g z_{\alpha g})e^{-r_fT} = \alpha\mathbb{E}(gz_{\alpha g})e^{-r_fT} \leqslant \alpha\mathbb{E}(gz_g)e^{-r_fT} = \alpha\pi_\rho(g).$$

Analogously,

$$\pi_\rho(g) = \pi_\rho\left(\frac{1}{\alpha}\alpha g\right) \leqslant \frac{1}{\alpha}\pi_\rho(\alpha g)$$

leads to $\alpha\pi_\rho(g) \leqslant \pi_\rho(\alpha g)$. For $\alpha = 0$ we only have to prove that $\pi_\rho(0) = 0$, but this equality is clear because otherwise

$$\pi_\rho(0) = \pi_\rho(2 \times 0) = 2\pi_\rho(0)$$

would lead to the contradiction $1 = 2$.

(C) Being π_ρ a convex function on L^p it is sufficient to see that π_ρ is continuous at $g = 0$ (Luenberger, 1969). Since ρ is continuous,[9] given $\varepsilon > 0$ there exists $\delta > 0$ such that $\|g\| \leqslant \delta \Rightarrow \rho(-g) \leqslant \varepsilon e^{r_fT}$ and therefore $\pi_\rho(g) \leqslant \varepsilon$ follows from Statement *A*). Besides, bearing in mind *B*) we have that

$$-\pi_\rho(g) \leqslant \pi_\rho(-g) \leqslant \varepsilon$$

because $\|-g\| \leqslant \delta$. Hence, $|\pi_\rho(g)| \leqslant \varepsilon$.

(D) If $y \in Y$ with the notations above we have $\pi_\rho(y) = \mathbb{E}(yz_y)e^{-r_fT}$, and $\mathbb{E}(yz_y) \leqslant \pi(y)e^{r_fT}$ because $z_y \in D_f$.

(E) The assumptions lead to

$$\pi(g) + \pi(-g) = 0,$$

so (33) implies that

$$\pi_\rho(g) + \pi_\rho(-g) = \pi(g) + \pi(-g) = 0.$$

Since Theorem 7*D* shows that $\pi_\rho(g) \leqslant \pi(g)$ and $\pi_\rho(-g) \leqslant \pi(-g)$ the equality above can only hold if both inequalities become equalities.

(F) If $g_1 \leqslant g_2 \in L^p$ then $y - g_1 \geqslant y - g_2$ and therefore $\rho(y-g_1) \leqslant \rho(y-g_2)$ for every $y \in Y$ because ρ is coherent and therefore decreasing. Consequently,

$$\text{Inf}\{\rho(y-g_1); y \in Y, \pi(y) \leqslant 0\} \leqslant \text{Inf}\{\rho(y-g_2); y \in Y, \pi(y) \leqslant 0\},$$

so the conclusion trivially holds. □

For imperfect markets π_ρ may strictly reduce the spread, and consequently it does not necessarily equal π on Y. Next let us characterize the equality $\pi(g) = \pi_\rho(g)$ and provide a very simple numerical example.

Proposition 8. *Consider* $g \in Y$ *and a dual solution* z^*. $\pi(g) = \pi_\rho(g)$ *holds if and only if* $\mathbb{E}(z^*g) = \pi(g)e^{r_fT}$.

Proof. The result trivially follows from

$$\pi_\rho(g) = \mathbb{E}(z^*g)e^{-r_fT}. \qquad \square$$

Remark 5. Example illustrating that $\pi_\rho(g) < \pi(g)$ may hold. Consider the same example (market) as in Remark 3 but suppose that

$$\Delta_\rho = \{(z_1, z_2);\ 0.1z_1 + 0.9z_2 = 1 \text{ and } 0 \leqslant z_i \leqslant 5,\ i = 1, 2\}.$$

It will be seen in Section 6 that Δ_ρ corresponds to the Conditional Value at Risk with $0.8 = 80\%$ as the level of confidence. $\pi_\rho(1,0)$ is the optimal value of

$$\begin{cases} \text{Max} \quad 0.1z_1, \\ 0.4 \leqslant 0.1z_1 \leqslant 0.7, \\ 0.1z_1 + 0.9z_2 = 1, \\ 0 \leqslant z_i \leqslant 5, \quad i = 1, 2. \end{cases}$$

Obviously, $\pi_\rho(1,0) = 0.5 < 0.7 = \pi(1,0)$.

Since π_ρ reduces the spread and satisfies the same properties as π (Theorem 7), one could use π_ρ to generate a new pricing rule π_ρ^* by applying the same method used to construct π_ρ from π. Next we will prove that $\pi_\rho^* = \pi_\rho$, so it is useless to extend the pricing rule two times. However, the equality $\pi_\rho^* = \pi_\rho$ shows that π_ρ may be an exact extension of π in particular situations, even if the market is imperfect.

Theorem 9. *The Stochastic Discount Factors of* (π, ρ) *and* (π_ρ, ρ) *coincide.*[10] *Consequently,*

[7] Consequently π_ρ "improves" the bid-ask spread (or the transaction costs) of π. Indeed, if we consider that $-\pi_\rho(-g)$ is the bid price of $g \in L^p$ and $\pi_\rho(g)$ is its ask price, then Theorem 7*B* shows that

$$\pi_\rho(g) + \pi_\rho(-g) \geqslant 0$$

i.e., the bid-ask spread cannot be negative. Analogously, (2) and (3) lead to $\pi(y) + \pi(-y) \geqslant 0$ for every $y \in Y$ such that $-y \in Y$. Furthermore

$$\pi(y) + \pi(-y) \geqslant \pi_\rho(y) + \pi_\rho(-y) \geqslant 0, \tag{33}$$

i.e., the bid-ask spread is improved by π_ρ.

[8] Almost all the statements above are going to be proved by using (29) as the expression generating π_ρ. Nevertheless, all of them may be also proved by using (32), *i.e.*, the Duality Theory of Section 3 is not needed to prove Theorem 7.

[9] Notice that the $\sigma(L^q, L^p)$-compactness of Δ_ρ and the fulfillment of (7) and (8) imply that ρ is continuous.

[10] In other words: If $z \in \Delta_\rho$ then

$$\mathbb{E}(yz) \leqslant \pi(y)e^{r_fT}$$

for every $y \in Y$ if and only if

$$\mathbb{E}(gz) \leqslant \pi_\rho(g)e^{r_fT}$$

for every $g \in L^p$.

$$\text{Max}\,\{\mathbb{E}(gz);\ z \in \Delta_\rho,\ \mathbb{E}(yz) \leqslant \pi(y)e^{r_f T}\ for\ every\ y \in Y\}$$
$$\text{Max}\,\{\mathbb{E}(gz);\ z \in \Delta_\rho,\ \mathbb{E}(yz) \leqslant \pi_\rho(y)e^{r_f T}\ for\ every\ y \in L^p\}$$
$$\text{Max}\,\{\mathbb{E}(gz);\ z \in \Delta_\rho,\ \mathbb{E}(yz) \leqslant \pi_\rho(y)e^{r_f T}\ for\ every\ y \in Y\},$$

i.e., if we construct a new pricing rule π_ρ^* from π_ρ and ρ then $\pi_\rho^* \quad \pi_\rho$.

Proof. If $z \in \Delta_\rho$ and $\mathbb{E}(yz) \leqslant \pi_\rho(y)e^{r_f T}$ for every $y \in L^p$ (or just for every $y \in Y$) then $z \in D_f$ owing to Theorem 7*D*. Conversely, suppose that $z \in D_f$ and take $y \in L^p$. Then $\pi_\rho(y)e^{r_f T}$ is the maximum value of $\mathbb{E}(yz')$ with $z' \in D_f$, so $\mathbb{E}(yz) \leqslant \pi_\rho(y)e^{r_f T}$. □

A very important consequence of the latter theorem is that natural assumptions on π prevent the existence of arbitrage for π_ρ.

Corollary 10. *Suppose that there exists $z^* \in D_f$ which is strictly positive, i.e.,*

$$\mathbb{E}(yz^*) > 0$$

for every $y \in L^p$ such that $y \geqslant 0$ and $y \neq 0$.[11] *Then π_ρ does not generate arbitrage opportunities, i.e., $g \geqslant 0$ and $\pi_\rho(g) \leqslant 0$ imply that $g \quad 0$ and $\pi_\rho(g) \quad 0$.*[12]

Proof. Suppose that $g \geqslant 0$ and $\pi_\rho(g) \leqslant 0$. Then $\mathbb{E}(gz^*) \geqslant 0$, with equality if and only if $g \quad 0$. Besides, the latter theorem implies that

$$\mathbb{E}(gz^*) \leqslant \pi_\rho(g)e^{r_f T} \leqslant 0,$$

so the equality holds. □

Finally, let us show that the proposed extension π_ρ also "improves" the "classical extension", usual in incomplete markets. So, consider $g \in L^p$ and the optimization problem

$$\begin{cases} \text{Min} \quad \pi(y), \\ y \geqslant g, \\ y \in Y, \end{cases}$$

and denotes by $\pi^*(g)$ the infimum of the problem above ($\pi^*(g) \quad \infty$ if the problem is not feasible). Then we have:

Proposition 11. *If ρ is coherent then $\pi^*(g) \geqslant \pi_\rho(g)$ holds for every $g \in L^p$.*

Proof. The conclusion is obvious if $\pi^*(g) \quad \infty$, so assume that $\pi^*(g) < \infty$. Take $n \in \mathbb{N}$ and $y_n \in Y$, $y_n \geqslant g$ such that

$$\pi^*(g) \geqslant \pi(y_n) \quad \frac{1}{n}.$$

Then, $y_n \geqslant g$ and Theorems 7*D* and 7*F* lead to

$$\pi^*(g) \geqslant \pi(y_n) \quad \frac{1}{n} \geqslant \pi_\rho(y_n) \quad \frac{1}{n} \geqslant \pi_\rho(g) \quad \frac{1}{n},$$

and the result trivially follows because $n \in \mathbb{N}$ is arbitrary. □

5. Dealing with deviations

If we consider a general lower semi continuous deviation mea sure σ, *i.e.*, a sub additive and homogeneous function satisfying (15) and (16), then, as indicated in the second section, (14) estab lishes a relationship between σ and a risk measure ρ for which we can construct the pricing rule π_ρ, denoted by $\pi_{\sigma \ \mathbb{E}}$ in this section owing to (14).

A particular interesting case, very used in finance, arises if $p \quad 2$ and $\sigma \quad \sigma_2$ is the Standard Deviation given by

$$\sigma_2(y) \quad \left(\int_\Omega (y \quad \mathbb{E}(y))^2\, d\mu\right)^{1/2}$$

for every $y \in L^2$. In such a case L^2 is a Hilbert space so, if one as sumes that the market is perfect, Y is closed and π is continuous, the Riesz Representation Theorem guarantees the existence of a un ique $y_0 \in Y$ such that

$$\pi(y) \quad \mathbb{E}(y_0 y) \tag{34}$$

holds for every $y \in Y$. The literature has often proposed extensions of π to the whole space L^2 by considering an element y_1 orthogonal to Y and defining

$$\pi_{y_0+y_1}(y) \quad \mathbb{E}[(y_0 + y_1)y]$$

for every $y \in L^2$.[13] A particular interesting example arises if $y_1 \quad 0$ since π_{y_0} becomes the composition of the orthogonal projection on Y and π, or, in other words, $\pi_{y_0}(y)$ coincides with $\pi(\Pi(y))$ for every $y \in L^2$, $\Pi(y)$ denoting the element in Y closest to y.

Obviously, the extensions above are specially useful when there exists y_1 orthogonal to Y and such that $y_0 + y_1 > 0$ almost surely (respectively, $y_0 > 0$ almost surely) because this inequality guaran tees the absence of arbitrage for the pricing rule $\pi_{y_0+y_1}$ (respec tively, π_{y_0}).

Actually, under the general assumptions above, as far as we were able to analyze the problem there were no clear relationships between the (non necessarily linear) extension $\pi_{\sigma_2 \ \mathbb{E}}$ and the extension $\pi_{y_0+y_1}$. However, for those cases such that both exten sions generate arbitrage free pricing rules (see Corollary 10) $\pi_{\sigma_2 \ \mathbb{E}}$ will be larger than $\pi_{y_0+y_1}$.

Proposition 12. *Suppose that the market is perfect, Y is closed and π is continuous. Consider the unique $y_0 \in Y$ such that (34) holds for every $y \in Y$. Suppose finally that there exists $z^* \in L^2$ such that*

(a) *$\mathbb{E}(z^*) \quad 0$, $\sigma_2(z^*) \leqslant 1$ and $1 + z^* > 0$ almost surely.*
(b) *$(1 + z^*)e^{\ r_f T} \quad y_0$ is orthogonal to Y. Then $\pi_{\sigma_2 \ \mathbb{E}}$ and $\pi_{(1+z^*)e^{\ r_f T}}$ do not generate arbitrage opportunities and*

$$\pi_{\sigma_2 \ \mathbb{E}}(y) \geqslant \pi_{(1+z^*)e^{\ r_f T}}(y)$$

holds for every $y \in L^2$.

Proof. It is shown in Rockafellar et al. (2006a) that

$$\Delta_{\sigma_2 \ \mathbb{E}} \quad \{1 + z;\ z \in L^2,\ \mathbb{E}(z) \quad 0 \text{ and } \sigma_2(z) \leqslant 1\}. \tag{35}$$

Hence Condition *a*) imposes that $1 + z^*$ is strictly positive and be longs to $\Delta_{\sigma_2 \ \mathbb{E}}$. Moreover, Condition *b*) leads to

$$\mathbb{E}((1 + z^*)y) \quad \mathbb{E}(y_0 e^{r_f T} y) \quad e^{r_f T}\pi(y)$$

for every $y \in Y$, which implies that $1 + z^*$ is in D_f (see (25)). Conse quently Corollary 10 implies that $\pi_{\sigma_2 \ \mathbb{E}}$ does not generate arbitrage opportunities. Similarly, $1 + z^* > 0$ almost surely leads to the ab sence of arbitrage opportunities for $\pi_{(1+z^*)e^{\ r_f T}}$. Finally, (29) and (24) lead to

$$\pi_{\sigma_2 \ \mathbb{E}}(y) \quad e^{\ r_f T} Max\,\{\mathbb{E}(yz);\ z \in D_f\} \geqslant e^{\ r_f T}\mathbb{E}((1 + z^*)y)$$
$$\pi_{(1+z^*)e^{\ r_f T}}(y)$$

for every $y \in L^2$. □

[11] Or equivalently, $z^* > 0$ almost surely.

[12] Bearing in mind (13) with a similar proof one can see that if ρ is coherent then Assumption 1 prevents the existence of the so called "strong" or "second type" arbitrage (Jaschke and Küchler, 2001), *i.e.*, the existence of $g \in L^p$ such that $g \geqslant 0$ and $\pi_\rho(g) < 0$.

[13] See, among others, Schweizer (1995) and Luenberger (2001).

Remark 6. A very particular case arises if $(1+z^*)e^{\ \ r_fT} \quad y_0$, *i.e.*, if $\pi_{(1+z^*)e^{\ \ r_fT}}$ is the composition of π and the orthogonal projection Π. This situation appears if $y_0 > 0$ almost surely, $\mathbb{E}(y_0) \quad e^{\ \ r_fT}$ and $\sigma_2(y_0) \leqslant e^{\ \ r_fT}$, in which case $\pi_{\sigma_2 \ \ \mathbb{E}}$ and π_{y_0} do not generate arbitrage opportunities and $\pi_{\sigma_2 \ \ \mathbb{E}} \geqslant \pi_{y_0}$ holds.

6. Using the Conditional Value at Risk

In this section, we will focus on the Conditional Value at Risk, since it is becoming a very well known Coherent and Expectation Bounded Risk Measure that respects the second order Stochastic Dominance (Ogryczak and Ruszczynski, 2002). In particular, this risk function has been used, amongst many others, by Wang (2000) in some insurance linked problems, Alexander et al. (2006) in portfolio choice problems involving derivatives, Mansini et al. (2007) in portfolio choice problems involving bonds and shares, or Balbás et al. (forthcoming) in optimal reinsurance problems.

If $0 < 1 \quad \mu_0 < 1$ represents the level of confidence then the $CVaR_{\mu_0}$ may be defined in L^1 and Rockafellar et al. (2006a) showed that

$$\Delta_{CVaR_{\mu_0}} \quad \left\{ z \in L^\infty;\ 0 \leqslant z \leqslant \frac{1}{\mu_0} \ \text{and} \ \mathbb{E}(z) \quad 1 \right\}.$$

Suppose the same hypotheses as in the second section as well as Assumption 1, *i.e.*, the existence of $P_0 \in \mathbb{R}$ and $g_0 \in L^1$ such that (17 P) is bounded, *i.e.*, the value of $CVaR_{\mu_0}(y)$ cannot tend to $\quad \infty$. According to Theorem 4 there are *SDF* of $(\pi, CVaR_{\mu_0})$, *i.e.*, D_f is non void.

The following result characterizes primal and dual solutions for $\rho \quad CVaR_{\mu_0}$, as well as it allows us to compute the value $\pi_{CVaR_{\mu_0}}(g)$ for $g \in L^1$ in practical applications.

Theorem 13. *Consider $g \in L^1$ and suppose that (19) attains it optimal value for g.*[14] *Consider also $z^* \in D_f$. Then, z^* solves (24) if and only if there exist a partition*

$$\Omega \quad \Omega_0 \cup \Omega^* \cup \Omega_{\mu_0}$$

of Ω composed of measurable sets and $y^* \in Y$ such that:

(A) *$z^* \quad 0$ on Ω_0 and $z^* \quad \frac{1}{\mu_0}$ on Ω_{μ_0}.*
(B) *$y^* \leqslant g$ on Ω_0, $y^* \quad g$ on Ω^* and $y^* \geqslant g$ on Ω_{μ_0}.*
(C) *$\mathbb{E}(z^* y^*) \quad \pi(y^*)e^{r_fT}$.*

Furthermore, in the affirmative case we have that $(\pi(y^*)e^{r_fT}, y^*)$ solves (17) and

$$\pi_{CVaR_{\mu_0}}(g) \quad \mathbb{E}(z^* g)e^{\ \ r_fT}. \tag{36}$$

Proof. Fix $P_1 \in \mathbb{R}$ and take (θ, y^*) solving (19) for P_1. If z^* solves (24) then (26) shows that C) must hold and z^* must solve

$$\begin{cases} \text{Min}\, \mathbb{E}(y^* z) \quad \mathbb{E}(gz) \\ \mathbb{E}(z) \quad 1 \\ z \leqslant \frac{1}{\mu_0}, \\ \quad z \leqslant 0, \\ z \in L^\infty. \end{cases} \tag{37}$$

The Slater Qualification holds since $z \quad 1$ belongs to $\Delta_{CVaR_{\mu_0}}$ and satisfies the two inequalities in strict terms. Then z^* must satisfy the optimality conditions. Bearing in mind that the dual space of L^∞ is composed of μ continuous finitely additive measures on $\mathscr{F}$ with bounded variation, there exists a couple of non negative measures (α_1, α_2) and a real number α such that

$$\begin{cases} y^* \quad g \quad \alpha + \alpha_1 \quad \alpha_2, \\ \int_\Omega \left(z^* \quad \frac{1}{\mu_0} \right) d\alpha_1 \quad 0, \\ \int_\Omega z^* \, d\alpha_2 \quad 0. \end{cases}$$

Denote by Ω_0 the set where z^* vanishes and by Ω_{μ_0} the set where $z^* \quad \frac{1}{\mu_0}$. The second and the third condition, along with $0 \leqslant z^* \leqslant \frac{1}{\mu_0}$, lead to $\alpha_1 \quad 0$ out of Ω_{μ_0} and $\alpha_2 \quad 0$ out of Ω_0. Thus, $\alpha_1 \quad y^* \quad g \quad \alpha$ on Ω_{μ_0} and $\alpha_2 \quad \quad y^* + g + \alpha$ on Ω_0, which shows that $\alpha_i \in L^1$, $i \quad 1, 2$.

If $\Omega^* \quad \Omega \setminus (\Omega_0 \cup \Omega_{\mu_0})$ then A) is obvious and B) holds as long as $\alpha \quad 0$. If $\alpha \neq 0$ then Proposition 6 guarantees that $y^* \quad \alpha$ solves (17 P)for $P_2 \quad P_1 \quad \alpha$, so take this new value for the P variable and rename $y^* \quad \alpha$ as y^*.

It only remains to prove (36). According to (29), and bearing in mind the objective function of (24),

$$\pi_{CVaR_{\mu_0}}(g) \quad \mathbb{E}(z^* g)e^{\ \ r_fT},$$

and (36) holds.

Conversely, suppose that the existence of the partition and $y^* \in Y$ is fulfilled. Take

$$\begin{cases} \alpha_1 \quad y^* \quad g & \text{on } \Omega_{\mu_0} \\ \alpha_1 \quad 0 & \text{otherwise} \end{cases}$$

and

$$\begin{cases} \alpha_2 \quad \quad y^* + g & \text{on } \Omega_0 \\ \alpha_2 \quad 0 & \text{otherwise} \end{cases}$$

and it is clear that z^* satisfies the optimality conditions of (37). Since this problem is linear z^* is optimal. Hence

$$\mathbb{E}(y^* z) \quad \mathbb{E}(gz) \geqslant \mathbb{E}(y^* z^*) \quad \mathbb{E}(gz^*)$$

if $z \in \Delta_{CVaR_{\mu_0}}$ leads to the fulfillment of the first and the second expressions in (26) if $\theta \quad \mathbb{E}(gz^*) \quad \mathbb{E}(y^* z^*)$.

Take $P \quad \pi(y^*)e^{r_fT}$ so as to guarantee the fulfillment of the third expression in (26). Then C) and $z^* \in D_f$ show that all the expressions in (26) hold and thus z^* solves (24). □

Another particular interesting case arises if (37) attains "bang bang" solutions. More accurately we have:

Corollary 14. Consider $g \in L^1$ and suppose that *(19) attains it optimal value for g. Consider $z^* \in D_f$ and suppose the existence of a partition $\Omega \quad \Omega_0 \cup \Omega_{\mu_0}$ such that $z^* \quad 0$ on Ω_0 and $z^* \quad \frac{1}{\mu_0}$ on Ω_{μ_0}. Then, z^* solves (24) if and only there exists $y^* \in Y$ such that:*

(A) *$y^* \leqslant g$ on Ω_0 and $y^* \geqslant g$ on Ω_{μ_0}.*
(B) *$\mathbb{E}(z^* y^*) \quad \pi(y^*)e^{r_fT}$.*

Furthermore, in the affirmative case we have that $(\pi(y^*)e^{r_fT}, y^*)$ solves (17) and (36) holds. □

Notice that the corollary above may be easily applied in practice. Indeed, on the one hand $\mathbb{E}(z^*) \quad 1$ leads to

$$\mu(\Omega_{\mu_0}) \quad \mu_0, \tag{38}$$

and $\mathbb{E}(z^* g)$ must be maximized on the other hand. Thus one must look for those measurable subsets Ω_{μ_0} satisfying (38) and making g as large as possible, and then check the fulfillment of A) and B) for some $y^* \in Y$.

7. Numerical examples and applications

This section will be devoted to illustrate how the pricing rule π_ρ may perform in practice when pricing a new security g. The two most important aspects are the quality of the optimal hedging, *i.e.*, how good the hedging portfolio y^* is so as to reduce the risk level $\rho(y^* \quad g)$, and the properties of the risk measure ρ.

[14] As in Proposition 6, if this property holds for a given $P_1 \in \mathbb{R}$ then it also holds for every $P \in \mathbb{R}$.

With respect to the properties of the risk measure ρ there are many interesting previous studies reflecting its degree of compatibility with the Second Order Stochastic Dominance (*SOSD*) and the usual Utility Functions. For instance, Ogryczak and Ruszczynski (1999) show, among many other properties, that the Standard Deviation is not compatible with the *SOSD* if asymmetric returns are involved. Other interesting contributions are, among many others, Ogryczak and Ruszczynski (2002), where the good properties of the *CVaR* and other risk functions are proved, and Biagini and Fritelli (2005), where relationships between risk functions and utility functions are given.

Since the properties of the risk functions have been broadly discussed in previous literature, let us focus on the optimal risk level $\rho(y^* - g)$ that the trader must face if he sells g, y^* being the solution of (17 $-$ P) for $P = 0$ (see also (32)). Obviously, the value of $\rho(y^* - g)$ will be closely related to the correlation level of g and those attainable or super-replicable pay-offs $y \in Y$. If there are elements $y \in Y$ very correlated with g then the protection generated by y^* will be high, the price $\pi_\rho(g)$ will be "reasonable" and connected with the market (Y, π), and the bid/ask spread $\pi_\rho(g) + \pi_\rho(-g)$ will be low. On the contrary, it the level of correlation between g and the elements of Y remains close to zero, then $\rho(y^* - g)$ will be high, the protection provided by y^* will be scant, the (ask) price $\pi_\rho(g)$ will be "unrealistic" and high, and the bid/ask spread $\pi_\rho(g) + \pi_\rho(-g)$ will be large. The trader must sell g for a "expensive price" due to the lack of appropriate hedging portfolios. Only the bid/ask average value

$$\frac{1}{2}\left(\pi_\rho(g) - \pi_\rho(-g)\right)$$

will reflect suitable levels.

In order to clarify the ideas above let us provide a couple of examples. The first one is a simple numerical exercise pointing out that low correlations provoke large spreads. In the next subsection, we will show that closer relationships among the involved securities lead to very interesting prices and spreads.

Consider two available securities, the risky asset and the riskfree one. The interest rate vanishes, whereas the risky asset current price equals zero and its final pay-off is a standard normal distribution S. Suppose that a new asset arises in the market. Its final pay-off g is also a standard normal distribution, and S and g are independent. Suppose also that a trader is interested in selling g and he uses $\rho = \sigma_2 - \mathbb{E}$. Then, bearing in mind (32),

$$\pi_\rho(g) = \text{Min}\left\{\sqrt{y_1^2 + 1} - y_0; y_0 \leqslant 0,\ y_1 \in \mathbb{R}\right\} = 1.$$

Moreover, the solution (optimal hedging portfolio) is $y_0 = y_1 = 0$.

It is clear that there are no reasons to pay one dollar for g if it is similar to S and the price of S vanishes. But the trader cannot adequately hedge his position if he sells for zero dollars. Notice that the optimal hedging strategy does not use S ($y_1 = 0$), which implies that S is useless if one wants to draw on it in order to hedge the sale of g. The independence between S and g makes it impossible to reduce the risk level of the sale of g by using S.[15]

Similarly, it is easy to see that $\pi_\rho(-g) = 1$, *i.e.*, the bid price of g is $-\pi_\rho(-g) = -1$. Consequently, if the trader is interested in buying g he will accept to pay -1 dollars. If he pays more his protection is not guaranteed, since the final value of g could be very negative, and he could lose a lot of money.

Summarizing, the null correlation between S and g provokes real quotes of g given by "*bid* $= -1$" and "*ask* $= 1$", and the huge "*bid/ask_spread* $= 2$".[16] Only the bid/ask average value vanishes, and therefore it equals the price of S.

It is important to remark that there are no asymmetries or fat tails involved in the example, so $\sigma_2 - \mathbb{E}$ is a "good" measure of risk, in the sense that it respects the *SOSD* (Ogryczak and Ruszczynski, 1999). However, $\sigma_2 - \mathbb{E}$ is not coherent in the sense of Artzner et al. (1999), since Δ_ρ, given by (35), does not satisfy (13). Actually $\sigma_2 - \mathbb{E}$ is not decreasing, which means that higher wealth does not necessarily imply lower risk, and this might be a drawback in some applications.

However, the huge spread above is not provoked by the risk function we are using. Indeed, consider the same data but take $\rho = CVaR_{\mu_0}$. Thus we are dealing with a Coherent and Expectation Bounded risk measure that also respects the *SOSD* and the classical Utility Functions (Ogryczak and Ruszczynski, 2002; Rockafellar et al., 2006a; Mansini et al., 2007, etc.). Without loss of generality we can consider that $\Omega = (0,1)^2$, μ is the Lebesgue measure, $S = N^{-1}(\omega_1)$ and $g = N^{-1}(\omega_2)$,

$$N : \mathbb{R} \to (0,1)$$

being the cumulative distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$$

of the standard normal distribution. In order to compute $\pi_\rho(g)$ let us apply Corollary 14. So, take $\Omega_0 = (0,1) \times (\mu_0, 1)$, $\Omega_{\mu_0} = (0,1) \times (0, \mu_0)$, and

$$z^* = \begin{cases} 0 & (\omega_1, \omega_2) \in \Omega_0, \\ 1/\mu_0 & (\omega_1, \omega_2) \in \Omega_{\mu_0}. \end{cases}$$

y^* is the constant (zero-variance) random variable $y^* = N^{-1}(\mu_0)$. It is easy to see that $z^* \in D_f$. Indeed, $\mathbb{E}(z^*) = 1$ is obvious, and

$$0 = \mathbb{E}(S) = \int_0^1 N^{-1}(\omega_1) d\omega_1$$

leads to

$$\mathbb{E}(z^* S) = \frac{1}{\mu_0} \int_0^{\mu_0} \left(\int_0^1 N^{-1}(\omega_1) d\omega_1 \right) d\omega_2 = 0.$$

To check the conditions of Corollary 14 it only remains to show that $y^* \leqslant g$ on Ω_0 and $y^* \geqslant g$ on Ω_{μ_0}. If $(\omega_1, \omega_2) \in \Omega_0$ then $\omega_2 > \mu_0$, which implies that $N^{-1}(\omega_2) > N^{-1}(\mu_0)$, *i.e.*, $g > y^*$. The other inequality is analogous, and therefore $(\pi(y^*), y^*)$ solves (17). Then, since $\pi(y^*) = y^*$ (the risk free rate vanishes), (30) implies that

$$\pi_\rho(g) = CVaR_{\mu_0}(N^{-1}(\mu_0) - g) + N^{-1}(\mu_0) = CVaR_{\mu_0}(g).$$

Once again g is more expensive than S, despite they have the same distribution. Moreover the optimal hedging strategy does not use S. It is also possible to prove that $\pi_\rho(-g) = CVaR_{\mu_0}(g)$, and thus the bid/ask spread $2CVaR_{\mu_0}(g)$ is "too high" for a "realistic" value of the level of confidence $1 - \mu_0$. Only the bid/ask average value vanishes and equals the price of S. We have parallel results for both the Standard Deviation and the Conditional Value at Risk. Furthermore, if we drew on (26) rather than Corollary 14 then we could show that the properties of the example are very robust with respect to the risk function ρ. We will not do that in order to shorten the exposition.

7.1. Pricing variance swaps

Let us analyze a second example reflecting a close relationship between those securities in Y and the new asset g to be priced and hedged. Variance and volatility linked derivatives are becoming very used in practice because they provide investors with new ways to diversify their portfolios. Besides, they are useful when facing market turmoils. Interesting studies may be found in Demeterfi et al. (1999) and Broadie and Jain (2008), among many others.

[15] Recall that we are using risk functions that can be interpreted in monetary terms, or as initial capital requirements.

[16] It is easy to show that the spread becomes higher as the variance of g grows.

Important particular cases are the variance and the volatility swap. Mainly, agents fix the underlying asset and the period $[0,T]$. At $t = 0$ they buy (or sell) the realized variance (volatility) for a given price W_0. At $t = T$ they know the trajectory reflected by the underlying asset, so they can compute the realized variance (volatility) W_T. If they bought then they will receive $\Phi(W_T - W_0)$ dollars (this quantity may be negative), $\Phi > 0$ being a known "price per variance (volatility) point".

There is a growing interest in new methods allowing us to price and hedge these products. With respect to the variance swap Demeterfi et al. (1999) used classical arbitrage arguments and proved that the variance swap current price must equal the price of the "log contract", *i.e.*, the price of the asset paying at T the amount

$$g(X_T) = \frac{\Phi}{T}\left[\frac{X_T}{F_0^T} - 1 - L\left(\frac{X_T}{F_0^T}\right)\right], \tag{39}$$

X_T being the final (at T) underlying asset price, and F_0^T being the price at $t = 0$ of the forward contract with maturity at T. If X_0 is the current underlying asset price and there are no dividends in the indicated period then $F_0^T = X_0 e^{r_f T}$, expression that must be slightly modified when dividends are considered. The result of Demeterfi et al. (1999) holds under quite general assumptions, and does not depend on the underlying asset behavior. It applies for the Black and Scholes model, for the Heston model, and for many other stochastic volatility models that do not reflect jumps in the volatility. Since g is obviously two times differentiable, the final pay off $g(X_T)$ may be replicated in a static framework. The replica will incorporate infinitely many European options because we will have to deal with all the strikes lying within the interval $(0,\infty)$. The number of options per strike depends on the second derivative $g''(X_T)$, and Demeterfi et al. (1999) showed that (39) is replicated by

$$\begin{aligned}&\text{Purchasing } \frac{\Phi}{T}\frac{1}{k^2}\,dk \text{ European Puts with strike } k, \quad 0 < k < F_0^T,\\ &\text{Purchasing } \frac{\Phi}{T}\frac{1}{k^2}\,dk \text{ European Calls with strike } k, \quad F_0^T < k < \infty.\end{aligned} \tag{40}$$

The put/call parity points out that (39) may be also replicated with a static strategy containing only puts or calls, along with the underlying asset (or the forward contract) and the riskless security.

The strategy above has significant advantages. It does not depend of the model for the underlying asset dynamic behavior, it provides a hedging portfolio composed of European options, and the price of the variance swap may be computed by using real market data rather than model linked parameters, since real quotes of European options are usually available. However, there is a caveat since it is impossible to buy "infinitely many options". Thus, Demeterfi et al. (1999) provided a pseudo replica that draws on the available options, though they did not compute the quality of the approximation.

The approach of Demeterfi et al. (1999) is extended in Broadie and Jain (2008) (among others), where the authors use and minimize the Standard Deviation so as to measure the degree of approximation between the variance swap and the strategy of (finitely many) European options. Moreover, these authors extend the discussion and show that the volatility swap may be also priced and hedged by using infinitely many options, though in this new case they must assume the Heston model to explain the underlying asset evolution. Once again they use the standard deviation in order to hedge the volatility swap with the available (finite) options.

The approach of Broadie and Jain (2008) is very interesting but there are three ideas that may be considered. Firstly, the Standard Deviation does not provide information in monetary terms, that will be only given by $\sigma_2 - \mathbb{E}$. Secondly, and much more importantly, the variance (volatility) swaps and calls and puts obviously present asymmetric returns (and heavy tails) which implies that the standard deviation is not compatible with the *SOSD* and the usual utility functions. Thirdly, as said above, $\sigma_2 - \mathbb{E}$ is not coherent.

The theory developed in this paper may overcome the caveats above, since one can use a risk function ρ reflecting capital requirements,[17] compatible with the *SOSD*, and coherent.[18] Despite (17) may be quite complex due to the absence of differentiability (see Rockafellar et al., 2006b, and Ruszczynski and Shapiro, 2006), and we have the same problem if we use (31) or (32), (26) provides necessary and sufficient optimality conditions that will apply. Moreover, (24) will be often linear, which allows us to solve it by using several algorithms even if we deal with continuous distributions and sets Ω composed of infinitely many states (see Balbás et al., 2009). Obviously, the enormous bid/ask spread of the example of the previous subsection will not arise here, since (40) shows a close relationship between the variance swap contract and the options, and therefore there is a strong dependence between their behaviors.

Obviously, the valuation of volatility linked securities is beyond the scope of this paper, which is in the realm of theoretical finance and applications of optimization theory. Nevertheless, for illustrative reasons, we have checked a numerical example. For the sake of simplicity let us assume that $\Phi = T = 1$ and X_T can only achieve an integer value lying within the interval [1,10]. Thus Ω is composed of those integers ω such that $1 \leqslant \omega \leqslant 10$. The probability μ is given by $\mu(\omega) = 0.1$. There are six available securities, the riskless asset, the underlying asset, and four European calls with maturity at $T = 1$ and strikes 3, 5, 7 and 9 respectively. The interest rate vanishes and the market is risk neutral, that is, the underlying asset price equals 5.5 and the option prices are 2.8, 1.5, 0.6 and 0.1 respectively. Then, if $\rho = CVaR_{0.34}$ and g is given by (39), we have that (24) is a linear problem that may be easily solved by a simplex method. Its solution provides the ask price of the variance swap that equals 0.422. Besides, the solution of (24) if $-g$ replaces g provides a bid price of g equal to 0.359. The bid/ask spread equals 0.063.

8. Conclusions

This paper has proposed a new method to extend pricing rules in both incomplete and imperfect markets by using general risk functions, with special focus on Expectation Bounded Risk Measures and General Deviation Measures.

The pricing rule extension draws on a mathematical programming problem that takes the point of view of the trader and minimizes the cost of the hedging strategy plus the initial capital requirement indicated by the selected risk function, *i.e.*, we minimize the initial capital needed by the trader.

Since the minimization of risk functions may be a very complicated problem due to the lack of differentiability, the paper has also presented a duality theory that solves this caveat for the optimization problem we have to deal with. Primal and dual solutions have been characterized by practical conditions, as it has been illustrated with numerical examples.

The proposed pricing rule presents some properties that may deserve to be pointed out. Firstly, it is sub additive and homogeneous. Secondly, it reduces the bid/ask spread if the initial market reflects frictions, and it is a genuine extension of the initial pricing rule otherwise. Thirdly, the proposed pricing rule prevents the

[17] In this case it may be very useful to measure the committed error in monetary terms (potential losses), since it is impossible to reach a perfect hedging with the available options.

[18] The *CVaR* satisfies these requirements, but it is not the only one.

existence of arbitrage under weak assumptions about the initial market.

Some special attention has been paid to the Expectation Bounded Risk Measure generated by the Standard Deviation. Some relationships with other pricing rule extensions presented in the literature and related to the Standard Deviation have been also analyzed.

The major findings of the paper have been particularized for the Conditional Value at Risk (*CVaR*), since it is becoming a risk mea sure very frequently used in Finance. Moreover, some applications to price volatility linked derivatives have been discussed.

Most of the developed theory strongly depends on the duality properties of the Convex Optimization Theory in Banach spaces, so the paper points out once again how Mathematical Program ming may play a crucial role in Asset Pricing and Hedging, two ma jor topics in Finance.

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The usual caveat applies.

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